Semigroup elements associated to conformal nets and boundary quantum field theory

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Introduction

- Algebraic quantum field theory: A family of algebras containing all local observables associated to space-time regions.
- Many structural results, recently also construction of interesting models
- Conformal field theory (CFT) in 1 and 2 dimension described by AQFT quite successful, e.g. partial classification results (e.g. $c < 1$) (Kawahigashi and Longo, 2004)
- Boundary Conformal Quantum Field Theory (BCFT) on Minkowski half-plane: (Longo and Rehren, 2004)
- Boundary Quantum Field Theory (BQFT) on Minkowski half-plane: (Longo and Witten, 2010)
Conformal Nets

Nets on Minkowski half-plane

Standard subspaces

Conformal nets associated to lattices

Semigroup elements
**Conformal Nets**

\( \mathcal{H} \) Hilbert space, \( \mathcal{I} = \text{family of proper intervals on } S^1 \cong \mathbb{R} \)

\[ \mathcal{I} \ni I \mapsto \mathcal{A}(I) = \mathcal{A}(I)^{''} \subset \mathcal{B}(\mathcal{H}) \]

**A.** Isotony. \( I_1 \subset I_2 \implies \mathcal{A}(I_1) \subset \mathcal{A}(I_2) \)

**B.** Locality. \( I_1 \cap I_2 = \emptyset \implies [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\} \)

**C.** Möbius covariance. There is a unitary representation \( U \) of the Möbius group (\( \cong \text{PSL}(2, \mathbb{R}) \)) on \( \mathcal{H} \) such that

\[ U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI). \]

**D.** Positivity of energy. \( U \) is a positive-energy representation, i.e. generator \( L_0 \) of the rotation subgroup (conformal Hamiltonian) has positive spectrum.

**E.** Vacuum. \( \ker L_0 = \mathbb{C}\Omega \) and \( \Omega \) (vacuum vector) is a unit vector cyclic for the von Neumann algebra \( \bigvee_{I \in \mathcal{I}} \mathcal{A}(I) \).

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**Consequences**

Marcel Bischoff (Uni Roma II)

Semigroup elements associated to conformal nets and BQFT

Paris, 1 June 2011
Outline

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Some consequences

- Irreducibility. $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I) = B(\mathcal{H})$
- Reeh-Schlieder theorem. $\Omega$ is cyclic and separating for each $\mathcal{A}(I)$.
- Bisognano-Wichmann property. The Tomita-Takesaki modular operator $\Delta_I$ and and conjugation $J_I$ of the pair $(\mathcal{A}(I), \Omega)$ are
  \[
  U(\Lambda(-2\pi t)) = \Delta^it, \quad t \in \mathbb{R}
  \]
  \[
  U(r_I) = J_I
  \]
  (Gabbiani and Fröhlich, 1993), (Guido and Longo, 1995)
- Haag duality. $\mathcal{A}(I') = \mathcal{A}(I)'$.
- Factoriality. $\mathcal{A}(I)$ is $\text{III}_1$-factor (in Connes classification)
- Additivity. $I \subset \bigcup_i I_i \implies \mathcal{A}(I) \subset \bigvee_i \mathcal{A}(I_i)$ (Fredenhagen and Jörß, 1996).
Complete rationality

Completely rational conformal net (Kawahigashi, Longo, Müger 2001)

- **Split property.** For every relatively compact inclusion of intervals \( \exists \) intermediate **type I factor** \( M \)

\[
\mathcal{A}(\bigcirc) \subset M \subset \mathcal{A}(\bigcirc)
\]

- **Strong additivity.** Additivity for touching intervals:

\[
\mathcal{A}(\bigcirc) \lor \mathcal{A}(\bigcirc) = \mathcal{A}(\bigcirc)
\]

- **Finite \( \mu \)-index:** finite Jones index of subfactor

\[
\mathcal{A}(\bigcirc) \lor \mathcal{A}(\bigcirc) \subset (\mathcal{A}(\bigcirc) \lor \mathcal{A}(\bigcirc))'
\]

where the intervals are splitting the circle.

**Consequences**

- Only finite sectors, each sector has finite statistical dimension
- **Modularity:** The category of DHR sectors is modular, i.e. non degenerated braiding.
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Example

$G$ compact Lie group

**Loop group:** $LG = C^\infty(S^1, G)$ (point wise multiplication)

Projective representations $\leftrightarrow$ representations of a central extension

\[
1 \rightarrow T \rightarrow \tilde{LG} \rightarrow LG \rightarrow 1
\]

$\pi_{0,k}$ projective **positive-energy** and **vacuum** representation (classified by the level $k$)

\[
I \mapsto A_{G,k}(I) = \pi_{0,k}(L_I G)''
\]

is a **conformal net**; $L_I G$ loops supported in $I$.

Example

$G = SU(n)$ gives completely rational conformal net (Xu, 2000)
Loop group net

Example

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Nets on the real line

- Conformal net on the **real line** identifying $S^1 \setminus \{-1\} \cong \mathbb{R}$

```
Conformal net on $S^1$ \[\rightarrow\] Conformal net on $\mathbb{R}$
```

$\{\}$
Minkowski half-plane $M_+$

- **Minkowski half-plane** $x > 0$, $ds^2 = dt^2 - dx^2$

- **Double cone** $\mathcal{O} = I_1 \times I_2$ where $I_1$, $I_2$ disjoint intervals
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**Boundary conformal quantum field theory** *(Longo and Rehren, 2004)*

\[ \mathcal{A}_+(\mathcal{O}) = \mathcal{A}(I_1) \vee \mathcal{A}(I_2) \]

**Boundary quantum field theory** *(Longo and Witten, 2010)*

\[ \mathcal{A}_V(\mathcal{O}) = \mathcal{A}(I_1) \vee V \mathcal{A}(I_2)V^* \]

*V* unitary on *H*

- \([V, T(t)] = 0\), i.e. commutes with translation *T(t)*
- \(V \mathcal{A}(\mathbb{R}_+)V^* \subseteq \mathcal{A}(\mathbb{R}_+)\)
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A local (time) translation covariant net on Minkowski half-plane on a Hilbert space $\mathcal{H}$ is a map $\mathcal{K}_+ \ni \mathcal{O} \mapsto \mathcal{B}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ which fulfills:

1. **Isotony.** $\mathcal{O}_1 \subset \mathcal{O}_2$ implies $\mathcal{B}(\mathcal{O}_1) \subset \mathcal{B}(\mathcal{O}_2)$.

2. **Locality.** If $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}_+$ are mutually space-like separated then $[\mathcal{B}(\mathcal{O}_1), \mathcal{B}(\mathcal{O}_2)] = \{0\}$. 
Local nets on Minkowski half-plane

A local (time) translation covariant net on Minkowski half-plane on a Hilbert space $\mathcal{H}$ is a map $\mathcal{K}_+ \ni \mathcal{O} \mapsto \mathcal{B}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ which fulfills:

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![Diagram showing the relationship between operators and regions in Minkowski half-plane](image)
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3. **Time-translation covariance.** There exists an unitary one-parameter group $T(t) = e^{itP}$ with positive generator $P$ such that:

$$T(t)\mathcal{B}(\mathcal{O})T(t)^* = \mathcal{B}(\mathcal{O}_t), \quad \mathcal{O} \in \mathcal{K}_+, \quad \mathcal{O}_t = \mathcal{O} + (t, 0)$$
A local (time) translation covariant net on Minkowski half-plane on a Hilbert space $H$ is a map $\mathcal{K}_+ \ni \mathcal{O} \mapsto B(\mathcal{O}) \subset B(H)$ which fulfills:

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4. **Vacuum.** $\Omega \in H$ is a up to the multiple unique $T$ invariant vector and cyclic and separating for every $B(\mathcal{O})$ for $\mathcal{O} \in \mathcal{K}_+$. 
Semigroup $E(A)$ associated to a conformal net $A$

Semigroup $E(A)$ of unitaries on $\mathcal{H}$ (associated to $A$)

- $[V, T(t)] = 0$, i.e. commutes with translation $T(t)$
- $VA(\mathbb{R}_+)V^* \subset A(\mathbb{R}_+) \sim VA(a + \mathbb{R}_+)V^* \subset A(a + \mathbb{R}_+)$

Trivial examples of elements in $E(A)$:

- $V = T(t)$ $t > 0$ positive translations
- $V$ inner symmetry, i.e. $VA(I)V^* = A(I)$ for all proper $I$

Construction

\[ \begin{array}{c}
\text{Conformal net} \\
A \text{ on } \mathbb{R}
\end{array} + \begin{array}{c}
\text{semigroup element} \\
V \in E(A)
\end{array} \longrightarrow \begin{array}{c}
\text{local net } A_V \\
on M_+
\end{array} \]
Outline

Conformal Nets

Nets on Minkowski half-plane

Standard subspaces

Conformal nets associated to lattices

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Standard subspaces

\( \mathcal{H} \) complex Hilbert space, \( H \subset \mathcal{H} \) real subspace. Symplectic complement:

\[
H' = \{ x \in \mathcal{H} : \text{Im}(x, H) = 0 \} = iH^\perp
\]

**Standard subspace:** closed, real subspace \( H \subset \mathcal{H} \) with \( \overline{H} + iH = \mathcal{H} \) and \( H \cap iH = \{0\} \).

Define antilinear unbounded closed involutive \((S^2 \subset 1)\) operator

\[
S_H : x + iy \mapsto x - iy \text{ for } x, y \in H.
\]

Conversely \( S \) densely defined closed, antilinear involution on \( \mathcal{H} \), \( H_S = \{ x \in \mathcal{H} : Sx = x \} \) is a standard subspace:

| standard subspaces \( H \) | 1:1 | densely defined, closed, antilinear involutions \( S \) |

**Modular Theory:** Polar decomposition \( S_H = J_H \Delta_H^{1/2} \)

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Standard subspaces and inner functions

**Standard pair.** \((H, T)\)

- \(H \subset \mathcal{H}\) standard subspace with
- \(T(t) = e^{itP}\) one-param. group with **positive generator** \(P\)
- \(T(t)H \subset H\) for \(t \geq 0\)

**Theorem (Borchers Theorem for standard subspaces)**

Let \((H, T)\) be a standard pair, then

\[
\Delta^i_s T(t) \Delta^{-i_s}_H = T(e^{-2\pi s t}) \quad (s, t \in \mathbb{R})
\]

\[
J_H T(t) J_H = T(-t) \quad (t \in \mathbb{R})
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Standard subspaces and inner functions

\[ \mathcal{E}(H) = \text{unitaries } V \text{ on } \mathcal{H} \text{ such that } VH \subset H \text{ and } [V, T(t)] = 0. \]

Analog of the Beurling-Lax theorem.

**Characterization of \( \mathcal{E}(H) \).** (Longo and Witten, 2010)

If \((H, T)\) irreducible standard pair, then are equivalent

1. \( V \in \mathcal{E}(H) \), i.e. \( VH \subset H \) with \( V \) unitary on \( \mathcal{H} \) commuting with \( T \).

2. \( V = \varphi(P) \) with \( \varphi \) boundary value of a symmetric inner analytic \( L^\infty \) function \( \varphi : \mathbb{R} + i\mathbb{R}_+ \to \mathbb{C} \), where
   - symmetric \( \overline{\varphi(p)} = \varphi(-p) \) for \( p \geq 0 \)
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Net of free bosons.

**Net of standard subspaces (prequantised theory)**

- \( L \mathbb{R} = C^\infty(S^1, \mathbb{R}) \) yields a Hilbert space \( \mathcal{H} = \overline{L \mathbb{R}}\|\cdot\| \) using
  - semi-norm. \( \|f\| = \sum_{k>0} k|\hat{f}_k| \)
  - complex-structure. \( \mathcal{J}: \hat{f}_k \mapsto -i \text{sign}(k) \hat{f}_k \)
  - symplectic form. \( \omega(f, g) = \text{Im}(f, g) = 1/(4\pi) \int g df \)

- Local spaces: \( L_I \mathbb{R} = \{ f \in L \mathbb{R} : \text{supp} f \subset I \} \)
  \[ I \mapsto H(I) = \overline{L_I \mathbb{R}} \subset \mathcal{H} \]

**Conformal net of a free boson**

- Second quantization. Conformal net on the symmetric Fock space \( e^\mathcal{H} \) by CCR functor (Weyl unitaries):
  \[ I \mapsto A(I) := \text{CCR}(H(I))'' \subset B(e^\mathcal{H}) \]

- Weyl unitaries
  \[ W(f)W(g) = e^{-i\omega(f,g)}W(f + g), \]

- Vacuum state
  \[ \phi(W(f)) = (\Omega, W(f)\Omega) = e^{-1/2\|f\|^2} \]
Net of free bosons.

Net of standard subspaces (prequantised theory)

- \( L_\mathbb{R} = C^\infty(S^1, \mathbb{R}) \) yields a Hilbert space \( \mathcal{H} = \overline{L_\mathbb{R}} \| \cdot \| \) using
  - semi-norm. \( \| f \| = \sum_{k>0} k |\hat{f}_k| \)
  - complex-structure. \( \mathcal{J} : \hat{f}_k \mapsto -i \text{sign}(k) \hat{f}_k \)
  - symplectic form. \( \omega(f,g) = \text{Im}(f,g) = 1/(4\pi) \int g df \)

- Local spaces: \( L_I \mathbb{R} = \{ f \in L_\mathbb{R} : \text{supp} f \subset I \} \)
  \[ I \mapsto H(I) = \overline{L_I \mathbb{R}} \subset \mathcal{H} \]

Conformal net of a free boson

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Localized automorphisms

Conformal net of $n$ free bosons

$$A_n(I) = A_1^\otimes n(I) = CCR(H(I) \oplus \cdots \oplus H(I))$$

Local endomorphisms (representations) of $A_n = A^\otimes n$

$\ell : S^1 \rightarrow \mathbb{R}^n$ smooth with compact support in $I \in \mathcal{I}$ gives localized automorphism

$$\rho_\ell(W(f)) = e^{-\frac{i}{2\pi} \int \langle \ell, f \rangle_{\mathbb{R}^n} W(f)}$$

Charge:

$$q_\ell = \frac{1}{2\pi} \int_{S^1} \ell \in \mathbb{R}^n \quad \rho_\ell \simeq \rho_m \iff q_\ell = q_m$$

Statistics operator:

$$\epsilon(\rho_\ell, \rho_m) = e^{\pm i\pi \langle q_\ell, q_m \rangle_{\mathbb{R}^n}}$$

Local extension: If $\langle q_\ell, q_\ell \rangle \in 2\mathbb{Z}$ then $\epsilon(\rho_\ell, \rho_\ell) = 1 \sim$ local extension (by cross product).
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Let $Q$ be an (positive-definite) **even lattice** (eg. root lattice) of rank $n$

- $\forall \alpha \in Q$: $\langle \alpha, \alpha \rangle \in 2\mathbb{N} \implies$ integral $\forall \alpha, \beta \in Q$: $\langle \alpha, \beta \rangle \in \mathbb{Z}$.

- **dual lattice** (characters)

  $Q^* = \{ \alpha \in E_Q : \langle \alpha, Q \rangle \in \mathbb{Z} \} \subset E_Q \equiv Q \otimes_{\mathbb{Z}} \mathbb{R}$. (eg. weight lattice in case of root lattices).

\[
A_2 \leftrightarrow SU(3)
\]

**corresponding torus**

$T_Q = E_Q/Q$
Even lattices

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\[ T_Q = E_Q / Q \]
Conformal nets associated to lattices

Local extension. For a lattice $Q$ of rank $n$ there is $\mathcal{A}_Q \supset \mathcal{A} \otimes n$ containing of the net $\equiv \mathcal{A} \otimes n$ of $n$ free bosons. Locally

$$\mathcal{A}_Q(I) = (\mathcal{A}(I) \otimes \ldots \otimes \mathcal{A}(I)) \rtimes Q$$

(Buchholz, Mack, Todorov 1988) ($n = 1$) (Staszkiewicz, 1995) (Dong and Xu, 2006)

Construction

- Conformal nets corresponding to Lattice Vertex Operator Algebras.

Some properties:

- Sectors finite group $Q^*/Q$, each sector statistical dimension 1.
- Completely rational net $\mu = |Q^*/Q|$ (Dong and Xu, 2006).
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**Construction**

| even lattice $Q$ of rank $n$ | Completely rational conformal net $\mathcal{A}_Q$ |

- Conformal nets corresponding to **Lattice Vertex Operator Algebras**.

**Some properties:**

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Simply laced groups and root lattices

$G$ simply-connected **simple-laced** Lie group, e.g.

- **A** $SU(n + 1), \ n \geq 1 \leftrightarrow A_n:\ 
  \begin{array}{c}
  -
  \end{array}$

- **D** $Spin(2n), \ n \geq 3 \leftrightarrow D_n:\ 
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  -
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- **E** Exceptional Lie Groups $E_6, E_7, E_8:\ 
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**Q** root lattice spanned by simple roots $\{\alpha_1, \ldots, \alpha_n\}$

Cartan matrix $(C_{ij})\ \langle \alpha_i, \alpha_j \rangle = C_{ij} = \begin{cases} 2 & i = j \\ -1 & i \rightarrow j \\ 0 & \end{cases}$
Simply laced groups and root lattices

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**Q root lattice** spanned by simple roots \( \{\alpha_1, \ldots, \alpha_n\} \)

**Maximal torus** \( (Q \otimes \mathbb{Z} \mathbb{R})/Q \cong T \subset G \sim \mathcal{A}_{T,1} \equiv \mathcal{A}_Q \)

(Conjectured) equivalence (proofed in case \( G = \text{SU}(n) \) (Xu, 2009))

\[
\begin{array}{ccc}
\text{loop group net} & = \mathcal{A}_{G,1} \leftrightarrow \sim & \mathcal{A}_Q \\
\text{for such } G \text{ at level } 1 & & \text{conformal net associated at } Q
\end{array}
\]
Outline

Conformal Nets

Nets on Minkowski half-plane

Standard subspaces

Conformal nets associated to lattices

Semigroup elements
Second quantization unitaries

\( \mathcal{H} \) one-particle space of a bosons (completion of \( L\mathbb{R} \) \( H(\mathbb{R}_+) \) standard subspace localized in \( \mathbb{R}_+ \)

\( \varphi : \mathbb{R} \rightarrow \mathbb{C} \) inner function, then

\[
V_0 = \varphi(P_0) \implies V_0 H(\mathbb{R}_+) \subset H(\mathbb{R}_+), \ [V_0, e^{itP_0}] = 0
\]

\( P_0 \) generator of translation.

By second quantization \( \mathcal{A}(I) = \text{CCR}(H(I))'' \).

\[
V = \Gamma(V_0) \implies V \in \mathcal{E}(\mathcal{A})
\]
More general for $n$ bosons

$$\mathcal{A}_n(\mathbb{R}_+) \cong \mathcal{A}(\mathbb{R}_+) \otimes^n = \text{CCR}(H(\mathbb{R}_+) \oplus \cdots \oplus H(\mathbb{R}_+))$$

**Theorem (Prequantized semigroup reducible case (Longo and Witten, 2010))**

$V_0 \in \mathcal{E}(H(\mathbb{R}_+) \oplus \cdots \oplus H(\mathbb{R}_+))$, then $V_0 = \varphi_{kl}(P_0)$ matrices of functions such that $\varphi_{kl}(p)$ unitary matrix for almost all $p > 0$, $\varphi_{kl}$ boundary value of a $L^\infty$ function analytic on the upper half-plane which is symmetric $\varphi_{kl}(p) = \varphi_{kl}(-p)$.

**Theorem**

$V = \Gamma(V_0) \in \mathcal{E}(\mathcal{A}_n)$ for the second quantization of $V_0$ given above.
More general for $n$ bosons

$$\mathcal{A}_n(\mathbb{R}_+) \cong \mathcal{A}(\mathbb{R}_+)^{\otimes n} = \text{CCR}(H(\mathbb{R}_+) \oplus \cdots \oplus H(\mathbb{R}_+))$$

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Question

Which elements of the semigroup $\mathcal{E}(\mathcal{A}_n)$ extend to the local extensions by lattices?

$$\mathcal{A}_Q(I) = \mathcal{A}_n(I) \rtimes Q$$

where $Q$ even lattice of rank $n$
Induction for local extension by free abelian groups

**Extension** of the endomorphism $\eta = \text{Ad} V$ of $\mathcal{A}_n(\mathbb{R}_+)$ with $V \in \mathcal{E}(\mathcal{A}_n)$ to

$$\mathcal{A}_Q(\mathbb{R}_+) = \mathcal{A}_n(\mathbb{R}_+) \rtimes \beta_i Q$$

$\beta_i$ localized in $\mathbb{R}_+$

Assume $\eta$ and $\beta_i$ commute up to some cocycle $z_i \in \mathcal{A}_n(\mathbb{R}_+)$

$$z_i \in \text{Hom}(\eta \beta_i, \beta_i \eta) \iff z_i \beta_i(\eta(x)) = \eta(\beta_i(x))z_i \text{ for all } x \in \mathcal{A}_n(\mathbb{R}_+)$$

and the **compatibility condition**

$$z_i \beta_i(z_j) = z_j \beta_j(z_i)$$

then $\eta$ extends to $\tilde{\eta} = \text{Ad} \tilde{V}$.

$$V \in \mathcal{E}(\mathcal{A}_n) \xrightarrow{\text{extends}} \tilde{V} \in \mathcal{E}(\mathcal{A}_Q)$$
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Existence of \( z_i \in \text{Hom}_{\mathcal{A}_n(\mathbb{R}_+)}(\eta/\beta_i, \beta_i \eta) \) with the above properties in our model ensure

\[
V = \Gamma(\varphi_{ik}(P_0)) \in \mathcal{E}(\mathcal{A}_n) \xrightarrow{\text{extends?}} \tilde{V} \in \mathcal{E}(\mathcal{A}_Q)
\]

**Restrictions.** Such \( z_i \) can be constructed if

- **Algebraic obstruction.** The “inner function matrix” has to be constant on every component of the lattice
- **Analytical obstruction.** The “inner function” need to be Hölder continuous at 0, i.e.

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\frac{|1 - \varphi(p)|^2}{|p|} \text{ locally integrable at } p = 0
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Semigroup elements for lattice models

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Results

Theorem

Let $\mathcal{A}$ be conformal net of the family

$\mathcal{A}_Q$ associated to an even irreducible lattice $Q$

$\mathcal{A}_{G,1}$ for $G = SU(n)$ ($G$ simple, simply connected, simple-laced)

$\mathcal{A}$ and $\varphi$ Hölder cont. $\rightarrow V \in \mathcal{E}(\mathcal{A}) \rightarrow$ local net $\mathcal{A}_V$ on Minkowski half-plane

Further

$U$ inner symmetry $V \in \mathcal{E}(\mathcal{A}) \implies Vu \in \mathcal{E}(\mathcal{A})$

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Other constructions using $\mathcal{E}(\mathcal{A})$

Models in 2D Minkowski space

If there is a one-parameter group $V_t$ with $V_t \in \mathcal{E}(\mathcal{A})$ for $t \geq 0$ with **negative** generator

$\leadsto$ local Poincaré covariant net on 2D Minkowski space (Longo).

$\leadsto$ wedge-local Poincaré covariant net on 2D Minkowski space with non-trivial scattering (Tanimoto).

Example

For $\mathcal{A}$ the net of free boson ($U(1)$-current) and the inner function $\varphi_t(p) = e^{-it/P}$ we have $V_t = \Gamma(\varphi_t(P_0))$ like above and the construction yields the free massive scalar boson on 2D Minkowski space.

- Works for all “free field construction”
- But because of the mentioned Hölder continuity this does not work for extensions by lattices.
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Extensions by the lattice $\mathbb{Z}^n$ (not even!) yields Fermi (=twisted local) net $\mathcal{F} = \text{Fer}_\mathbb{C} \otimes^n$. Even part $\mathcal{A} := \mathcal{F}_{\mathbb{Z}^2}$ local conformal net, i.e.

$$\text{Fer}_\mathbb{C} \otimes^n = \mathcal{A}_{\mathbb{Z}^n}$$

But $\text{Fer}_\mathbb{C}$ can be realized on antisymmetric Fock space (CAR). Using second quantization...

...we have two methods to construct elements in $\mathcal{E}(\mathcal{F})$ (and $\mathcal{E}(\mathcal{A})$).

- $\mathcal{E}(\mathcal{F})_{\text{CCR}}$ : constructed as extensions by the lattice
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Boson-Fermion correspondence

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...we have two methods to construct elements in $\mathcal{E}(\mathcal{F})$ (and $\mathcal{E}(\mathcal{A})$).

- $\mathcal{E}(\mathcal{F})_{\text{CCR}}$: constructed as extensions by the lattice
- $\mathcal{E}(\mathcal{F})_{\text{CAR}}$: second quantization unitaries in the CAR algebra (analogous like the CCR case).

$$\mathcal{E}(\mathcal{F})_{\text{CAR}} \cap \mathcal{E}(\mathcal{F})_{\text{CCR}} = \text{trivial elements}.$$
Extensions by the lattice $\mathbb{Z}^n$ (not even!) yields Fermi (=twisted local) net $\mathcal{F} = \text{Fer}_C^\otimes n$. Even part $\mathcal{A} := \mathcal{F}^\mathbb{Z}_2$ local conformal net, i.e.

$$\text{Fer}_C^\otimes n = \mathcal{A}_{\mathbb{Z}^n}$$

But $\text{Fer}_C$ can be realized on antisymmetric Fock space (CAR). Using second quantization.

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$$\mathcal{E}(\mathcal{F})_{\text{CAR}} \cap \mathcal{E}(\mathcal{F})_{\text{CCR}} = \text{trivial elements.}$$
Summary

We have constructed

▶ Elements of the semigroup $E(A)$ for a large class of rational conformal field theories is found

→ New models of boundary quantum field theory.

Open questions

▶ Loop group nets at higher level (Coset construction/Orbifold)
▶ Restriction of a net of free fermions (semigroup elements by second quantization) should give more examples.
▶ Construction of 1+1D massive models one-parameter semigroup. Until yet just examples from free field construction.
Merci beaucoup!!

Semigroup elements associated to conformal nets and boundary quantum field theory

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