A planar algebraic description of defect lines in conformal field theory

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*based on work with R. Longo, Y. Kawahigashi and K.-H. Rehren
In binding together elements long-known but heretofore scattered and appearing unrelated to one another, it suddenly brings order where there reigned apparent chaos — Henri Poincaré
Question

Do all (finite index finite depth) subfactors arise from conformal field theory?
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Desired construction

| Subfactor \( N \subset M \) | $\mapsto$ | Conformal net \( \mathcal{A}_{N \subset M} \) |

such that \( \text{Rep}(\mathcal{A}_{N \subset M}) \cong D(N \subset M) \) (\( = \) quantum double*)

“Royal road”: [\ldots] extract from the subfactor the Boltzmann weights of a critical two-dimensional lattice model then construct a quantum field theory from the scaling limit of the \( n \)-point functions. (Jones 2015)

* \( D(N \subset M) \) = unitary modular tensor category obtained from Ocneanu’s asymptotic inclusion, Popa’s SE algebra, Longo–Rehren subfactor, Drinfeld center
Definition (Conformal Net)

\[ S_1 \supset I \mapsto \mathcal{A}(I) = \mathcal{A}(I)^{''} \subset \mathcal{B} (\mathcal{H}) , \text{ such that} \]

1. Isotony: \( I \subset J \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J) \)

2. Locality: \( [\mathcal{A}(I), \mathcal{A}(J)] = \{0\} \) if \( I \cap J = \emptyset \).

3. Covariance: \( \mathcal{U} \) is a unitary \textbf{positive-energy} representation of the Möbius group, s.t. \( \mathcal{U}(g)\mathcal{A}(I)\mathcal{U}(g)^* = \mathcal{A}(gI) \).

4. Vacuum: \( \exists \Omega \) is a (up to a phase) unique vector invariant under the Möbius group, s.t. \( \vee_I \mathcal{A}(I)\Omega = \mathcal{H} \).
Definition (Conformal Net)

\[ S_1 \supset I \mapsto A(I) = A(I)^{''} \subset B(H), \text{ such that} \]

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3. **Covariance:** \( U \) is a unitary **positive-energy** representation of the Möbius group, s.t. \( U(g)A(I)U(g)^* = A(gI) \).

4. **Vacuum:** \( \exists \Omega \) is a (up to a phase) unique vector invariant under the Möbius group, s.t. \( \vee_I A(I)\Omega = \mathcal{H} \).

It follows that \( A(I) \) are type III\(_1\) factors and \( \Omega \) is cyclic and separating for \( A(I) \). Let

\[ N := A \begin{array}{c} \circ \end{array} \subset M := A \begin{array}{c} \circ \end{array} \supset \tilde{N} := A \begin{array}{c} \circ \end{array} \]

Then \( (N \subset M, \tilde{N} \subset N' \cap M, \Omega) \) is a complete invariant and every such triple gives a conformal net (possibly reducible) provided \( \Omega \) is cyclic for \( N, \tilde{N} \) and separating for \( M \) and \( \sigma^{t}_{(M,\Omega)}(N) \subset N \) for \( t \leq 0 \).
A conformal net $\mathcal{A}$ is **completely rational** if:

- **Strong additivity.** $N \vee \tilde{N} = M$:
  
  $$\mathcal{A}(\bigcirc) \vee \mathcal{A}(\bigcirc) = \mathcal{A}(\bigcirc)$$

  (Then $\tilde{N} = N' \cap M$ holds. $(N \subset M, \Omega)$ is a complete invariant.)

- **Finite $\mu$-index**: finite Jones index of subfactor
  
  $$\mu_\mathcal{A} = \left[ \mathcal{A}(\bigcirc)' : \mathcal{A}(\bigcirc) \right] < \infty \quad \mathcal{A}(\bigcirc) := \mathcal{A}(\bigcirc) \vee \mathcal{A}(\bigcirc)$$

- **Split property.** For every inclusion $\bar{I} \subset J$ of intervals $\exists$ intermediate type I factor $S$, e.g.:

  $$\mathcal{A}(\bigcirc) \subset S \subset \mathcal{A}(\bigcirc)$$

  This holds if $\text{Tr}(e^{-\beta L_0}) < \infty$ for all $\beta > 0$, where $L_0$ is the generator of rotations: $U(z \mapsto e^{it}z) = e^{itL_0}$. 
Representation of $\mathcal{A} = \{\mathcal{A}(I)\}_{I \subset S^1}$ is a family:

$$\pi = \{\pi_I : \mathcal{A}(I) \to B(\mathcal{H}_\pi)\},$$

which is compatible, i.e. $\pi_J \upharpoonright \mathcal{A}_0(I) = \pi_I$ for $I \subset J$. 

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- Every $\pi$ unitarily equivalent to a localized endomorphism $\rho \in \text{End}(\mathcal{A}(I))$.
- Statistical dimension $d = [\mathcal{A}(I) : \rho(\mathcal{A}(I))]^{\frac{1}{2}}$.
- Tensor product: composition of localized endomorphisms.
- $\exists$ natural braiding $\{\varepsilon_{\rho,\sigma} : \rho \circ \sigma \rightarrow \sigma \circ \rho\}$ (Fredenhagen, Rehren, Schroer (1989)).
Theorem (Kawahigashi, Longo, Müger (2001))

Let $A_0$ be a completely rational conformal net. Then $\text{Rep}(A_0)$ is a modular $C^*$-tensor category = unitary modular tensor category (UMTC).

Example: Loop group net of $SU(N)$ at level $k$:
(see Wassermann '98)

$A_{SU(N),k}(I) = \pi(L_{SU(N)})''$ (completely rational (Xu '00))

with $\pi$ level $k$ vacuum PER of loop group $LSU(N) = C^\infty(S^1, SU(N))$. 

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Extensions of nets are characterized by one local extension

We write if $\mathcal{B} \supset \mathcal{A}$ if

- $\mathcal{A}(J) \subset \mathcal{B}(J)$ for all $J \in S^1 \setminus \{-1\}$ finite index, irreducible.
- Relatively local extension:

$$[\mathcal{A}(I_1), \mathcal{B}(I_2)] = \{0\} \text{ if } I_1 \cap I_2 = \emptyset$$

Completely characterized by $M := \mathcal{B}(I) \supset N := \mathcal{A}(I)$ for a fixed $I$. 
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**Theorem ((Longo–Rehren ’95))**

Conversely, every $M \supset N := \mathcal{A}(I)$, finite index, irreducible overfactor $M$, such that $N M N$ (actually the dual canonical endomorphism $\theta : N \to N$) is a representation of $\mathcal{A}$ localized in $I$.

$\mathcal{B}$ is a local net iff multiplication $N M N \otimes N M M \to N M N$ is commutative (with respect to the braiding of $\text{Rep}(\mathcal{A})$).
One can define a **conformal net** on **Minkowski space** by

\[ \mathcal{A}_2(O) = \mathcal{A}(I_1) \otimes \mathcal{A}(I_2) \]

where \( \mathcal{A}_\pm \) are conformal nets on \( \mathbb{R} \).
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**Full CFTs** based on \( \mathcal{A} \) are given by maximal local extensions

\[ \mathcal{B}(O) \supset \mathcal{A}_2(O) \equiv \mathcal{A}(I_1) \otimes \mathcal{A}(I_2) . \]

**Locality.** \( [\mathcal{B}(O_1), \mathcal{B}(O_2)] = \{0\} \) if \( O_1 \) and \( O_2 \) are space like separated.
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**Theorem (B–Kawahigashi–Longo ’14)**

There is a one-to-one correspondence between

- **Full CFTs** \( \mathcal{B}_2 \supset \mathcal{A}_2 \)
- \( M \supset N = \mathcal{A}(I) \) with \( NM_N \in \text{Rep}(\mathcal{A}) \) up to Morita equivalence.
Two different full CFTs $\mathcal{B}_a, \mathcal{B}_b \supset \mathcal{A}_2$ divided by a defect line

\[
\begin{align*}
\{ O_a \} & \quad \xrightarrow{\mathcal{B}_a(O_a), \mathcal{D}(O_x), \mathcal{B}_b(O_b)} \quad \supset \mathcal{A}_2(O.) \\
\{ O_\times \} & \quad \supset \mathcal{D}_\pm \subset \mathcal{D} \supset \mathcal{D}_{-} \\
\{ O_b \} & \quad \supset \mathcal{B}_a \quad \cup \quad \mathcal{B}_b \\
\end{align*}
\]

where $\mathcal{D}_\pm$ left/right center:

$\mathcal{A}_2 \subset \mathcal{D}_\pm \subset \mathcal{D}$

maximal intermediate local nets.
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\begin{align*}
\{O_a, O_x, O_b\} & \hookrightarrow \{\mathcal{B}_a(O_a), \mathcal{D}(O_x), \mathcal{B}_b(O_b)\} \supset \mathcal{A}_2(O).
\end{align*}
\]

\[
\mathcal{D}_+ \subset \mathcal{D} \supset \mathcal{D}_-
\]

where $\mathcal{D}_\pm$ left/right center:

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\mathcal{A}_2 \subset \mathcal{D}_\pm \subset \mathcal{D}
\]

maximal intermediate local nets.

- Defect line invisible for the subnet $\mathcal{A}_2$ (conserves symmetries prescribed by $\mathcal{A}$)
- Different realization $\leftrightarrow$ different boundary conditions
- $\mathcal{A}$-topological $\mathcal{B}_a$-$\mathcal{B}_b$ defect line.
Fusion of defect lines

\[ \mathcal{F} = \mathcal{D} \otimes_{\mathcal{B}_a} \mathcal{E} \] can be defined Connes' fusion over wedge algebra or using the braiding.
Classification of irreducible defect lines

Fuse trivial defects over $\mathcal{A}_2$ (not a full CFT!!) $\sim$ non-factorial “defect line”

\[
\mathcal{B}_a \quad \mathcal{B}_b \\
\mathcal{B}_a \quad \mathcal{A}_2 \quad \mathcal{A}_2 \quad \mathcal{B}_b \\
\mathcal{A}_2 \quad \mathcal{A}_2 \quad \mathcal{A}_2
\]

\[
\mathcal{D}_+ \subset \mathcal{D}_u \supset \mathcal{D}_-
\]

\[
\mathcal{B}_a \quad \mathcal{A}_2 \quad \mathcal{B}_b
\]

Theorem ((B–Kawahigashi–Longo–Rehren ’14))

1. $\mathcal{D}^u(O)' \cap \mathcal{D}^u(O) = \mathcal{A}_2(O)' \cap \mathcal{D}^u(O)$, i.e. $\mathcal{D}^u(O)$ has finite center.
2. Every minimal central projection $p \in \mathcal{D}(O)$ yields an irreducible $\mathcal{A}$-topological $\mathcal{B}_a$–$\mathcal{B}_b$-defect $\mathcal{D}_p \cong \mathcal{D}_p$.
3. Every irreducible $\mathcal{A}$-topological $\mathcal{B}_a$–$\mathcal{B}_b$-defect arises this way.
4. Minimal projections $\xrightarrow{1:1}$ irreducible $M_a$–$M_b$ sectors related to $\text{Rep}(\mathcal{A})$. 
Intertwiner between configurations of defect lines

Bounded maps $t : \mathcal{H}_F \to \mathcal{H}_G$, which are $B_a, B_b$ equivariant, i.e. $ta = at$ for $a \in B_a, B_b$. 

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A planar algebraic description of defect lines in CFT  
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Fix $\mathcal{D}$ and $\mathcal{A}$-topological $\mathcal{B}_a - \mathcal{B}_b$-defect and $\overline{\mathcal{D}}$ the dual $\mathcal{B}_b - \mathcal{B}_a$-defect.

Theorem (B. unpublished)

On obtains a subfactor planar algebra. This is the planar algebra of the subfactor related to $\text{Rep}(\mathcal{A})$ which characterizes $\mathcal{D}$. 
Sketch of proof:

\[ N := \mathcal{A}(I), \quad N\mathcal{C}_N := \text{Rep}^I(\mathcal{A}), \text{then } B_\bullet \supset A_2 \text{ are characterized by subfactors } M_\bullet \supset N \text{ with } N\mathcal{M}_\bullet N \in N\mathcal{C}_N. \mathcal{D} \text{ is characterized by a sector } \beta : M_a \to M_b \in M_b\mathcal{C}_{M_a}. \]

\[ A_2(O) \subset B_\bullet(O) \text{ is conjugated to a Longo–Rehren subfactor and yields a Morita equivalence: } \]

\[ N\mathcal{C}_N \boxtimes N\mathcal{C}_N^{\text{op}} \sim M_\bullet \mathcal{C}_M \boxtimes M_\bullet \mathcal{C}_M^{\text{op}}. \]

\[ B_a(O) \subset \mathcal{D}(O) \text{ yields a Morita equivalence: } \]

\[ M_a \mathcal{C}_{M_a} \boxtimes M_a \mathcal{C}_{M_a}^{\text{op}} \sim M_b \mathcal{C}_{M_b} \boxtimes M_a \mathcal{C}_{M_a}^{\text{op}}. \]

\[ B_a(O) \subset \mathcal{D}(O) \text{ is actually conjugated to } \beta(M_a) \subset M_b. \]

\[ \text{The inclusion } \]

\[ B_a(O) \subset \mathcal{D}(O) \subset (\mathcal{D} \boxtimes \mathcal{D})(O) \subset (\mathcal{D} \boxtimes \mathcal{D} \boxtimes \mathcal{D} \subset)(O) \subset \cdots \]

is essentially the Jones tower and every element in the relative commutant is already a defect intertwiner (full inclusion) and vice versa.
Which subfactor (planar algebras) are known so far to arise this way?

\[ [M : N] < 4 \] (Classified by Jones, Ocneanu, ...)
- \(A_k, D_{2n}, E_6, E_8\) (in pairs): All arise in the corresponding ADE classification of SU(2) CFTs of Cappelli, Itzykson and Zuber (1987), observed by Ocneanu.

\[ [M : N] = 4 \] (Classified by Popa)
- Affine Dynkin diagrams: ?

\(4 < [M : N] < 5\)
- GHJ: related to \(E_6\) above
- 2221: \(G_{2,3} \subset E_{6,1}\)
- Haagerup: Conjectured by Evans-Gannon.
- Aseda–Haagerup: ?
- Extended Haagerup: ?

\[ [M : N] > 5 \] ...
Let $N := \mathcal{A}(I)$, $\mathcal{N} := \text{Rep}^I(\mathcal{A})$. There is a functor from the

- 2-category of
  Morita classes of subfactors $[N \subset M_a]$ based on $\mathcal{N}$, Morphisms $\beta: M_a \to M_b$ based on $\mathcal{N}$ and intertwiner $t \in \text{Hom}(\beta_1, \beta_2)$

- 2-category of
  full CFTs based on $\mathcal{A}$, $\mathcal{A}$-topological defects and interwiners.

which is an equivalence.
Higher structure of quantum double/center which maps $\mathcal{N} \mapsto Z(\mathcal{N})$.

*First quantization is a mystery, but second quantization is a functor! – Edward Nelson*
Theorem

Let $\mathcal{A}$ be a conformal net with $\text{Rep}(\mathcal{A}) = D(N \subset M)$, then the planar algebra of $N \subset M$ prescribes a certain topological defect line of full CFTs based on $\mathcal{A}_{N \subset M}$.

Desired construction

\[
\begin{array}{c}
\text{Subfactor} \quad N \subset M \\
\text{Conformal net} \quad \mathcal{A}_{N \subset M}
\end{array}
\]

such that $\text{Rep}(\mathcal{A}_{N \subset M}) \cong D(N \subset M)$ ( = quantum double)
Thank you!
Example $G = \text{SU}(2)$: Irreducible representations $\{0, \frac{1}{2}, 1, \ldots, \frac{k}{2}\}$. \[\text{Rep}(\mathcal{A}_{\text{SU}(2), k})\] is generated by $\frac{1}{2}$-representation $\rho$ and $\cup \in \text{Hom}(\text{id}, \rho \rho)$:

\[
\begin{align*}
\text{circle} &= -d \\
\text{cross} &= \quad = \\
\text{box} &= \quad = 
\end{align*}
\]

with $\cap \in \text{Hom}(\rho \rho, \text{id})$ and braiding defined by Kaufmann bracket

\[
\begin{align*}
\text{cross} &:= - \quad \text{circle}^* \\
\text{box} &= q^{\frac{1}{2}} \quad \text{box} + q^{-\frac{1}{2}} \\
\end{align*}
\]

where $q = e^{\frac{i\pi}{k+2}}$, $d = q + q^{-1} = 2 \cos \left(\frac{\pi}{k+2}\right)$. 