Subfactors and Topological Defects in Conformal Quantum Field Theory*

Marcel Bischoff
http://www.math.vanderbilt.edu/~bischom

Department of Mathematics
Vanderbilt University
Nashville, TN

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*based on work with R. Longo, Y. Kawahigashi and K.-H. Rehren arXiv:1405.7863, arXiv:1407.4793, (see also: arXiv:1410.8848)

- Algebraic quantum field theory: A family of von Neumann algebras containing all local observables/operations associated with space-time regions.
- Conformal Quantum Field Theory (CQFT) in 1 and 2 dimension described by AQFT quite successful, e.g. partial classification results (e.g. $c < 1$ (Kawahigashi, Longo '04)).
- Topological Field Theory (TFT) construction of CFT on Riemann surfaces with boundaries/defects (Fuchs, Runkel, Schweigert '02+).

Conformal Nets

Boundaries / Defects

Classification (modular case)

Conformal net $\mathcal{A}_0$ on $S^1 \cong \overline{\mathbb{R}}$ (compactified light-ray):

$$S_1 \supset I \mapsto \mathcal{A}_0(I) \subset \mathrm{B}(\mathcal{H}), \qquad \mathcal{H} \text{ fixed Hilbert space}$$

1. Isotony: $I \subset J \Rightarrow \mathcal{A}_0(I) \subset \mathcal{A}_0(J)$
2. Locality: $[\mathcal{A}_0(I), \mathcal{A}_0(J)] = \{0\}$ if $I \cap J = \emptyset$.
3. Covariance: U is a unitary **positive-energy** representation of the Möbius group/diffeomorphism group, s.t. $U(g)\mathcal{A}_0(I)U(g)^* = \mathcal{A}_0(gI)$.
4. Vacuum: $\exists\ \Omega$ is a (up to a phase) unique vector invariant under the Möbius group.

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Representation of $\mathcal{A}_0 = \{\mathcal{A}_0(I)\}_{I\subset S^1}$ is a family:

$$\pi = \{\pi_I\colon \mathcal{A}_0(I) \to \mathrm{B}(\mathcal{H}_\pi)\}\,,$$

which is compatible, i.e. $\pi_J \restriction \mathcal{A}_0(I) = \pi_I$ for $I \subset J$.

- For all I_0 exists a $\rho \cong \pi$ on $\mathcal{H}$, s.t. $\rho_I = \mathrm{id}_{\mathcal{A}_0(I)}$ for $I \cap I_0 = \emptyset$.
- ρ_J is localized DHR endomorphism: $\rho_J(\mathcal{A}_0(J)) \subset \mathcal{A}_0(J)$ for all $J \supset I_0$.
- Tensor product: composition of localized endomorphisms.
- $\mathrm{Rep}^I(\mathcal{A}_0) \subset \mathrm{End}(\mathcal{A}_0(I))$ (full and replete).
- $\exists$ natural braiding $\{c_{\rho,\sigma}\colon \rho\circ\sigma \to \sigma\circ\rho\}$ (Fredenhagen, Rehren, Schroer (1989)).

Theorem ((Kawahigashi, Longo, Müger (2001)))

Let $\mathcal{A}_0$ be a **completely rational** *conformal net.* ‣ Complete Rationality
Then $\mathrm{Rep}(\mathcal{A}_0)$ is a **modular C*-tensor category** = **unitary modular tensor category (UMTC)**.

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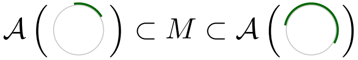
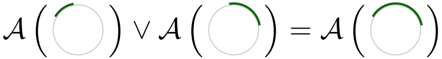
Then $\mathrm{Rep}(\mathcal{A}_0)$ is a **modular $\mathbf{C}^*$-tensor category = unitary modular tensor category (UMTC)**.

Completely rational conformal net (Kawahigashi, Longo, Müger (2001))

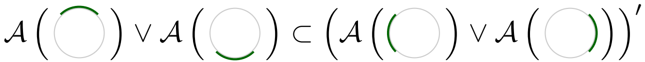
- Split property. For every relatively compact inclusion of intervals $\exists$ intermediate **type** I **factor** M



- Strong additivity. Additivity for touching intervals:



- Finite μ-index: finite Jones index of subfactor



where the intervals are splitting the circle.

Loop group net of $\mathrm{SU}(N)$ at level k: (see (Wassermann '98))

$$\mathcal{A}_{\mathrm{SU}(N),k}(I) = \pi(\mathrm{L}_I\mathrm{SU}(N))'' \qquad \text{(completely rational (Xu '00))}$$

with π level k vacuum PER of **loop group** $\mathrm{LSU}(N) = \mathrm{C}^\infty(S^1, \mathrm{SU}(N))$.

Example $G = \mathrm{SU}(2)$**:** Irreducible representations $\{0, \frac{1}{2}, 1, \ldots, \frac{k}{2}\}$.
$\mathrm{Rep}(\mathcal{A}_{\mathrm{SU}(2),k})$ is generated by $\frac{1}{2}$-representation ρ and $\cup \in \mathrm{Hom}(\mathrm{id}, \rho\rho)$:

 $= -d$ $=$ $=$

with $\cap \in \mathrm{Hom}(\rho\rho, \mathrm{id})$ and **braiding** defined by **Kaufmann bracket**

 $:= -$ * $= q^{\frac{1}{2}}$ $+ q^{-\frac{1}{2}}$

where $q = e^{\frac{\mathrm{i}\pi}{k+2}}$, $d = q + q^{-1} = 2\cos\left(\frac{\pi}{k+2}\right)$.

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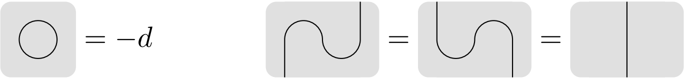

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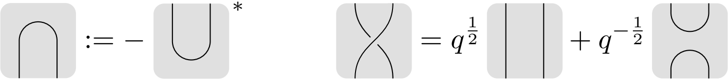

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Theorem (well-known: Popa, ..., Hayashi, Yamagami)

Every finite rigid C^* tensor category $\mathcal{C}$ is realizable as $\mathcal{C} \subset \mathrm{End}_0(N)$ with N a III_1 factor.*

The strict rigid C^* tensor category $\mathrm{End}_0(N)$, with N a type III factor.

- Objects: Endomorphisms $\rho\colon N \to N$ with finite index $[N : \rho(N)] < \infty$.
- Morphisms: $t\colon \rho \to \sigma$ is a $t \in N$, such that $t\rho(x) = \sigma(x)t$ for all $x \in N$.
- Tensor product: $\rho \otimes \sigma := \rho \circ \sigma$ (composition of endomorphisms)
- Direct sums: $\rho \oplus \sigma = \mathrm{Ad}\, t_1 \circ \rho + \mathrm{Ad}\, t_2 \circ \sigma$ with $t_i \in N$, s.t. $\sum_{i=1}^{2} t_i t_i^* = 1$, $t_i^* t_j = \delta_{i,j}$ (Cuntz algebra).
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- Let $\mathcal{C} \subset \mathrm{End}_0(N)$ (full and replete) be a braided fusion category.
- Let $N \subset M$ be a finite index subfactor. Then exists the dual morphism $\bar{\iota}\colon M \to N$ of the canonical inclusion $\iota\colon N \to M$.
- If the **dual canonical endomorphism** $\theta = \bar{\iota} \circ \iota\colon N \to N$ is in $\mathcal{C}$, we call the pair $(N \subset M, \mathcal{C})$ a **braided subfactor**. *.
- θ has canonically the structure of a Q-system (algebra object) in $\mathcal{C}$. Actually, there is a one-to-one correspondence between (Longo '94):

$$\boxed{\text{Q-sytems in } \mathcal{C}} \longleftrightarrow \boxed{\text{braided subfactors } (N \subset M, \mathcal{C})}$$

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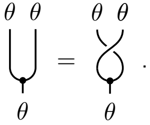

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$$N \begin{array}{c} \subset \\ \subset \end{array} \begin{array}{c} M_+ \\ M_- \end{array} \begin{array}{c} \subset \\ \subset \end{array} M$$

- Given $N \subset M_a, M_b$ there is a von Neumann algebra with finite center $M = M_a \times^{\pm}_{N} M_b \supset N$:

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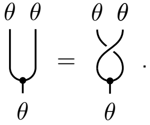

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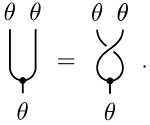
- $(N \subset M, \mathcal{C})$ is called **local** if the Q-system is commutative:



- $(N \subset M, \mathcal{C})$ then exist local subfactors $(N \subset M_\pm, \mathcal{C})$, s.t.

$$N \begin{array}{c} \subset \\ \subset \end{array} \begin{array}{c} M_+ \\ \\ M_- \end{array} \begin{array}{c} \subset \\ \subset \end{array} M$$

- Given $N \subset M_a, M_b$ there is a von Neumann algebra with finite center $M = M_a \times_N^\pm M_b \supset N$:

$$N \begin{array}{c} \subset \\ \subset \end{array} \begin{array}{c} M_a \subset M_\pm \\ \\ M_b \subset M_\mp \end{array} \begin{array}{c} \subset \\ \subset \end{array} M \equiv M_a \times_N^\pm M_b$$

Given a (conformal) net $I \mapsto \mathcal{A}_0(I)$.

There is a one-to-one correspondence (up to equivalence) between:

- Finite index **(non-local) extensions** $\mathcal{B}_0$, i.e. a net $\mathcal{B}_0(I) \supset \mathcal{A}_0(I)$, which is **relatively locality** w.r.t. $\mathcal{A}$, i.e. $[\mathcal{A}_0(I_1), \mathcal{B}_0(I_2)] = \{0\}$ for I_1, I_2 disjoint.
- Braided subfactors $(N \subset M, \mathcal{C})$, where $N = \mathcal{A}(I)$ and $\mathcal{C} = \mathrm{Rep}^I(\mathcal{A}_0)$.

$\boxed{\mathcal{B}_0 \text{ is a } \textbf{local} \text{ net}} \iff \boxed{(N \subset M, \mathcal{C}) \text{ is } \textbf{local}}$

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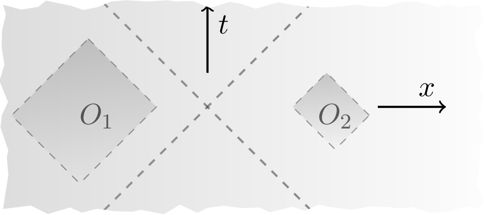
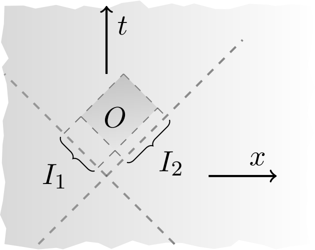
Conformal net $\mathcal{A}$ on Minkowski space $\mathbb{R}^{1,1}$:

$$\mathbb{R}^{1,1} \supset O \longmapsto \mathcal{A}(O) \subset \mathrm{B}(\mathcal{H}_\mathcal{A}), \qquad \mathcal{H}_\mathcal{A} \text{ fixed Hilbert space}$$

1. Isotony: $O_0 \subset O_1 \Rightarrow \mathcal{A}(O_0) \subset \mathcal{A}(O_1)$
2. Locality: $[\mathcal{A}(O_1), \mathcal{A}(O_2)] = \{0\}$ if O_1 and O_1 spacelike separated:



3. Covariance: $U_\mathcal{A}$ is a unitary **positive-energy** representation of the "2D conformal group", s.t. $U_\mathcal{A}(g)\mathcal{A}(O)U_\mathcal{A}(g)^* = \mathcal{A}(gO)$.
4. Vacuum: $\exists\ \Omega$ is unique translation invariant vector.



One can define a **chiral conformal net** on **Minkowski space** by

$$\mathcal{A}(O) = \mathcal{A}_+(I_1) \otimes \mathcal{A}_-(I_2)$$

where $\mathcal{A}_\pm$ are conformal nets on $\mathbb{R}$.

Non-chiral nets are given by local extensions

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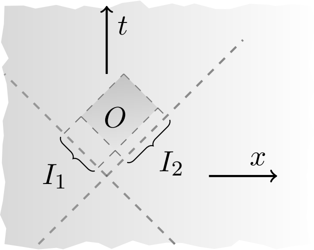

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Left observables $\mathcal{B}_{\mathrm{L}} \supset \mathcal{A}$. Right observables $\mathcal{B}_{\mathrm{R}} \supset \mathcal{A}$.

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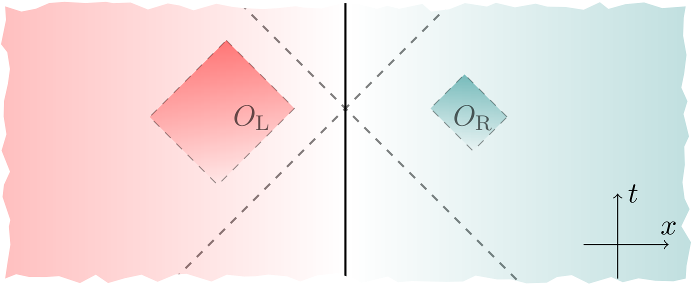

- Locality: $[\mathcal{B}_{\mathrm{L}}(O_{\mathrm{L}}), \mathcal{B}_{\mathrm{R}}(O_{\mathrm{R}})] = \{0\}$ for O_{L} spacelike left of O_{R}.
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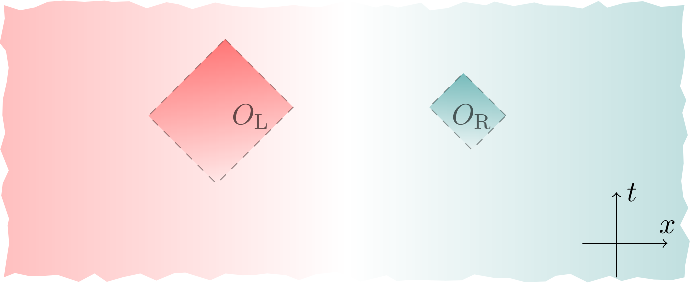

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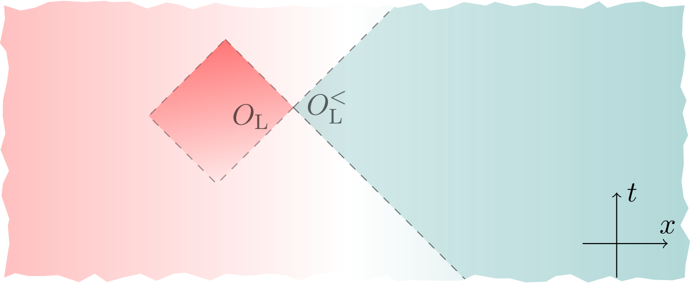

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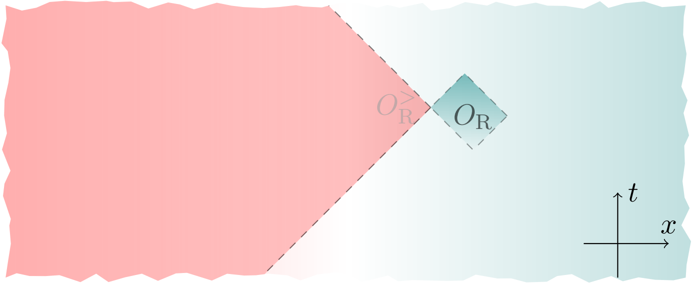

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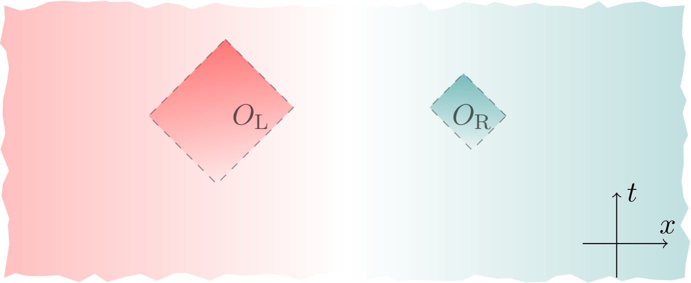

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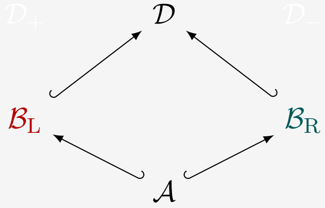
An $\mathcal{A}$–topological $\mathcal{B}_{\mathrm{L}}$–$\mathcal{B}_{\mathrm{R}}$-defect $\mathcal{D}$ is a (non-local) extension $\mathcal{D} \supset \mathcal{A}$ such that:



$$\mathcal{C}_{\mathrm{L}}(O)\mathcal{D}(O) \cap \mathcal{D}(O^<)' \equiv \mathcal{D}(O)_+ \qquad \mathcal{C}_{\mathrm{R}}(O)\mathcal{D}(O) \cap \mathcal{D}(O^>)' \equiv \mathcal{D}(O)_-$$

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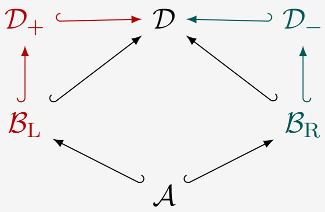

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Do such defects exist in general? Can we classify them?

Universal construction

Theorem

If $\mathcal{B}_{\mathrm{L}} \supset \mathcal{A}$ and $\mathcal{B}_{\mathrm{R}} \supset \mathcal{A}$ are local, irreducible, finite index extensions then there exists a unique $\mathcal{A}$–topological $\mathcal{B}_{\mathrm{L}}$–$\mathcal{B}_{\mathrm{R}}$ defect $\mathcal{D}_{\mathrm{univ}}$ with the above properties. Its central decomposition gives all irreducible defects.

- Universal $\mathcal{A}$–topological $\mathcal{B}_{\mathrm{L}}$–$\mathcal{B}_{\mathrm{R}}$ defect $\mathcal{D}_{\mathrm{univ}}$ is given by the extension $\mathcal{A} \subset \mathcal{B}_{\mathrm{L}}, \mathcal{B}_{\mathrm{R}} \subset \mathcal{D}_{\mathrm{univ}}$ characterized by (braided product):

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Conformal Nets

Boundaries / Defects

Classification (modular case)

Now we choose $\mathcal{A}_\pm = \mathcal{A}_0$ (parity symmetry) and $\mathcal{A}_0$ to be a **completely rational** conformal net. Further, we now choose the irreducible local extensions

$$\mathcal{B}_{\mathrm{L}}, \mathcal{B}_{\mathrm{R}} \supset \mathcal{A} = \mathcal{A}_0 \otimes \mathcal{A}_0$$

to be **maximal**. We call $\mathcal{B}_{\mathrm{L}}, \mathcal{B}_{\mathrm{R}}$ full CFT.

Theorem ((Rehren, Müger, Kawahigashi–Longo, Kong–Runkel, B–K–L, ...))

Let $\mathcal{A} \equiv \mathcal{A}_0 \otimes \mathcal{A}_0$ with $\mathcal{A}_0$ completely rational. There is a one-to-one correspondence:

- *Full CFTs $\mathcal{B} \supset \mathcal{A}$, and*
- *Morita equivalence classes of a braided subfactor $(N \subset M, \mathcal{C})$, with $N = \mathcal{A}_0(I)$, $\mathcal{C} = \mathrm{Rep}^I(\mathcal{A}_0)$.*

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Let $N = \mathcal{A}_0(I)$ and $\mathcal{C} = \mathrm{Rep}^I(\mathcal{A}_0)$. Then the full CFTs $\mathcal{B} \supset \mathcal{A}$ are given by Morita equivalence classes of braided subfactors $(N \subset M, \mathcal{C})$.

Example (A, D, E classification for $\mathrm{SU}(2)_k$ (Ocneanu))

Let $\mathcal{A}_0 = \mathcal{A}_{\mathrm{SU}(2),k}$, then the Morita equivalence classes of braided subfactors $(N \subset M, \mathcal{C})$ are in correspondence given by A, D, E Coxeter diagrams with Coxeter number $k + 1$.
A_n (trivial subfactor), D_{even} and $E_{6,8}$ are realizable by local subfactors.

Let $N = \mathcal{A}_0(I)$ and $\mathcal{C} = \mathrm{Rep}^I(\mathcal{A}_0) \subset \mathrm{End}(N)$

- Given $(N \subset M_\mathrm{L}, \mathcal{C})$ and $(N \subset M_\mathrm{R}, \mathcal{C})$ braided subfactor.
- They give full CFTs $\mathcal{B}_\mathrm{L}, \mathcal{B}_\mathrm{R} \supset \mathcal{A} \equiv \mathcal{A}_0 \otimes \mathcal{A}_0$ on Minkowski space.
- $\iota_\bullet \colon N \to M_\bullet$ inclusion map, $\bar\iota_\bullet \colon M_\bullet \to N$ conjugate

Theorem

There is a one-to-one correspondence between: minimal central projections $e \in \mathcal{D}_\mathrm{univ}(O)' \cap \mathcal{D}_\mathrm{univ}(O)$ and irreducible sectors $[\beta]$ with

$$\beta \prec \iota_\mathrm{L} \circ \rho \circ \bar\iota_\mathrm{R} \colon M_\mathrm{L} \leftarrow M_\mathrm{R}\,, \qquad \rho \in \mathcal{C}$$

Classification by **braided subfactor data:** $M_\mathrm{L} \supset N \subset M_\mathrm{R}$, $\mathcal{C} \equiv \mathrm{Rep}^I(\mathcal{A}_0)$.

- *# boundaries* $= \mathrm{tr}(Z_\mathrm{L} Z_\mathrm{R}^t)$
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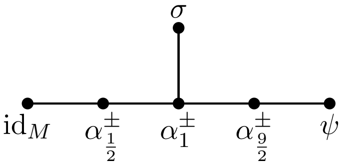
$M_{\mathrm{L}} = M_{\mathrm{R}} = M$ with

$$N = \mathcal{A}_{\mathrm{SU}(2),10}(I) \subset \mathcal{A}_{\mathrm{Spin}(5),1}(I) = M\,.$$

Then there are 12 irreducible defects corresponding to the sectors:

$$\underbrace{\left\{\mathrm{id}_M, \psi, \sigma, \alpha^{\pm}_{\frac{1}{2}}\right\}}_{\mathrm{Rep}^I(\mathcal{A}_{\mathrm{Spin}(5),1})} \cup \left\{\alpha^{\pm}_{1}, \alpha^{\pm}_{\frac{9}{2}}, \alpha^{+}_{\frac{1}{2}}\alpha^{-}_{\frac{1}{2}}, \alpha^{+}_{\frac{1}{2}}\alpha^{-}_{1} = \alpha^{+}_{1}\alpha^{-}_{\frac{1}{2}}, \alpha^{+}_{\frac{9}{2}}\alpha^{-}_{1} = \alpha^{+}_{1}\alpha^{-}_{\frac{9}{2}}\right\}$$



Summary

- Boundaries: **locality** and inivisiblity for subnet $\mathcal{A}$ (**conservation**).
- Existence of **universal defect** for finite index case.
- Classification of irreducible boundaries by "**chiral data**" in the rational maximal case.

Open problems

- **Fusion** of phase boundaries and defects.
- Phase boundaries without assuming conformal symmetry.
- **Deformations** of defects and relations to integrable QFT.

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  Nuclear Phys. B, 646(3):353–497, 2002.

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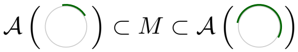
Commun. Contemp. Math., 2(3):307–347, 2000.

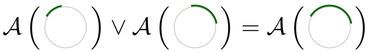
Complete rationality

Completely rational conformal net (Kawahigashi, Longo, Müger (2001))

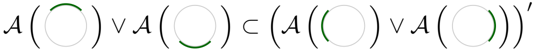
- Split property. For every relatively compact inclusion of intervals $\exists$ intermediate **type** I **factor** M



- Strong additivity. Additivity for touching intervals:



- Finite μ-index: finite Jones index of subfactor



  where the intervals are splitting the circle.

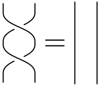
Consequences

- Only finite sectors, each sector has finite statistical dimension
- **Modularity:** The category of DHR sectors is modular, i.e. non degenerated braiding.

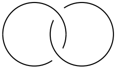
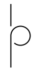
return

If the net $\mathcal{A}$ is completely rational then $\mathrm{Rep_f}(\mathcal{A})$ is a modular C*-tensor category (unitary MTC):

1. **Finite #** of **sectors**.
2. The **braiding** is **non-degenerated**, i.e.

$$\varepsilon(\rho,\sigma)\varepsilon(\sigma,\rho) \equiv$$  $$= 1 \text{ for all } \rho \quad \Longrightarrow \quad [\sigma] = N[\mathrm{id}]$$

identity is the only *transparent* object, with respect to the braiding or equivalently S-matrix (Rehren) is unitary:

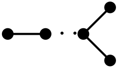
$$S_{\rho\sigma} \sim \rho$$  $$\sigma\ ; \qquad T_{\rho\rho} \sim$$  $$= \text{conformal spin}$$

$$SS^* = TT^* = 1, \qquad (ST)^3 = S^2, \qquad S^4 = 1$$

Unitary representation of the "modular group" $\mathrm{SL}(2,\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$.

A_n :

D_n :



$E_{\{6,7,8\}}$: