Generalized Orbifolds in Algebraic Conformal QFT

Marcel Bischoff

http://www.math.vanderbilt.edu/~bischom

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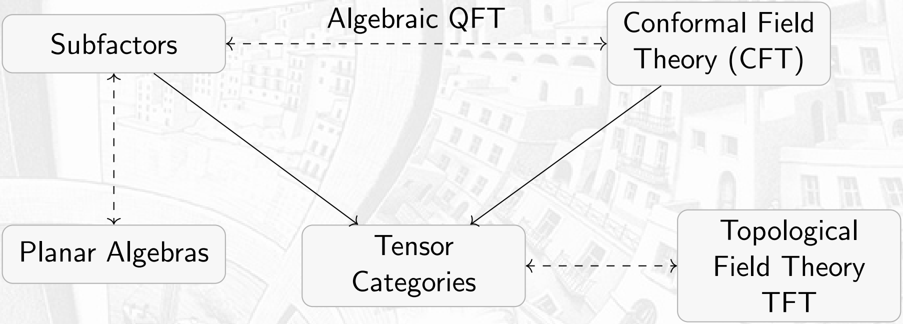

In binding together elements long-known but heretofore scattered and appearing unrelated to one another, it suddenly brings order where there reigned apparent chaos — Henri Poincaré

Problem

Given a local net $\mathcal{B}$ characterize all irreducible subnets $\mathcal{A} \subset \mathcal{B}$, i.e. subnets with $\mathcal{A}(O)' \cap \mathcal{B}(O) = \mathbb{C}$.

Higher dimensions (Doplicher–Haag–Roberts): Given $\mathcal{B}$

- $\mathrm{Rep}(\mathcal{B})$ is a rigid symmetric C*–tensor category and by Doplicher–Roberts duality braided equiavalent to $\mathrm{Rep}^k(G)$.
- (G, k) supergroup, i.e. G compact group, with $k \in Z(G)$ and $k^2 = e$.
- There is a unique $\mathbb{Z}_2$-graded "field net" $\mathcal{F} \supset \mathcal{B}$, with an action of G, such that $\mathcal{B} = \mathcal{F}^G$ and $\mathcal{B}(O)' \cap \mathcal{F}(O) = \mathbb{C}$.
- Given $\mathcal{A} \subset \mathcal{B}$ irreducible, $\mathcal{F}$ is also a field net for $\mathcal{A}$ and it follows, that $\mathcal{A} = \mathcal{F}^H$ for some compact group H with $G \subset H \subset \mathrm{Aut}(\mathcal{F})$.

Since $\mathcal{A}$ and $\mathcal{B}$ have the same field net they describe "similar" physics.

Space-time dimension $d \leq 2$:

- $\mathrm{Rep}(\mathcal{B})$ is a braided (sometimes rigid, basically never symmetric) C*–tensor category but there is no nice duality theory (sometimes weak C*–Hopf algebra).
- There is no gauge group.
- There is no field net (sometimes a field bundle).
- Non-integer dimensions $d \in \{2\cos(\pi/n) : n \geq 3\} \cup [2, \infty]$.

Example

Let $\mathcal{B}$ with $\mathrm{Rep}(\mathcal{B}) \cong \langle \mathrm{id} \rangle$ (vacuum is the only sector) and G a finite group in $\mathrm{Aut}(\mathcal{B})$:

$d > 2$ $\mathrm{Rep}(\mathcal{B}^G) \cong \mathrm{Rep}(G)$, in particular $\dim(\mathrm{Rep}(\mathcal{B}^G)) = \sum_{\rho \in \mathrm{Irr}} d\rho^2 = |G|$,

$d \leq 2$ $\mathrm{Rep}(\mathcal{B}^G) \cong \mathrm{Rep}(D^\omega(G))$ for some $[\omega] \in H^3(G, \mathbb{T})$, in particular not determined by G and $\dim(\mathrm{Rep}(\mathcal{B}^G)) = \sum_{\rho \in \mathrm{Irr}} d\rho^2 = |G|^2$.

The strict rigid C^*–tensor category $\mathrm{End}_0(N)$, with N a type III factor.

- Objects: Endomorphisms $\rho\colon N \to N$ with finite index $[N : \rho(N)] < \infty$.
- Morphisms: $t\colon \rho \to \sigma$ is a $t \in N$, such that $t\rho(x) = \sigma(x)t$ for all $x \in N$.
- Tensor product: $\rho \otimes \sigma := \rho \circ \sigma$ (composition of endomorphisms)
- $d\rho = [N : \rho(N)]^{\frac{1}{2}}$ with $[N : \rho(N)]$ the **Jones index**.
- Unitarity $\rightsquigarrow$ (ess. unique) spherical structure (Longo–Roberts '1997).

Theorem (well-known: (Ocnneanu '88), (Popa '95), . . . , (Hayashi–Yamagami '00))

Every (amenable) **rigid C^*-tensor category** $\mathcal{C}$, *is realizable as a full subcategory of* $\mathrm{End}_0(N)$ *with* N *a (hyperfinite) type* III_1 *factor*

$\mathcal{C}$ is called a **unitary fusion category (UFC)** if $\#\,\mathrm{Irr}(\mathcal{C}) < \infty$.

A **conformal net** associates with every interval $I \subset S^1$ a von Neumann algebra on a fixed Hilbert space $\mathcal{H}$:

$$S^1 \supset I \longmapsto \mathcal{A}(I) \subset \mathrm{B}(\mathcal{H})$$

1. Isotony: $I \subset J \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)$

We often additionally assume **complete rationality** in the sense of (Kawahigashi–Longo–Müger '01)

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3. Covariance: $U\colon G \to \mathcal{U}(\mathcal{H})$, s.t. $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$.

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4. Vacuum: Unique (up to phase) G-invariant unit vector $\Omega \in \mathcal{H}$, s.t. $\overline{\vee_I \mathcal{A}(I)\Omega} = \mathcal{H}$.

We often additionally assume **complete rationality** in the sense of (Kawahigashi–Longo–Müger '01)

Loop group net of G at level k. G compact Lie group, $\mathrm{L}G = C^\infty(S^1, G)$

$$\mathcal{A}_{G_k}(I) = \pi_{0,k}(\mathrm{L}_I G)'', \qquad \mathrm{L}_I G = \{\gamma \in \mathrm{L}G : \operatorname{supp}\gamma \subset I\}$$

Net associated with even lattice $\Gamma \subset \mathbb{R}^n$ = torus loop group $\mathrm{L}\mathbb{T}$ with $\mathbb{T} = \mathbb{R}^n/\Gamma$

$$\mathcal{A}_L(I) = \mathcal{A}_{\mathbb{T}}(I)$$

Virasoro net with $c \in \left\{1 - \frac{6}{m(m+1)} : m = 3, 4, \dots\right\} \cup [1, \infty)$:

$$\mathrm{Vir}_c(I) = \pi_{c,0}\left(\mathrm{Diff}^+_I(S_1)\right)''$$

Constructions:

- $\otimes$-product
- Coset construction
- **Orbifold construction** = fixed point by finite group G
- Mirror extensions

A **representation** $\pi \in \mathrm{Rep}(\mathcal{A})$ of $\mathcal{A} = \{\mathcal{A}(I)\}_{I \subset S^1}$ is a family:

$$\pi = \{\pi_I \colon \mathcal{A}(I) \to \mathrm{B}(\mathcal{H}_\pi)\}\,,$$

which is compatible, i.e. $\pi_J \restriction \mathcal{A}_0(I) = \pi_I$ for $I \subset J$.

- Every π unitarily equivalent to a Doplicher–Haag–Roberts localized endomorphism $\rho \in \mathrm{End}(\mathcal{A}(I))$.
- $\mathrm{Rep}^I(\mathcal{A}) \subset \mathrm{End}(\mathcal{A}(I))$ full and replete.
  $\rightsquigarrow$ **C*-tensor category**
- Assume finite **dimensions** $d\rho \equiv [\mathcal{A}(I) : \rho(\mathcal{A}(I))]^{\frac{1}{2}} \overset{!}{<} \infty$
  $\rightsquigarrow$ **rigidity + semisimplicty**
- $\exists$ natural braiding $\{\varepsilon_{\rho,\sigma} \colon \rho \otimes \sigma \to \sigma \otimes \rho\}$ (Fredenhagen–Rehren–Schroer '89) $\varepsilon_{\rho,\sigma} =$ 

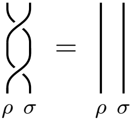
$\rightsquigarrow$ **unitary ribbon category**

Theorem ((Kawahigashi–Longo–Müger ’01))

Let $\mathcal{A}$ be a completely rational conformal net, then all irreducible representations have **finite dimensions** *and* $\mathcal{C} := \mathrm{Rep}^I(\mathcal{A})$ *is a unitary modular tensor category.*

$\mathcal{C}$ braided UFC is **modular** $\Longleftrightarrow$ trivial **Müger center**

$$\mathcal{C}' \cap \mathcal{C} = \left\{ \rho \in \mathcal{C} : \right.$$



$$\left. \text{for all } \sigma \in \mathcal{C} \right\} = \langle \mathrm{id} \rangle$$

$\rightsquigarrow$ unitary representations of the **modular group** $\mathrm{SL}(2, \mathbb{Z})$.

On the opposite **symmetric** means $\mathcal{C}' \cap \mathcal{C} = \mathcal{C}$.

Theorem ((Longo–Rehren '95),(Müger),(B–Kawahigashi–Longo '14))

There is a one-to-one correspondence between

- *commutative algebras θ in $\mathcal{C}$ and*
- *local, finite index extensions $\mathcal{B} \supset \mathcal{A}$.*

If $\mathcal{A}$ is rational, then $\mathrm{Rep}(\mathcal{B}) \cong \mathrm{Mod}^0_{\mathrm{Rep}(\mathcal{A})}(\theta)$, the category of local θ-modules.

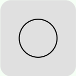
Note: In higher dimensions, $\mathrm{Rep}(\mathcal{A})$ is symmetric and $\mathrm{Mod}^0_{\mathrm{Rep}(\mathcal{A})}(\theta) = \mathrm{Mod}_{\mathrm{Rep}(\mathcal{A})}(\theta)$.

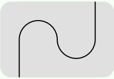
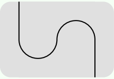
Example ($\mathrm{Rep}(\mathcal{A}_{\mathrm{SU}(2)_k})$)

Irreducible representations $\{0, \frac{1}{2}, 1, \ldots, \frac{k}{2}\}$:

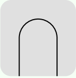
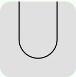
$$[i] \times [j] = \bigoplus_{n=|i-j|}^{\min(i+j, k-i-j)} [n]$$

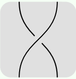
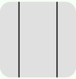
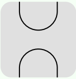
$\mathrm{Rep}(\mathcal{A}_{\mathrm{SU}(2)_k})$ is generated by $\frac{1}{2}$-representation ρ and $\cup \in \mathrm{Hom}(\mathrm{id}, \rho\rho)$:

 $= -d$

 $=$  $=$

with $\cap \in \mathrm{Hom}(\rho\rho, \mathrm{id})$ and **braiding** defined by the **Kaufmann bracket**

 $:= -$  *

 $= q^{\frac{1}{2}}$  $+ q^{-\frac{1}{2}}$ 

where $q = \mathrm{e}^{\frac{\mathrm{i}\pi}{k+2}}$, $d = q + q^{-1} = 2\cos\left(\frac{\pi}{k+2}\right)$.

Example ((Capelli–Itzykson–Zuber '87),(Ocneanu))

$\mathcal{B} \supset \mathcal{A}_{\mathrm{SU}(2)_k} \overset{1:1}{\longleftrightarrow} A, D, E$ Coxeter–Dynkin diagrams with Coxeter number $k+2$.

- A_{k+1} $\mathcal{A}_{\mathrm{SU}(2)_k} \subset \mathcal{A}_{\mathrm{SU}(2)_k}$
- D_{2n} $\mathcal{A}_{\mathrm{SU}(2)_{4n-4}} \subset \mathcal{A}_{\mathrm{SO}(3)_{2n-2}} \equiv \mathcal{A}_{\mathrm{SU}(2)_{4n-4}} \rtimes \mathbb{Z}_2$ with representations $\frac{1}{2}D_{2n}$
- E_6, E_8 $\mathcal{A}_{\mathrm{SU}(10)_k} \subset \mathcal{A}_{\mathrm{Spin}(5)_1}$ and $\mathcal{A}_{\mathrm{SU}(2)_{28}} \subset \mathcal{A}_{\mathrm{G}_{2,1}}$, with Ising and Fib representations.

A completely rational net $\mathcal{B}$ is called **holomorphic** if $\mathrm{Rep}(\mathcal{B}) = \langle \mathrm{id} \rangle$, i.e. $\mathcal{B}$ has only the vacuum sector.

Example

- Conformal net associated with even self-dual lattice, i.e. $L \overset{!}{=} L^* := \{x \in \mathbb{R}^n : \langle x, L \rangle \in \mathbb{Z}\}$. E.g. $E_8, E_8 \oplus E_8, D_{16}^+$, Leech, ...
- Loop group net of E_8 at level 1 $\mathcal{A}_{\mathrm{E}_{8,1}}$ and $\mathcal{A}_{\mathrm{SO}(32)_1}$ (not s.c.).
- Moonshine net $\mathcal{A}^\sharp$

Let G be a finite group acting properly on a net $\mathcal{B}$ then the fixed point net $\mathcal{B}^G$ is called the G-**orbifold net**.

Question (Evans–Gannon '11)

> *Can we orbifold a VOA [or conformal net] by something more general than a group?*

Theorem (probably well-known, (B. '16))

Let $\mathcal{B}^{\mathbb{K}} \subset \mathcal{B}$ be a finite index subnet for a Kac algebra (finite C-Hopf algebra) $\mathbb{K}$, then $\mathbb{K}$ is a finite group.*

Let (M, Ω) be a **non-commutative propability space**,i.e.

- $M = M'' \subset \mathrm{B}(\mathcal{H})$ a von Neumann algebra.
- $\Omega \in \mathcal{H}$ a cyclic and separating vector, i.e. $\overline{M\Omega} = \overline{M'\Omega} = \mathcal{H}$.

A linear map $\phi\colon (M_1, \Omega_1) \longrightarrow (M_2, \Omega_2)$ is called a **stochastic map** if

- $\phi\colon M_1 \to M_2$ is **completely positive**, i.e. $\phi \otimes 1_{\mathbb{M}_n(\mathbb{C})}\colon M_1 \otimes \mathbb{M}_n(\mathbb{C}) \to M_2 \otimes \mathbb{M}_n(\mathbb{C})$ is positive for all $n \geq 1$.
- **State-preserving:** $(\Omega_2, \phi(\,\cdot\,)\Omega_2) = (\Omega_1, \,\cdot\,\Omega_1)$.
- **Normal:** Continuous with respect to the ultraweak topology.
- **Unital:** $\phi(1_{M_1}) = 1_{M_2}$.

An **adjoint** is a stochastic map $\phi^\sharp\colon (M_2, \Omega_2) \to (M_1, \Omega_1)$, such that

$$(\phi^\sharp(m_2)\Omega_1, m_1\Omega_1) = (m_2\Omega_2, \phi(m_1)\Omega_2), \qquad m_i \in M_i$$

Stochastic maps are quantum operations preserving a state.

Theorem (Stinespring, Connes)

M type purely infinite and $\phi\colon M \to M$ stochastic map, then there is an isometry $v \in M$ and an endomorphism $\beta \in \mathrm{End}(M)$, such that

$$\phi(\,\cdot\,) = v^*\beta(\,\cdot\,)v\,.$$

Conversely, v an isometry and $\beta \in \mathrm{End}(M)$, such that $(v\Omega, \beta(\,\cdot\,)v\Omega) = (\Omega, \cdot\,\Omega)$, then $\phi(\,\cdot\,) = v^*\beta(\,\cdot\,)$ is a stochastic map.

Given a net $(\{\mathcal{A}(I)\},\Omega)$ a family of stochastic maps $\{\phi_k^I \colon (\mathcal{A}(I),\Omega) \colon (\mathcal{A}(I),\Omega) : I \in \mathcal{I}, k = 0,\ldots,n\}$, such that

- **compatible:** $\phi_k^I \restriction \mathcal{A}(\tilde{I}) = \phi_k^{\tilde{I}}$
- **unital:** $\phi_0^I = \mathrm{id}_{\mathcal{A}}(I)$
- **closed under adjoints:** $k \mapsto \bar{k}$ on $\{1,\ldots n\}$, such that $\phi_k^\sharp = \phi_{\bar{k}}$
- **closed under composition:** $\phi_i \circ \phi_j = \sum_k C_{ij}^k \phi_k$ for some $(C_{ij}^k \geq 0)$.
- **dual/weak inverse:** $C_{ij}^0 > 0$ if and only if $j = \bar{i}$.
- **simplex:** The ϕ_k are extremal and affine independent.

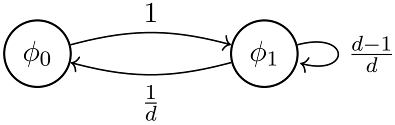
is called a **hypegroup action**. The set $K = \{c_0,\ldots,c_k\}$ with the coefficients C_{ij}^k is called a **hypergroup**. The span $\mathbb{C}K$ has the structure of a finite C*-algebra, generalizing the group algebra $\mathbb{C}G$.

Example (Hypergroup)

$K_D = \{\phi_0 = \mathrm{id}, \phi_1\}$ with $d \in [1, \infty)$ and

$$\phi_1 \circ \phi_1 = \frac{1}{d}\phi_0 + \frac{d-1}{d}\phi_1 \qquad \phi_0 \circ \phi_1 = \phi_1 = \phi_1 \circ \phi_0, \quad \phi_0 \circ \phi_0 = \phi_0 \,.$$

$\phi_1 \circ \cdot : \mathrm{Conv}(K) \to \mathrm{Conv}(K)$ defines a **Markov chain**:



Example

If $d = 1$ then $K \cong \mathbb{Z}_2$.

Haar element (invariant measure) $E \in \mathrm{Conv}(K)$:

$$E(\,\cdot\,) := \frac{1}{d+1}\left[\phi_0(\,\cdot\,) + d\phi_1(\,\cdot\,)\right] \quad \rightsquigarrow \quad E \circ \phi_k = E = \phi_k \circ E, \quad E \circ E = E$$

Example (Hypergroup action)

Let G be a finite group of **(first kind) gauge automorphism** $\{\alpha_g^I\colon \mathcal{A}(I) \to \mathcal{A}(I) : I \in \mathcal{I}, g \in G\}$ (**internal symmetries**).

Theorem (B.)

Let K be a **hypergroup** *action on $\mathcal{B}$. Then $\mathcal{B}^K$ defined by the* **fixed-point**

$$\mathcal{B}^K(I) := \mathcal{B}(I)^K = \{b \in \mathcal{B}(I) : \phi_k^I(b) = b \text{ for all } k \in K\}$$

*is a finite index subnet of $\mathcal{B}$, called the K-***orbifold** *net of $\mathcal{B}$.*

Example (Doplicher–Haag–Roberts, Rehren, Xu, Müger)

If $K = G$ is a group, then $\mathcal{B}^G$ is the G-orbifold[a] net of $\mathcal{B}$.

[a] the name **orbifold** has roots in string theory, where the fixed point with respect to a finite group has geometrical meaning.

- When is $\mathcal{A} = \mathcal{B}^K$ for some hypergroup action?

- How does the hypergroups K abstractly look like?

- How does $\mathrm{Rep}(\mathcal{B}^K)$ looks like in terms of $\mathrm{Rep}(\mathcal{B})$?

Theorem

Let $\mathcal{A} \subset \mathcal{B}$ with $[\mathcal{B} : \mathcal{A}] < \infty$, then there is a canonical hypergroup action K, such that $\mathcal{A} = \mathcal{B}^K$.
There is a Galois like correspondence between intermediate nets $\mathcal{B}^K \subset \tilde{\mathcal{A}} \subset \mathcal{B}$ and subhypergroups $L \subset K$.
In this case $\mathcal{A} = \tilde{\mathcal{A}}^{K//L}$, where $K//L = L\backslash K/L$ are the double cosets.

Proof.

There is a unique conditional expectation $E\colon \mathcal{B}(I) \to \mathcal{A}(I) \subset \mathcal{B}(I)$ and $E(\,\cdot\,) = v^*\gamma(\,\cdot\,)v$. The canonical endomorphisms $[\gamma] = \bigoplus_i [\beta_i]$ with sector $\{\beta_i\}$ pairwise different (no multiplicities). It follows that

$$E(\,\cdot\,) = \frac{1}{[\mathcal{B} : \mathcal{A}]} \sum_i d\beta_i \cdot \underbrace{w_i^* \beta_i(\,\cdot\,) w_i}_{=:\phi_i(\,\cdot\,)}$$

and $K = \{\phi_i\}$ gives the hypergroup action on $\mathcal{B}(I)$ which extends naturally to all $\tilde{I} \in \mathcal{I}$. □

- ☺ When is $\mathcal{A} = \mathcal{B}^K$ for some hypergroup action?

- How does the hypergroups K abstractly look like?

- How does $\mathrm{Rep}(\mathcal{B}^K)$ looks like in terms of $\mathrm{Rep}(\mathcal{B})$?

Example

Let $\mathcal{F}$ be a UFC and $\mathrm{Rep}(\mathcal{A}) \cong Z(\mathcal{F})$.

- The the Longo–Rehren inclusion gives a local net $\mathcal{B} \supset \mathcal{A}$, with $\mathrm{Rep}(\mathcal{B}) = \langle \mathrm{id} \rangle$
- Can be seen as generalized crossed-product $\mathcal{B} = \mathcal{A} \rtimes \hat{\mathcal{F}}$. and $\mathcal{A} = \mathcal{B}^F$
- $F \cong \mathrm{Irr}(\mathcal{F})$ forms a hypergroup with:

$$C_{[\rho],[\sigma]}^{[\tau]} = \frac{d\rho \cdot d\sigma}{d\tau} N_{[\rho],[\sigma]}^{[\tau]} \qquad [\rho] \times [\sigma] = \sum N_{[\rho],[\sigma]}^{[\tau]} [\tau]$$

If G is a finite group and $\mathcal{F} = \langle \alpha_g : g \in G \rangle$, where $\alpha \colon G \to \mathrm{Out}(M)$ is a G-kernel (characterized by $[\omega] \in H^3(G, \mathbb{T})$), then $F = G$ and $\mathcal{B} = \mathcal{A} \rtimes \hat{G}$.

Theorem

Let $\mathcal{B}$ be **holomorphic**, *i.e.* $\mathrm{Rep}(\mathcal{B}) = \langle \mathrm{id} \rangle$. *If K is a hypergroup action, then there is a fusion category $\mathcal{F}$ whose fusion rules give K and* $\mathrm{Rep}(\mathcal{B}^K) \cong Z(\mathcal{F})$.

1. In other words, $\mathcal{F}$ is a categorification of K.
2. Wide open mathematical problem: which hypergroups (fusion rings) permit a categorification?
3. If $K = G$ is a group, categorifications are given by $[\omega] \in H^3(G, \mathbb{T})$ or equivalently a G-kernel $\alpha\colon G \to \mathrm{Out}(M)$.

- ☺ When is $\mathcal{A} = \mathcal{B}^K$ for some hypergroup action?
  - ☺ If $[\mathcal{B} : \mathcal{A}] < \infty \rightsquigarrow K$ finite hypergroup.

- – How does the hypergroups K abstractly look like?
  - ☺ If $\mathcal{B}$ is **holomorphic** $\rightsquigarrow K$ is a categorifiable fusion ring.
  - – If $\mathcal{B}$ is **competely rational**?

- – How does $\mathrm{Rep}(\mathcal{B}^K)$ looks like in terms of $\mathrm{Rep}(\mathcal{B})$?
  - ☺ If $\mathcal{B}$ is **holomorphic** $\rightsquigarrow \mathrm{Rep}(\mathcal{B}) \cong Z(\mathcal{F})$ for some $\mathcal{F}$ with $\mathrm{Irr}(\mathcal{F}) = K$.
  - – If $\mathcal{B}$ is **competely rational**?

Theorem

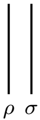
Let $\mathcal{B}$ be **completely rational** *(then $\mathrm{Rep}(\mathcal{B})$ is a UMTC). If a hypergroup K is acting properly.*

1. *$\exists$ a (canonical) hypergroup $\tilde{K}$, such that $K = \tilde{K}//K_{\mathrm{Rep}(\mathcal{B})}$.*
2. *$\tilde{K}$ is categorifable, i.e. $\exists$ a fusion category $\mathcal{F} \supset \overline{\mathrm{Rep}(\mathcal{B})}$* **centrally** *with $K_{\mathcal{F}}$.*
3. *$\mathrm{Rep}(\mathcal{B}^K) \cong \overline{\mathrm{Rep}(\mathcal{B})}' \cap Z(\mathcal{F})$, i.e.* **Müger's centralizer** *of $\overline{\mathrm{Rep}(\mathcal{B})}$ in the* **Drinfeld center** *$Z(\mathcal{F})$.*

Müger's centralizer $\mathcal{D} \subset \mathcal{C}$ where $\mathcal{C}$ is braided fusion category:

$$\mathcal{D}' \cap \mathcal{C} = \left\{ \rho \in \mathcal{C} : \right.$$

$$=$$



$$\left. \text{for all } \sigma \in \mathcal{D} \right\}$$

Let $\mathcal{F} \supset \mathcal{D}$ be an extension of a UMTC $\mathcal{D}$. Then the inclusion $\iota\colon \mathcal{D} \to \mathcal{F}$ is **central**, if there is a **braided** $\otimes$-functor $\tilde{\iota}\colon \mathcal{D} \to Z(\mathcal{F})$, such that the following diagram commutes:

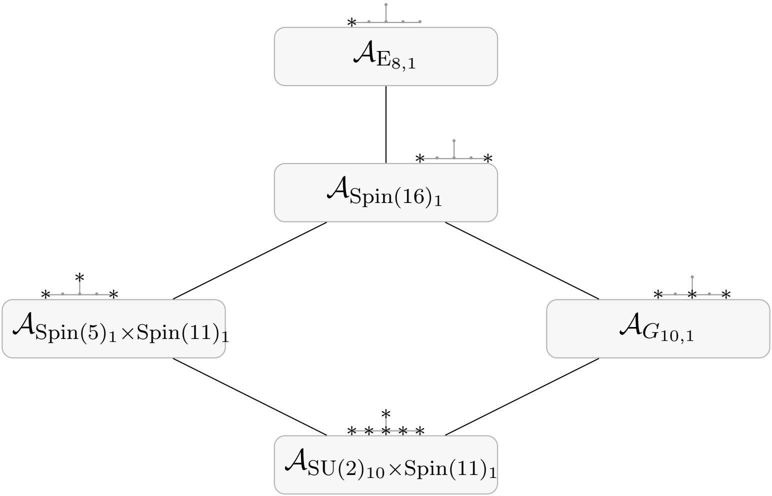
$$\begin{array}{ccc} & & Z(\mathcal{F}) \\ & \overset{\tilde{\iota}}{\nearrow} & \downarrow F \\ \mathcal{D} & \xrightarrow{\iota} & \mathcal{F} \end{array}$$

where $F\colon Z(\mathcal{F}) \to \mathcal{F}$ is the **forgetful functor** (forgetting the half-braiding).

- When is $\mathcal{A} = \mathcal{B}^K$ for some hypergroup action?
  - If $[\mathcal{B} : \mathcal{A}] < \infty \rightsquigarrow K$ finite hypergroup.

- How does the hypergroups K abstractly look like?
  - If $\mathcal{B}$ is **holomorphic** $\rightsquigarrow K$ is a categorifiable fusion ring.
  - If $\mathcal{B}$ is **competely rational**?

- How does $\mathrm{Rep}(\mathcal{B}^K)$ looks like in terms of $\mathrm{Rep}(\mathcal{B})$?
  - If $\mathcal{B}$ is **holomorphic** $\rightsquigarrow \mathrm{Rep}(\mathcal{B}) \cong Z(\mathcal{F})$ for some $\mathcal{F}$ with $\mathrm{Irr}(\mathcal{F}) = K$.
  - If $\mathcal{B}$ is **competely rational**?



$G = (\mathrm{SU}(2) \times \mathrm{Spin}(11))/\mathbb{Z}_2$ and $\mathrm{Rep}(\mathcal{A}_{G_{10,1}})$ realizes the Drinfeld center of the even part of the E_6 subfactor (B. 2015).

- Twisted representations?
- Generalization of G-crossed braided tensor categories.
- Relation to phase boundaries and defects.
- Harmonic analysis, Fourier transformation, generalized S-matrices.
- Construction/existence of Haagerup net (VOA) $\mathcal{A}_{\mathrm{Hg}}$ conjectured by (Evans–Gannon '11).
- Actions on loop group models, action on vertex operator algebras, e.g.. affine Lie algebras.
- Infinite index inclusions (semi-compact/discrete), e.g. $\mathrm{Vir}_c \subset \mathcal{B}$ for $c > 1$.
  $\rightsquigarrow$ a lot of analysis.
- Fixed point for lattice models and models of topological phase of matter (opposite of condensation).

Thank you for your attention!

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