Generalized Orbifolds in Algebraic Conformal QFT

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In binding together elements long-known but heretofore scattered and appearing unrelated to one another, it suddenly brings order where there reigned apparent chaos — Henri Poincaré
Motivation

Problem

Given a local net $B$ characterize all irreducible subnets $A \subset B$, i.e. subnets with $A(O)' \cap B(O) = \mathbb{C}$.

Higher dimensions (Doplicher–Haag–Roberts): Given $B$

- $\text{Rep}(B)$ is a rigid symmetric $C^*$–tensor category and by Doplicher–Roberts duality braided equivalent to $\text{Rep}^k(G)$.
- $(G, k)$ supergroup, i.e. $G$ compact group, with $k \in Z(G)$ and $k^2 = e$.
- There is a unique $\mathbb{Z}_2$-graded “field net” $\mathcal{F} \supset B$, with an action of $G$, such that $B = \mathcal{F}^G$ and $B(O)' \cap \mathcal{F}(O) = \mathbb{C}$.
- Given $A \subset B$ irreducible, $\mathcal{F}$ is also a field net for $A$ and it follows, that $A = \mathcal{F}^H$ for some compact group $H$ with $G \subset H \subset \text{Aut}(\mathcal{F})$.

Since $A$ and $B$ have the same field net they describe “similar” physics.
Space-time dimension $d \leq 2$:

- $\text{Rep}(\mathcal{B})$ is a braided (sometimes rigid, basically never symmetric) $C^*$–tensor category but there is no nice duality theory (sometimes weak $C^*$–Hopf algebra).
- There is no gauge group.
- There is no field net (sometimes a field bundle).
- Non-integer dimensions $d \in \{2 \cos(\pi/n) : n \geq 3\} \cup [2, \infty]$.

**Example**

Let $\mathcal{B}$ with $\text{Rep}(\mathcal{B}) \cong \langle \text{id} \rangle$ (vacuum is the only sector) and $G$ a finite group in $\text{Aut}(\mathcal{B})$:

- $d > 2 \quad \text{Rep}(\mathcal{B}^G) \cong \text{Rep}(G)$, in particular \[ \dim(\text{Rep}(\mathcal{B}^G)) = \sum_{\rho \in \text{Irr}} d\rho^2 = |G|, \]
- $d \leq 2 \quad \text{Rep}(\mathcal{B}^G) \cong \text{Rep}(D^\omega(G))$ for some $[\omega] \in H^3(G, \mathbb{T})$, in particular not determined by $G$ and \[ \dim(\text{Rep}(\mathcal{B}^G)) = \sum_{\rho \in \text{Irr}} d\rho^2 = |G|^2. \]
The strict rigid \( C^\ast \)-tensor category \( \text{End}_0(N) \), with \( N \) a type III factor.

- **Objects:** Endomorphisms \( \rho : N \to N \) with finite index \( [N : \rho(N)] < \infty \).
- **Morphisms:** \( t : \rho \to \sigma \) is a \( t \in N \), such that \( t\rho(x) = \sigma(x)t \) for all \( x \in N \).
- **Tensor product:** \( \rho \otimes \sigma := \rho \circ \sigma \) (composition of endomorphisms)
- **Unitarity** \( \sim \) (ess. unique) spherical structure (Longo–Roberts ’1997).

**Theorem (well-known: (Ocneanu ’88), (Popa ’95), . . . , (Hayashi–Yamagami ’00))**

*Every (amenable) rigid \( C^\ast \)-tensor category \( C \), is realizable as a full subcategory of \( \text{End}_0(N) \) with \( N \) a (hyperfinite) type III\(_1\) factor*

\( C \) is called a **unitary fusion category (UFC)** if \( \# \text{Irr}(C) < \infty \).
Conformal net axioms

A **conformal net** associates with every interval $I \subset S^1$ a von Neumann algebra on a fixed Hilbert space $\mathcal{H}$:

$$S^1 \ni I \mapsto \mathcal{A}(I) \subset B(\mathcal{H})$$

1. **Isotony:** $I \subset J \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)$

We often additionally assume **complete rationality** in the sense of (Kawahigashi–Longo–Müger ’01)
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3. **Covariance**: $U : G \to \mathcal{U}(\mathcal{H})$, s.t. $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$.

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3. **Covariance:** \( U : G \rightarrow U(\mathcal{H}) \), s.t. \( U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI) \).
4. **Vacuum:** Unique (up to phase) \( G \)-invariant unit vector \( \Omega \in \mathcal{H} \), s.t.

\[
\bigwedge_I \mathcal{A}(I)\Omega = \mathcal{H}.
\]

We often additionally assume **complete rationality** in the sense of (Kawahigashi–Longo–Müger ‘01)
Conformal net by example

**Loop group net** of $G$ at level $k$. $G$ compact Lie group, $L_G = C^\infty(S^1, G)$

$$A_{G_k}(I) = \pi_{0,k}(L_I G)'',$$  \quad $L_I G = \{ \gamma \in L_G : \text{supp} \gamma \subset I \}$

**Net associated with even lattice** $\Gamma \subset \mathbb{R}^n = \text{torus loop group } L_T$ with $T = \mathbb{R}^n / \Gamma$

$$A_L(I) = A_T(I)$$

**Virasoro net** with $c \in \left\{ 1 - \frac{6}{m(m+1)} : m = 3, 4, \ldots \right\} \cup [1, \infty)$:

$$\text{Vir}_c(I) = \pi_{c,0} \left( \text{Diff}_I^+(S_1) \right)''$$

**Constructions:**

- $\otimes$-product
- Coset construction
- **Orbifold construction** = fixed point by finite group $G$
- Mirror extensions
A representation $\pi \in \text{Rep}(A)$ of $A = \{A(I)\}_{I \subset S^1}$ is a family:

$$\pi = \{\pi_I : A(I) \to B(H_\pi)\},$$

which is compatible, i.e. $\pi_J \upharpoonright A_0(I) = \pi_I$ for $I \subset J$.

- Every $\pi$ unitarily equivalent to a Doplicher–Haag–Roberts localized endomorphism $\rho \in \text{End}(A(I))$.
- $\text{Rep}^I(A) \subset \text{End}(A(I))$ full and replete.
- $\leadsto$ C*-tensor category

- Assume finite dimensions $d_\rho \equiv [A(I) : \rho(A(I))]^{\frac{1}{2}} \leq \infty$
- $\leadsto$ rigidity + semisimplicity

- $\exists$ natural braiding $\left\{\varepsilon_{\rho,\sigma} : \rho \otimes \sigma \to \sigma \otimes \rho\right\}$ (Fredenhagen–Rehren–Schroer '89)

$$\varepsilon_{\rho,\sigma} = \bigotimes_{\rho \sigma} \leadsto \text{unitary ribbon category}$$
Complete rationality

**Theorem ((Kawahigashi–Longo–Müger ’01))**

Let $\mathcal{A}$ be a completely rational conformal net, then all irreducible representations have **finite dimensions** and $\mathcal{C} := \text{Rep}^I(\mathcal{A})$ is a unitary modular tensor category.

$\mathcal{C}$ braided UFC is **modular** $\iff$ trivial Müger center

\[
\mathcal{C}' \cap \mathcal{C} = \left\{ \rho \in \mathcal{C} : \begin{array}{c}
\rho \\
\rho \sigma
\end{array} = \begin{array}{c}
\sigma \\
\rho \sigma
\end{array} \text{ for all } \sigma \in \mathcal{C} \right\} = \langle \text{id} \rangle
\]

$\sim$ unitary representations of the **modular group** $\text{SL}(2, \mathbb{Z})$.

On the opposite **symmetric** means $\mathcal{C}' \cap \mathcal{C} = \mathcal{C}$. 
Extensions

**Theorem (Longo–Rehren ’95, Müger, B–Kawahigashi–Longo ’14)**

There is a one-to-one correspondence between

- commutative algebras $\theta$ in $\mathcal{C}$ and
- local, finite index extensions $\mathcal{B} \supset \mathcal{A}$.

If $\mathcal{A}$ is rational, then $\text{Rep}(\mathcal{B}) \cong \text{Mod}^{0}_{\text{Rep}(\mathcal{A})}(\theta)$, the category of local $\theta$-modules.

Note: In higher dimensions, $\text{Rep}(\mathcal{A})$ is symmetric and $\text{Mod}^{0}_{\text{Rep}(\mathcal{A})}(\theta) = \text{Mod}_{\text{Rep}(\mathcal{A})}(\theta)$.
Example \((\text{Rep}(\mathcal{A}_{\text{SU}(2)_k}))\)

Irreducible representations \(\{0, \frac{1}{2}, 1, \ldots, \frac{k}{2}\}\):

\[
[i] \times [j] = \bigoplus_{n=|i-j|} [n]
\]

\(\text{Rep}(\mathcal{A}_{\text{SU}(2)_k})\) is generated by \(\frac{1}{2}\)-representation \(\rho\) and \(\cup \in \text{Hom}(\text{id}, \rho \rho)\):

\[
\begin{align*}
\bigcirc & = -d \\
\begin{array}{c}
\hline
\end{array} & = \begin{array}{c}
\hline
\end{array} \\
\begin{array}{c}
\hline
\end{array} & = \begin{array}{c}
\hline
\end{array}
\end{align*}
\]

with \(\cap \in \text{Hom}(\rho \rho, \text{id})\) and braiding defined by the Kaufmann bracket

\[
\begin{align*}
\begin{array}{c}
\hline
\end{array} & := - \begin{array}{c}
\hline
\end{array}^* \\
\begin{array}{c}
\hline
\end{array} & = q^{\frac{1}{2}} \begin{array}{c}
\hline
\end{array} + q^{-\frac{1}{2}} \begin{array}{c}
\hline
\end{array}
\end{align*}
\]

where \(q = e^{\frac{i\pi}{k+2}}\), \(d = q + q^{-1} = 2 \cos \left(\frac{\pi}{k+2}\right)\).
Example ((Capelli–Itzykson–Zuber ’87), (Ocneanu))

\[ \mathcal{B} \supset \mathcal{A}_{\text{SU}(2)_k} \leftrightarrow^{1:1} \mathcal{A}, D, E \] Coxeter–Dynkin diagrams with Coxeter number \( k + 2 \).

- \( A_{k+1} \) \( \mathcal{A}_{\text{SU}(2)_k} \subset \mathcal{A}_{\text{SU}(2)_k} \)
- \( D_{2n} \) \( \mathcal{A}_{\text{SU}(2)_{4n-4}} \subset \mathcal{A}_{\text{SO}(3)_{2n-2}} \equiv \mathcal{A}_{\text{SU}(2)_{4n-4}} \times \mathbb{Z}_2 \) with representations \( \frac{1}{2} D_{2n} \)
- \( E_6, E_8 \) \( \mathcal{A}_{\text{SU}(10)_k} \subset \mathcal{A}_{\text{Spin}(5)_1} \) and \( \mathcal{A}_{\text{SU}(2)_{28}} \subset \mathcal{A}_{\text{G}_2,1} \), with Ising and Fib representations.
A completely rational net $\mathcal{B}$ is called **holomorphic** if $\text{Rep}(\mathcal{B}) = \langle \text{id} \rangle$, i.e. $\mathcal{B}$ has only the vacuum sector.

**Example**

- Conformal net associated with even self-dual lattice, i.e. $L \overset{!}{=} L^* := \{ x \in \mathbb{R}^n : \langle x, L \rangle \in \mathbb{Z} \}$. E.g. $E_8$, $E_8 \oplus E_8$, $D_{16}^+$, Leech, ...
- Loop group net of $E_8$ at level 1 $\mathcal{A}_{E_8,1}$ and $\mathcal{A}_{SO(32),1}$ (not s.c.).
- Moonshine net $\mathcal{A}^\#$
Let $G$ be a finite group acting properly on a net $\mathcal{B}$ then the fixed point net $\mathcal{B}^G$ is called the $G$-\textbf{orbifold net}.

\textbf{Question (Evans–Gannon ’11)}

Can we orbifold a VOA [or conformal net] by something more general than a group?

\textbf{Theorem (probably well-known, (B. ’16))}

Let $\mathcal{B}^K \subset \mathcal{B}$ be a finite index subnet for a Kac algebra (finite $C^*$-Hopf algebra) $K$, then $K$ is a finite group.
Stochastic Maps

Let \((M, \Omega)\) be a \textbf{non-commutative probability space}, i.e.

\begin{itemize}
  \item \(M = M'' \subset B(\mathcal{H})\) a von Neumann algebra.
  \item \(\Omega \in \mathcal{H}\) a cyclic and separating vector, i.e. \(M\Omega = M'\Omega = \mathcal{H}\).
\end{itemize}

A linear map \(\phi: (M_1, \Omega_1) \rightarrow (M_2, \Omega_2)\) is called a \textbf{stochastic map} if

\begin{itemize}
  \item \(\phi: M_1 \rightarrow M_2\) is \textbf{completely positive}, i.e. \(\phi \otimes 1_{M_n(\mathbb{C})}: M_1 \otimes M_n(\mathbb{C}) \rightarrow M_2 \otimes M_n(\mathbb{C})\) is positive for all \(n \geq 1\).
  \item \textbf{State-preserving}: \((\Omega_2, \phi(\cdot)\Omega_2) = (\Omega_1, \cdot \Omega_1)\).
  \item \textbf{Normal}: Continuous with respect to the ultraweak topology.
  \item \textbf{Unital}: \(\phi(1_{M_1}) = 1_{M_2}\).
\end{itemize}

An \textbf{adjoint} is a stochastic map \(\phi^\#: (M_2, \Omega_2) \rightarrow (M_1, \Omega_1)\), such that

\[(\phi^\#(m_2)\Omega_1, m_1\Omega_1) = (m_2\Omega_2, \phi(m_1)\Omega_2), \quad m_i \in M_i\]
Stochastic maps are quantum operations preserving a state.

**Theorem (Stinespring, Connes)**

$M$ type purely infinite and $\phi : M \to M$ stochastic map, then there is an isometry $v \in M$ and an endomorphism $\beta \in \text{End}(M)$, such that

$$\phi(\cdot) = v^* \beta(\cdot) v.$$ 

Conversely, $v$ an isometry and $\beta \in \text{End}(M)$, such that

$$(v\Omega, \beta(\cdot)v\Omega) = (\Omega, \cdot \Omega),$$

then $\phi(\cdot) = v^* \beta(\cdot)$ is a stochastic map.
Given a net \( \{ \mathcal{A}(I) \}, \Omega \) a family of stochastic maps \( \{ \phi_k^I : (\mathcal{A}(I), \Omega) : (\mathcal{A}(I), \Omega) : I \in \mathcal{I}, k = 0, \ldots, n \} \), such that

- **compatible:** \( \phi_k^I \upharpoonright \mathcal{A}(\tilde{I}) = \phi_{\tilde{I}}^I \)
- **unital:** \( \phi_0^I = \text{id}_{\mathcal{A}(I)} \)
- **closed under adjoints:** \( k \mapsto \overline{k} \) on \( \{1, \ldots, n\} \), such that \( \phi_k^\dagger = \phi_{\overline{k}} \)
- **closed under composition:** \( \phi_i \circ \phi_j = \sum_k C_{ij}^k \phi_k \) for some \( (C_{ij}^k \geq 0) \).
- **dual/weak inverse:** \( C_{ij}^0 > 0 \) if and only if \( j = \overline{i} \).
- **simplex:** The \( \phi_k \) are extremal and affine independent.

is called a **hypegroup action**. The set \( K = \{ c_0, \ldots, c_k \} \) with the coefficients \( C_{ij}^k \) is called a **hypergroup**. The span \( \mathbb{C}K \) has the structure of a finite \( C^* \)-algebra, generalizing the group algebra \( \mathbb{C}G \).
Example (Hypergroup)

\[ K_D = \{ \phi_0 = \text{id}, \phi_1 \} \text{ with } d \in [1, \infty) \text{ and} \]

\[ \phi_1 \circ \phi_1 = \frac{1}{d} \phi_0 + \frac{d - 1}{d} \phi_1 \quad \phi_0 \circ \phi_1 = \phi_1 = \phi_1 \circ \phi_0, \quad \phi_0 \circ \phi_0 = \phi_0. \]

\[ \phi_1 \circ \cdot : \text{Conv}(K) \to \text{Conv}(K) \text{ defines a Markov chain:} \]

Example

If \( d = 1 \) then \( K \cong \mathbb{Z}_2 \).

Haar element (invariant measure) \( E \in \text{Conv}(K) \):

\[ E(\cdot) := \frac{1}{d+1} [\phi_0(\cdot) + d\phi_1(\cdot)] \sim E \circ \phi_k = E = \phi_k \circ E, \quad E \circ E = E. \]
Example (Hypergroup action)

Let $G$ be a finite group of (first kind) gauge automorphism
\[ \{ \alpha^I_g : \mathcal{A}(I) \to \mathcal{A}(I) : I \in \mathcal{I}, g \in G \} \] (internal symmetries).

Theorem (B.)

Let $K$ be a hypergroup action on $\mathcal{B}$. Then $\mathcal{B}^K$ defined by the fixed-point

$$\mathcal{B}^K(I) := \mathcal{B}(I)^K = \{ b \in \mathcal{B}(I) : \phi_k^I(b) = b \text{ for all } k \in K \}$$

is a finite index subnet of $\mathcal{B}$, called the $K$-orbifold net of $\mathcal{B}$.

Example (Doplicher–Haag–Roberts, Rehren, Xu, Müger)

If $K = G$ is a group, then $\mathcal{B}^G$ is the $G$-orbifold\(^a\) net of $\mathcal{B}$.

\(^a\)the name orbifold has roots in string theory, where the fixed point with respect to a finite group has geometrical meaning.
– When is $\mathcal{A} = \mathcal{B}^K$ for some hypergroup action?

– How does the hypergroups $K$ abstractly look like?

– How does $\text{Rep}(\mathcal{B}^K)$ looks like in terms of $\text{Rep}(\mathcal{B})$?
Theorem

Let \( A \subset B \) with \([B : A] < \infty\), then there is a canonical hypergroup action \( K \), such that \( A = B^K \).

There is a Galois like correspondence between intermediate nets \( B^K \subset \tilde{A} \subset B \) and subhypergroups \( L \subset K \).

In this case \( A = \tilde{A}^{K//L} \), where \( K//L = L\setminus K/L \) are the double cosets.

Proof.

There is a unique conditional expectation \( E : B(I) \rightarrow A(I) \subset B(I) \) and \( E(\cdot) = v^*\gamma(\cdot)v \). The canonical endomorphisms \([\gamma] = \bigoplus_i [\beta_i] \) with sector \( \{\beta_i\} \) pairwise different (no multiplicities). It follows that

\[
E(\cdot) = \frac{1}{[B : A]} \sum_i d\beta_i \cdot w_i^* \beta_i(\cdot) w_i =: \phi_i(\cdot)
\]

and \( K = \{\phi_i\} \) gives the hypergroup action on \( B(I) \) which extends naturally to all \( \tilde{I} \in \mathcal{I} \).
When is $\mathcal{A} = \mathcal{B}^K$ for some hypergroup action?

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- How does $\text{Rep}(\mathcal{B}^K)$ looks like in terms of $\text{Rep}(\mathcal{B})$?
Example

Let $\mathcal{F}$ be a UFC and $\text{Rep}(\mathcal{A}) \cong Z(\mathcal{F})$.

- The Longo–Rehren inclusion gives a local net $\mathcal{B} \supset \mathcal{A}$, with $\text{Rep}(\mathcal{B}) = \langle \text{id} \rangle$.
- Can be seen as generalized crossed-product $\mathcal{B} = \mathcal{A} \rtimes \hat{\mathcal{F}}$. and $\mathcal{A} = \mathcal{B}^F$.
- $F \cong \text{Irr}(\mathcal{F})$ forms a hypergroup with:

$$C_{[\rho],[\sigma]}^{[\tau]} = \frac{d\rho \cdot d\sigma}{d\tau} N_{[\rho],[\sigma]}^{[\tau]} \quad [\rho] \times [\sigma] = \sum N_{[\rho],[\sigma]}^{[\tau]} [\tau]$$

If $G$ is a finite group and $\mathcal{F} = \langle \alpha_g : g \in G \rangle$, where $\alpha : G \to \text{Out}(M)$ is a $G$-kernel (characterized by $[\omega] \in H^3(G, \mathbb{T})$), then $F = G$ and $\mathcal{B} = \mathcal{A} \rtimes \hat{G}$. 
Theorem

Let $B$ be **holomorphic**, i.e. $\text{Rep}(B) = \langle \text{id} \rangle$. If $K$ is a hypergroup action, then there is a fusion category $\mathcal{F}$ whose fusion rules give $K$ and $\text{Rep}(B^K) \cong \mathbb{Z}(\mathcal{F})$.

1. In other words, $\mathcal{F}$ is a categorification of $K$.
2. Wide open mathematical problem: which hypergroups (fusion rings) permit a categorification?
3. If $K = G$ is a group, categorifications are given by $[\omega] \in H^3(G, \mathbb{T})$ or equivalently a $G$-kernel $\alpha : G \to \text{Out}(M)$.
When is $\mathcal{A} = \mathcal{B}^K$ for some hypergroup action?

- If $[\mathcal{B} : \mathcal{A}] < \infty \sim K$ finite hypergroup.

How does the hypergroups $K$ abstractly look like?

- If $\mathcal{B}$ is holomorphic $\sim K$ is a categorifiable fusion ring.
  - If $\mathcal{B}$ is competely rational?

How does $\text{Rep}(\mathcal{B}^K)$ looks like in terms of $\text{Rep}(\mathcal{B})$?

- If $\mathcal{B}$ is holomorphic $\sim \text{Rep}(\mathcal{B}) \cong Z(\mathcal{F})$ for some $\mathcal{F}$ with $\text{Irr}(\mathcal{F}) = K$.
  - If $\mathcal{B}$ is competely rational?
Theorem

Let $\mathcal{B}$ be completely rational (then $\text{Rep}(\mathcal{B})$ is a UMTC). If a hypergroup $\mathcal{K}$ is acting properly.

1. $\exists$ a (canonical) hypergroup $\tilde{\mathcal{K}}$, such that $\mathcal{K} = \tilde{\mathcal{K}} // K_{\text{Rep}(\mathcal{B})}$.

2. $\tilde{\mathcal{K}}$ is categorifiable, i.e. $\exists$ a fusion category $\mathcal{F} \supset \overline{\text{Rep}(\mathcal{B})}$ centrally with $K_{\mathcal{F}}$.

3. $\text{Rep}(\mathcal{B}^K) \cong \overline{\text{Rep}(\mathcal{B})}' \cap \mathcal{Z}(\mathcal{F})$, i.e. M"uger's centralizer of $\overline{\text{Rep}(\mathcal{B})}$ in the Drinfeld center $\mathcal{Z}(\mathcal{F})$.

M"uger’s centralizer $\mathcal{D} \subset \mathcal{C}$ where $\mathcal{C}$ is braided fusion category:

$$\mathcal{D}' \cap \mathcal{C} = \left\{ \rho \in \mathcal{C} : \begin{array}{c}
\begin{array}{c}
\rho \\
\sigma
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\rho \\
\sigma
\end{array}
\end{array} \text{ for all } \sigma \in \mathcal{D} \right\}$$
Let $\mathcal{F} \supset \mathcal{D}$ be an extension of a UMTC $\mathcal{D}$. Then the inclusion $\iota: \mathcal{D} \to \mathcal{F}$ is **central**, if there is a **braided** $\otimes$-functor $\tilde{\iota}: \mathcal{D} \to Z(\mathcal{F})$, such that the following diagram commutes:

\[
\begin{array}{ccc}
Z(\mathcal{F}) & \xrightarrow{\tilde{\iota}} & \mathcal{D} \\
\downarrow F & & \downarrow \iota \\
\mathcal{F} & \xrightarrow{\iota} & \mathcal{F}
\end{array}
\]

where $F: Z(\mathcal{F}) \to \mathcal{F}$ is the **forgetful functor** (forgetting the half-braiding).
When is $\mathcal{A} = \mathcal{B}^K$ for some hypergroup action?

- If $[\mathcal{B} : \mathcal{A}] < \infty \sim K$ finite hypergroup.

How does the hypergroups $K$ abstractly look like?

- If $\mathcal{B}$ is holomorphic $\sim K$ is a categorifiable fusion ring.
- If $\mathcal{B}$ is competely rational?

How does $\text{Rep}(\mathcal{B}^K)$ looks like in terms of $\text{Rep}(\mathcal{B})$?

- If $\mathcal{B}$ is holomorphic $\sim \text{Rep}(\mathcal{B}) \cong Z(\mathcal{F})$ for some $\mathcal{F}$ with $\text{Irr}(\mathcal{F}) = K$.
- If $\mathcal{B}$ is competely rational?
Quantum Galois Correspondence: $E_6$ example

$G = (SU(2) \times Spin(11))/\mathbb{Z}_2$ and $\text{Rep}(\mathcal{A}_{G_{10,1}})$ realizes the Drinfeld center of the even part of the $E_6$ subfactor (B. 2015).
Twisted representations?

Generalization of $G$-crossed braided tensor categories.

Relation to phase boundaries and defects.

Harmonic analysis, Fourier transformation, generalized $S$-matrices.

Construction/existence of Haagerup net (VOA) $\mathcal{A}_{Hg}$ conjectured by (Evans–Gannon '11).

Actions on loop group models, action on vertex operator algebras, e.g., affine Lie algebras.

Infinite index inclusions (semi-compact/discrete), e.g. $\text{Vir}_c \subset \mathcal{B}$ for $c > 1$.

$\sim$ a lot of analysis.

Fixed point for lattice models and models of topological phase of matter (opposite of condensation).
Thank you for your attention!


David E. Evans and Terry Gannon.
The exoticness and realisability of twisted Haagerup-Izumi modular data.

K. Fredenhagen, K.-H. Rehren, and B. Schroer.
Superselection sectors with braid group statistics and exchange algebras. I. General theory.

Tomohiro Hayashi and Shigeru Yamagami.
Amenable tensor categories and their realizations as AFD bimodules.

Y. Kawahigashi, Roberto Longo, and Michael Müger.
Multi-Interval Subfactors and Modularity of Representations in Conformal Field Theory.
Roberto Longo and Karl-Henning Rehren.
Nets of Subfactors.

R. Longo and J. E. Roberts.
A theory of dimension.

Adrian Ocneanu.
Quantized groups, string algebras and Galois theory for algebras.

Sorin Popa.
An axiomatization of the lattice of higher relative commutants of a subfactor.