Generalized fixed points of conformal nets

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2017 Joint Mathematics Meetings - Atlanta, GA
AMS Special Session on Advances in Operator Algebras – January 7th, 2017

*based on arXiv:1608.00253
In binding together elements long-known but heretofore scattered and appearing unrelated to one another, it suddenly brings order where there reigned apparent chaos — Henri Poincaré
Motivation

**Question [Evans–Gannon ’11]**

Can we orbifold\(^a\) a VOA [or conformal net] by something more general than a group?

\(^a\)orbifold = fixed point by a finite group of (gauge) automorphisms

- Completely positive maps naturally generalize gauge transformations.
- Hypergroups of completely positive maps are generalized symmetries of quantum field theory in low dimensions.
- Finite index subtheories can be described as fixed points by hypergroup actions.
Let \((M, \Omega)\) \textit{non-commutative probability space}:

- \(M\) von Neumann algebra,
- \(\Omega\) cyclic and separating vector with state \(\omega = (\Omega, \cdot \Omega)\).
Noncommutative probability spaces and Markov maps

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- stochastic map, i.e.
  - \(\phi: M \to M\) is a normal unital completely positive linear map
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Choi’s theorem and Kadison–Schwarz inequality imply

**Well-known fact**

The **fixed point** set \(M^\phi = \{m \in M : \phi(m) = m\}\) is a von Neumann algebra.
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\(\phi\) *extremal* if \(\phi = t\phi_1 + (1-t)\phi_2\) for \(t \in (0, 1)\), \(\phi_i\) Markov \(\Rightarrow \phi_i = \phi\).
Finite set $K = \{\phi_0 = \text{id}_M, \ldots, \phi_n\}$ of extremal Markov map, s.t.\n\n$\text{Conv}(K) = \{\sum_i \lambda_i \phi_i : \lambda_i \geq 0, \sum_i \lambda_i = 1\}$ is a $n$-simplex
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A basis $K = \{c_0 = 1, c_1, \ldots c_m\}$ of a finite-dim unital $\mathbb{C}^*$-algebra such that the elements in $K$ fulfill 1-3.
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Well-known examples, besides finite groups.

- The double cosets $G\backslash H := H \backslash G / H$ of finite groups $H \leq G$
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- The double cosets $K//L := L\setminus K/L$ of finite hypergroups $L \leq K$
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- Fusion algebras with $c_k = \frac{[\rho_k]}{\text{FPdim}(\rho_k)}$, e.g. $K_0(\mathcal{F})$ for $\mathcal{F}$ fusion category.
Proper actions of finite hypergroups by Markov maps II

\[ D(K) = \sum_{k=0}^{n} w_k \text{ global weight of } K, \text{ e.g. } D(G) = |G| \]
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**Haar element:** \( E \circ E = E^\# = E = \phi_k \circ E = E \circ \phi_k \) given by

\[
E(\cdot) = \frac{1}{D(K)} \sum_{k} w_k \phi_k(\cdot) \in \text{Conv}(K)
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**Example**

Finite group $G$ of outer gauge automorphisms $\{\alpha_g : g \in G\}$ with Haar element

$$E(\cdot) = \frac{1}{|G|} \sum_{g \in G} \alpha_g$$
Consider $N \subset M$ finite index (type III) subfactor with

- Jones tower $N \subset M \subset M_1 \subset M_2 \subset \cdots$
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- 2-box space = the relative commutant $M' \cap M_2$. 

Theorem

Consider $(N \subset M, \Omega)$ irreducible with commutative 2-box space and (unique) state-preserving conditional expectation $E: M \to N \subset M$. Then there exists a unique finite hypergroup $K$ of Markov maps, such that $N = M_K$. The global weight of $K$ is $D(K) \equiv \sum_i w_i = [M:N]$. 

Example

Easiest hypergroup $K = \{ \phi_0 = \text{id}_M, \phi_1 \}$ with $\phi_1 \phi_1 = 1$ $w_{\phi_0} + 1 - w_{\phi_1}$, for example $[M:N] = 2$; $w = 1$; $K \sim = \mathbb{Z}_2$.

$[M:N] = 3 + \sqrt{3}$ (E6 GHJ subfactor); $w = 2 + \sqrt{3}$ $\phi_0 \phi_1 1 (w - 1)/w - 1$.
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Example Easiest hypergroup $K = \{ \phi_0 = \text{id} M, \phi_1 \}$ with $\phi_1 \phi_1 = 1 w + 1 - w^2 w$, for example $[M : N] = 2; w = 1$; $K \cong = \mathbb{Z}_2$.

Marcel Bischoff (Vanderbilt)
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Easiest hypergroup $K = \{\phi_0 = \text{id}_M, \phi_1\}$ with $\phi_1 \phi_1 = \frac{1}{w} \phi_0 + \frac{1-w}{w} \phi_1$, for example.
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Galois correspondence

**Theorem ([B. '17])**

Consider subfactor $N = M^K \subset M$ as before. Then there is a one-to-one correspondence between

- intermediate factors $N \subset P \subset M$ and
- subhypergroups $L \subset K$.
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In this case, $P = M^L$ and $N = P^{K \parallel L}$

$$
\begin{align*}
M^K & \subset M^L \subset M \\
N & \subset P \subset M \\
N^{K \parallel L} & \subset P
\end{align*}
$$

where hypergroup $K \parallel L$ acts properly on $P$
Conformal nets

\(\pi_0\) vacuum PER (positive energy representation) of loop group \(LG = C^\infty(S^1, G)\) (\(G\) compact). Consider “net” \(S^1 \supset I \mapsto \pi_0(L^I G)\)” is a conformal net, e.g.
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- \( G = SU(N) \) conformal net \( \mathcal{A}_{SU(N)_k} \) of level \( k \in \mathbb{N} \) [Wassermann ’98][Xu ’00]
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- \( G = \mathbb{T}^n \), conformal net \( \mathcal{A}_L \) of “level” \( L \subset \mathbb{R}^n \) with \( L \) an even lattice \( (\langle x, x \rangle \in 2\mathbb{N} \text{ for all } x \in L) \) [Dong–Xu '06]
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**Definition**

A **conformal net** \( \mathcal{A} \) associates with every interval \( I \subset S^1 \) a von Neumann algebra on a fixed Hilbert space \( \mathcal{H} \), i.e. \( S^1 \ni I \mapsto \mathcal{A}(I) \subset B(\mathcal{H}) \)
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- $G = \mathbb{T}^n$, conformal net $\mathcal{A}_L$ of “level” $L \subset \mathbb{R}^n$ with $L$ an even lattice ($\langle x, x \rangle \in 2\mathbb{N}$ for all $x \in L$) [Dong–Xu '06]

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1. **Isotony**: $I \subset J \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)$
Conformal nets

$\pi_0$ vacuum PER (positive energy representation) of loop group $LG = C^\infty (S^1, G)$ ($G$ compact). Consider “net” $S^1 \ni I \mapsto \pi_0(L^I G)$” is a conformal net, e.g.

- $G = \text{SU}(N)$ conformal net $\mathcal{A}_{\text{SU}(N)_k}$ of level $k \in \mathbb{N}$ [Wassermann '98][Xu '00]
- $G = \mathbb{T}^n$, conformal net $\mathcal{A}_L$ of “level” $L \subset \mathbb{R}^n$ with $L$ an even lattice ($\langle x, x \rangle \in 2\mathbb{N}$ for all $x \in L$) [Dong–Xu '06]

Definition

A **conformal net** $\mathcal{A}$ associates with every interval $I \subset S^1$ a von Neumann algebra on a fixed Hilbert space $\mathcal{H}$, i.e. $S^1 \ni I \mapsto \mathcal{A}(I) \subset B(\mathcal{H})$

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**Definition**

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\(\sim A(I) \) is factor of type \( \text{III}_1 \) and \( Ω \) is cyclic and separating for \( A(I) \)
Remember: A finite hypergroup $K = \{c_0 = 1, c_1, \ldots, c_n\}$ acts properly on $(M, \Omega)$ if there is an injective affine map $\phi: \text{Conv}(K) \rightarrow \text{Markov}(M, \Omega)$ such that $\phi(\text{Conv}(K))$ a simplex with extreme points $\phi(K)$. 

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**Definition (proper hypergroup action (generalized orbifold))**

A finite hypergroup \( K \) acts properly on a conformal net \( \mathcal{A} \) if there is a family \( \phi = \{\phi_I: K \to \text{Markov}(\mathcal{A}(I), \Omega)\} \) of proper actions, which is compatible:

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Marcel Bischoff (Vanderbilt)  Generalized fixed points of conformal nets  01/06/17
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Let $\mathcal{A}$ be a conformal net and $K$ a hypergroup acting properly on $\mathcal{A}$.

Then $\mathcal{A}^K$ defined by $I \mapsto \mathcal{A}(I)^K$ is an irreducible finite index conformal subnet with index

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Finite index inclusions are generalized fixed points

**Theorem ([B. '17])**

Let \( B \subset A \) be a finite index subnet. Then:

1. \( B(I) \subset A(I) \) is irreducible with commutative 2-box space.
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Corollary (absence of non-trivial Kac actions on conformal nets) [B ’17]

If $\mathcal{B} \subset \mathcal{A}$ is a finite index subnet which is fixed point by an outer action of a Kac (finite-dim C∗-Hopf algebra) then $\mathcal{B} = \mathcal{A}^G$ for a finite group $G$.


Let $\mathcal{B} \subset \mathcal{A}$ be finite index subnet, then

$\exists$ subfactors $\{\mathcal{N}_i \subset \mathcal{M}_i\}_{n_i=1}^n [\mathcal{A}(I) : \mathcal{B}(I)] = 1 + n \sum_{i=1}^n [\mathcal{M}_i : \mathcal{N}_i] \in \{1, 2, 3, 4, 3 + \sqrt{3}, 5, \ldots\}$

due to Jones’ index rigidity $[\mathcal{M} : \mathcal{N}] \in \{4 \cos^2(\pi/n) : n = 3, 4, 5, \ldots\} \cup [4, \infty]$ and

$[\mathcal{M} : \mathcal{N}] \leq 5$ classification. New gap $(3 + \sqrt{3}, 5)$ cf. [Carpi–Kawahigashi–Longo ’10]
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**Unitary (braided) fusion categories**

**Unitary fusion category** $\mathcal{F} = \text{rigid semisimple C}^*$-$\otimes$-category with $|\text{Irr}(\mathcal{F})| < \infty$

- $\text{Rep}^k(G)$ for $G$ finite group $k \in Z(G)$ with $k^2 = e$
- $\text{Hilb}_G^\omega$ for some $[\omega] \in H^3(G, \mathbb{T})$ or equivalently a $G$-kernel
- **even part** $\mathcal{F}_{N \subset M}$ of a finite index finite depth subfactor $N \subset M$
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Müger’s centralizer \( \mathcal{D} \subset \mathcal{C} \) subcategory of \( \mathcal{C} \) braided fusion category:

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\mathcal{D}' \cap \mathcal{C} = \left\{ \rho \in \mathcal{C} : \varepsilon(\sigma, \rho)\varepsilon(\rho, \sigma) \equiv \frac{\sigma}{\rho} \frac{\rho}{\sigma} \equiv 1_{\rho \otimes \sigma} \text{ for all } \sigma \in \mathcal{D} \right\}
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- $\mathcal{C}$ **symmetric** if $\mathcal{C}' \cap \mathcal{C} = \mathcal{C} \sim \mathcal{C} \cong \text{Rep}^k (G)$ [Doplicher–Roberts ’89]
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- $\mathcal{C}$ non-degenerate braided if $\mathcal{C}' \cap \mathcal{C} \cong \text{Hilb} \sim \text{unitary modular tensor category} \sim 3\text{-manifold invariants, topological field theory, etc.}$
If $\mathcal{F}$ is a unitary fusion category, then the Drinfel’d center $Z(\mathcal{F})$ is a unitary modular tensor category.
Unitary (braided) fusion categories

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\[ \mu_{\mathcal{A}} = \left[ \mathcal{A} \left( \begin{array}{c} \circ \end{array} \right)^\prime : \mathcal{A} \left( \begin{array}{c} \circ \end{array} \right) \right] < \infty \]

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**Conjecture** [Kawahigashi, ...]

If $\mathcal{F}$ is a **unitary fusion category** there exists a completely rational conformal net $\mathcal{A}$, such that $\text{Rep}(\mathcal{A})$ is braided equivalent to $Z(\mathcal{F})$. 
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**Conjecture [Kawahigashi, ...]**

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\( \sim \) **Reconstruction Program** [Jones]. Examples: [B. '15][Xu '16][Evans–Gannon]

**Proposition [B. '16]**

Let \( N \subset M \) be a finite index finite depth subfactor. If there is a conformal net \( \mathcal{A} \) such that \( \text{Rep}(\mathcal{A}) \cong Z(\mathcal{F}_{N \subset M}) \), then \( N \subset M \) arises from \( \mathcal{A} \).
Theorem (classification of actions & categorification - holomorphic case [B. '17])

Let $F$ be a proper hypergroup acting on a holomorphic net $\mathcal{A}$.
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Let $F$ be a proper hypergroup acting on a holomorphic net $\mathcal{A}$.

Then there exist a fusion category $\mathcal{F}$ with Grothendieck ring $K_0(\mathcal{F})$ equal to $F$ and $\text{Rep}(\mathcal{A}^F)$ braided equivalent to the Drinfel'd center $Z(\mathcal{F})$. 

Interpretation (in analogy with finite groups):

▶ $F$ is obtained by $\alpha$-induction $\sim \text{K-twisted representations of } \mathcal{A}$

▶ $\text{Rep}(\mathcal{A}^F) = \text{K-equivariantization}$
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Case: $F = G$ a finite group then $\mathcal{F} \cong \text{Hilb}_G^{\omega}$ for some $[\omega] \in H^3(G, \mathbb{T})$ [Müger]
Hypergroup actions on holomorphic nets

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**Theorem (··· - completely rational case [B. ’17])**

Let $K$ be a proper hypergroup acting on a completely rational net $\mathcal{A}$.

Then there exists a fusion category $\mathcal{F} \supset \text{Rep}(\mathcal{A})$ and $K = K_0(\mathcal{F})/K_0(\text{Rep}(\mathcal{A}))$ and $\text{Rep}(\mathcal{A}^K)$ is braided equivalent to the Müger centralizer $\text{Rep}(\mathcal{A})' \cap Z(\mathcal{F})$.

Interpretation (in analogy with finite groups):

- $\mathcal{F}$ is obtained by $\alpha^+-$induction $\sim “K$-twisted representations of $\mathcal{A}”$
- $\text{Rep}(\mathcal{A}^K) = “\mathcal{F}^K \sim K$-equivariantization”
Consider unitary fusion category $\mathcal{F}$ with Grothendieck ring $F = K_0(\mathcal{F}) = G \cup \{\rho\}$ with $\rho^2 = \sum_{g \in G} g$ and $\rho g = g \rho = \rho$.

Then $G$ is abelian and $\mathcal{F}$ are given by a non-degenerate bicharacter (Fourier transformation) on $G$ and a sign (Frobenius–Schur indicator)

[Tambara–Yamagami '98]
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[Tambara–Yamagami '98]

**Theorem ([B. in preparation])**

**Let** \( \mathcal{F} \) **as above with** \( G \) **odd (abelian) group, then there is**

1. **and even lattice self-dual lattice** \( L \)
2. **an action of** \( F \) **on** \( \mathcal{A}^{\mathcal{F}}_{TL} = \mathcal{A}_L \)

**such that** \( \text{Rep}(\mathcal{A}^{\mathcal{F}}_{TL}) \) **is braided equivalent to** \( Z(\mathcal{F}) \).
Reconstruction I: Tambara–Yamagami categories

Consider unitary fusion category $\mathcal{F}$ with Grothendieck ring $F = K_0(\mathcal{F}) = G \cup \{\rho\}$ with $\rho^2 = \sum_{g \in G} g$ and $\rho g = g \rho = \rho$.

- Then $G$ is abelian and $\mathcal{F}$ are given by a non-degenerate bicharacter (Fourier transformation) on $G$ and a sign (Frobenius–Schur indicator)

[Tambara–Yamagami '98]

**Theorem ([B. in preparation])**

Let $\mathcal{F}$ as above with $G$ odd (abelian) group, then there is

1. and even lattice self-dual lattice $L$
2. an action of $F$ on $A_{TL} = A_L$

such that $\text{Rep}(A^F_{TL})$ is braided equivalent to $Z(\mathcal{F})$.

$\mathcal{F} \cong \text{Hilb}_G \rtimes \mathbb{Z}_2$ is a nilpotent fusion category $\sim F//G \cong \mathbb{Z}_2$

$A^F_L(I) \subset A_L(I) \sim R^{\mathbb{Z}_2} \subset R \rtimes \Delta(G)$ for action of $(G \rtimes _{-1} \mathbb{Z}_2) \times G$ on $R$

$\mathbb{Z}_2$-action on $A_{\tilde{L}}$ and choose $L = (\tilde{L} \times \tilde{L}') \oplus G$ and $\tilde{L}'$ mirror of $\tilde{L}$

Uses that $(G, q)$ lifts always lifts to a lattice $\tilde{L}$ [Nikulin '79] relation to real projective K3 surfaces???
Reconstruction II: Izumi–Xu near group categories and even part of $2^G 1$ subfactors

- **Izumi–Xu** unitary fusion categories $\mathcal{F}$, i.e. $F = K_0(\mathcal{F}) = G \cup \{\rho\}$ with $\rho^2 = |G|\rho + \sum_{g \in G} g$

- **Cuntz algebra approach** [Izumi '01][Evans–Gannon '13] $\sim$ polynomial eqs.

- Existence: solution only known for $|G| \leq 13$ (conjectured for all odd $G$)

### Conjecture

For every $G$ odd abelian group there exists an even lattice $L$ and an action of $K = F//G = \{\text{id}, \phi\}$ on $\mathcal{A}_L$, such that $\text{Rep}(\mathcal{A}_L^F) = Z(\mathcal{F})$ for some categorification $\mathcal{F}$ of $F$.

\[ a_{\text{conj}}: \text{involving lattice lifts of } (G, q) \text{ and } (G', q') \text{ with } |G'| = |G| + 4 \]

<table>
<thead>
<tr>
<th>$G$</th>
<th>$L$</th>
<th>$\mathcal{A}_L^K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}_1$</td>
<td>$E_8$</td>
<td>$\mathcal{A}<em>{G</em>{2,1} \times F_{4,1}}$</td>
</tr>
<tr>
<td>$\mathbb{Z}_2$</td>
<td>$D_8$</td>
<td>$\mathcal{A}<em>{\text{SU}(2)</em>{10} \times \text{Spin}(11)_{1} \times \mathbb{Z}_2}$</td>
</tr>
<tr>
<td>$\mathbb{Z}_3$</td>
<td>$E_6 \times A_2$</td>
<td>$\mathcal{A}<em>{G</em>{2,3} \times \text{SU}(2)_{1}}$</td>
</tr>
<tr>
<td>$\mathbb{Z}_3 \times \mathbb{Z}_3$</td>
<td>$(E_6 \times A_2)^2$</td>
<td>$\mathcal{A}<em>{\text{Hg} \otimes \mathcal{A}</em>{E_6 \times A_2}}$</td>
</tr>
</tbody>
</table>

[Dynkin '52][?][B. '16]  
[B. '16]  
[Dynkin '52][Xu unpublished] hypothetical [EvGa '11]

*these come from inclusion of corresponding Lie algebras already studied by Dynkin
Thank you for your attention!