
Outline:

(1) Introduction
(2) Two interval inclusions
(3) Modularity

Goal. Let \( \mathcal{A} \) be a completely rational conformal net. Orit showed the first few of these:

(1) **Semisimplicity**: Every separable non-degenerate rep is completely reducible.
(2) The number of unitary equiv. classes of irreducible reps is finite
(3) **Finite statistics**: Every separable irreducible representation has finite statistical dimension
(4) **Modularity**: \( \text{Rep}_f(\mathcal{A}) \) has a monoid structure with simple unit and duals (conjugates) and a maximally non-degenerate braiding, thus is modular.

1. **Introduction**

Assume \( \mathcal{A} \) is a completely rational conformal net, i.e.

\[ \mathcal{I} \ni I \mapsto \mathcal{A}(I) \subset B(H_0) \]

with \( H_0 \) the vacuum Hilbert space, \( \Omega \in H_0 \) the vacuum vector, \( U \rhd H_0 \) unitary positive energy representation of \( PSU(1, 1) \). These data fullfil some axioms (Corbett) plus the additional assumption of **complete rationality**:

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Available online at [http://math.mit.edu/~eep/CFTworkshop](http://math.mit.edu/~eep/CFTworkshop). Please email eep@math.mit.edu with corrections and improvements!
(1) strong additivity
(2) split property
(3) finite $\mu_2$ index

Recall a representation of $\mathcal{A}$ is a collection of reps $\{\pi_I\}_{I \in \mathcal{I}}$ with $\pi_I : \mathcal{A}(I) \to B(H)$ which are compatible. If $H$ separable (then we call $\pi$ a separable representation), for all $I \in \mathcal{I}$ there is $\rho \simeq \pi$ (we also write $\rho \in [\pi]$; the equivalence class $[\pi]$ is called sector) on $H_0$ with $\rho_I = id_{\mathcal{A}(I')}$. Thus the representation acts trivial outside $I$. $\rho$ then is called localized in $I$. One has a monoidal structure, given by composition of localized endomorphism (Yoh showed relation to Connes fusion).

Conjugates: Let $\pi \simeq \rho$ be a separable non-degenerate representation localized in $I$. Let $P, Q$ be two other intervals.

Let $r_Q \in PSU_{\pm}(1,1)$ reflection associated to the intervall $Q$, cf:

Then we can define another representation by

$$\tilde{\rho}_I(x) = J_P \rho_{r_Q(I)}(J_Q x J_Q) J_P$$

where $J_P$ is the modular conjugation for the algebra $\mathcal{A}(P)$, i.e. $J_P \mathcal{A}(P) J_P = \mathcal{A}(P)'$. Remember that we have Bisognano-Wichman property, telling us that $J_P x J_P = U(r_P) x U(r_P)^*$ holds, where $U$ is now the extended (anti) unitary representation of $PSU_{\pm}(1,1)$, i.e. $J_P$ acts geometrically by a reflection. This ensures the above formula is well defined.

It turns out the equivalence class $[\tilde{\rho}_I]$ does not depend on $P, Q$.

**Theorem 1.1.** If $\pi$ is separable and irreducible with finite statistical dimension, then there exists a conjugate representation $\tilde{\pi}$. If $\pi$ is Möbius covariant, then also $\tilde{\pi}$. In particular if $\rho \in [\pi]$ like above then $\tilde{\rho} \in [\tilde{\pi}]$

So the conjugate representation is given by the above formula up to some choice in the unitary equivalence class.
We begin with some fact from subfactor theory

**Fact.** Let \( N \subset M \) be an inclusion of type III factors, which is irreducible (ie \( N' \cap M = \mathbb{C}1 \)) and has finite index: \( [M : N] \leq \infty \). We assume we have a canonical endomorphism \( \gamma : M \to N, \gamma(x) = J_N J_M x J_M J_N \) for \( x \in M \). Then are equivalent:

1. \( \sigma \in \text{End}(N) : \sigma \prec \gamma|_N \), i.e. there is \( U \in N \) such that \( U\sigma(x) = \gamma(x)U \)
2. There is \( \psi \in M \) such that \( \psi x = \sigma(x)\psi \) for all \( x \in N \).

This we want to apply to the two interval inclusion \( \mathcal{A}(E) \subset \mathcal{A}(E) : = \mathcal{A}(E)' \)

with the canonical endomorphism \( \gamma_E : \mathcal{A}(E) \to \mathcal{A}(E) \).

Pick \( \pi_i \) an irreducible separable representation with finite index, \( \rho_i \in [\pi_i] \) localized on \( I_1 \).

Then exist a conjugate \( \tilde{\pi}_i \) and we pick \( \tilde{\rho}_i \in [\tilde{\pi}_i] \) localized in \( I_2 \).

There exist a up to constant unique intertwiner (think of co-evaluation map) \( R_i \in \text{Hom}(1, \rho_i \tilde{\rho}_i) \in \mathcal{A}(E) \), i.e. \( R_i(x) = \rho_i(\tilde{\rho}_i(x))R_i \).

Thus using \( \sigma = \rho_i \tilde{\rho}_i \) in the above fact we get \( \rho_i \tilde{\rho}_i \prec \lambda_E = \gamma_E|_{\mathcal{A}(E)} \). On the lefthand side we can even take a sum over mutually non-equivalent representations with finite index \( \Gamma_f \) and the inequivalence still holds:

\[
\bigoplus_{i \in \Gamma_f} \rho_i \tilde{\rho}_i \prec \lambda_E = \gamma_E|_{\mathcal{A}(E)}
\]
because the endomorphism are mutually inequivalent. It turns out by some further arguments:

$$\bigoplus_{i \in \Gamma_f} \rho_i \tilde{\rho}_i \simeq \lambda_E = \gamma_E|_{\mathcal{A}(E)}$$

Taking the index on both sides one can conclude:

$$\sum_{\Gamma_f} d(\rho_i)^2 = [\hat{\mathcal{A}}(E) : \mathcal{A}(E)] = \mu_2$$

We will use another fact from subfactor theory

**Fact.** Let $\gamma(x) = \sum U_i \sigma_i(x) U_i^*$ for $x \in N$ with $\sigma_i$ irreducible, $U_i$ partial isometries, such that $\sum U_i^* U_i = 1$, $U_j U_i^* = \delta_{ij} 1$. Then every $x \in M$ is of the form $x = \sum x_i \psi_i$ for unique $x_i \in N$.

So, for each $x \in \hat{\mathcal{A}}(E)$ we have a decomposition $x = \sum_{i \in \Gamma_f} x_i R_i$ with unique $x_i \in \mathcal{A}(E)$. Thus every element of the bigger factor can be written as elements of the smaller subfactor and intertwiner $\{R_i\}$:

$$\hat{\mathcal{A}}(E) = \mathcal{A}(E) \vee \{R_i\}'$$

The two-intervall inclusion is connected to the intertwiner $R_i$, thus connected to the representation theory of the net.

### 3. Modularity

**Proposition 3.1.** Every irreducible separable representation of $\mathcal{A}$ has finite statistical dimension.

**Proof.** Sketch: Let $\rho, \rho' \in [\pi]$ be localized in the two components of $E$ respectively and $u \in \text{Hom}(\rho, \rho') \subset \hat{\mathcal{A}}(E)$ their intertwiner. By the last fact we can uniquely write $u$ as $u = \sum u_i R_i$. Then exist an $i$ such that $u_i \neq 0$ and a short calculation shows that $u_i \in \text{Hom}(\rho_i \rho, id)$, i.e. there exist an non trivial intertwiner $\rho_i \rho$ with the vacuum representation for some $i$. Duality implies the existence of a non-trivial intertwiner between $\rho$ and $\tilde{\rho}_i$ given essentially by:

$$\rho \xrightarrow{\text{coun}_{\tilde{\rho}_i}} \tilde{\rho}_i \rho \rho \xrightarrow{1 \otimes u_i} \tilde{\rho}_i$$

and because $\rho, \tilde{\rho}_i$ both are irreducible this means $\rho \simeq \tilde{\rho}_i$. \qed
Next: what’s the braiding in this category? Braiding is given by a bijective morphism \( \epsilon(\rho, \eta) \in \text{Hom}(\rho \eta, \eta \rho) \) satisfying some identities.

The idea how to define \( \epsilon \) is to transport \( \rho \) and \( \eta \) in disjoint regions (so they commute), exchange the order, and than transport back. This does not depend one the explicit choice of the regions. One could for example transport \( \eta \) to the left or to the right, this gives in particular two (a priori) inequivalent choices.

So let \( \rho, \eta \) be localized in some intervalls, cf

\[
\eta_{L/R} \in [\eta] \quad \text{be two equivalent representations localized left and right from } \rho, \text{ respectively and } T_{L/R} \in \text{Hom}(\eta, \eta_{L/R}) \text{ intertwiners. Note that } \rho \eta_{R/L} = \eta_{R/L} \rho.
\]

Define \( \epsilon(\rho, \eta) \)

\[
\epsilon(\rho, \eta) \equiv \begin{array}{c}
\eta \\
\rho
\end{array}
\begin{array}{c}
\rho \\
\eta
\end{array}
:=
\begin{array}{c}
\eta \\
\rho
\end{array}
\begin{array}{c}
\rho \\
\eta
\end{array}
= T_{L}^* \rho(T_{L})
\]

Then

\[
\epsilon(\eta, \rho)\equiv \begin{array}{c}
\eta \\
\rho
\end{array}
\begin{array}{c}
\rho \\
\eta
\end{array}
:=
\begin{array}{c}
\eta \\
\rho
\end{array}
\begin{array}{c}
\rho \\
\eta
\end{array}
= T_{R}^* \rho(T_{R})
\]

thus is given by the other choice.
Note: \( T^*_L \rho(T_L) \) is indeed
\[
\rho \eta \xrightarrow{1 \otimes T_L \rho \eta \eta} \eta \rho \eta \xrightarrow{T^*_R \rho \eta \eta} \rho_i
\]
using that the categorical tensor product \( \rho \eta \equiv \rho \otimes \eta \) is the composition of localized endomorphism.

**Definition.** \( \rho \) and \( \eta \) have trivial monodromy if \( \epsilon(\rho, \eta) = \epsilon(\eta, \rho)^* \) or equivalently \( \epsilon_M(\rho, \eta) := \epsilon(\rho, \eta)\epsilon(\eta, \rho) = 1 \), i.e.

\[
\begin{array}{c}
\begin{array}{c}
\epsilon_M(\rho, \eta) = 1
\end{array}
\end{array}
\]

Note that \( \epsilon_M([\rho], [\eta]) = \epsilon_M(\rho, \eta) \) is well-defined, i.e. the monodromy just depends on sectors and not on the representations itself.

**Definition.** \( \pi \) separable, non-degenerate representation of \( \mathcal{A} \) is called finite if one of the following equivalent conditions holds

- \( \pi \) is a finite direct sum of irreps.
- \( \pi \) has finite statistical dimension
- \( \pi(C^*(\mathcal{A}))' \) is finite.

Let \( \text{Rep}_f(\mathcal{A}) \) be the category of all finite reps.

**Definition.** \( \rho \) is called degenerate with respect to braiding if \( \epsilon_M(\rho, \eta) = 1 \) for all \( \eta \in \text{Rep}_f(\mathcal{A}) \).

The center \( Z_2(\text{Rep}_f) \) is the set of degenerate w.r.t. braiding reps.

Note: in a modular category \( \mathcal{C} \), \( Z_2(\mathcal{C}) \) is trivial, i.e sums of 1. This is the most non-trivial fact to check.

We use two ingredients:

**Criterion for degeneracy:** \( \epsilon_M(\rho, \eta) = 1 \) iff \( \rho(T) = T \) for \( T \in \text{Hom}(\eta_L, \eta_R) \).

**Proof.** \( \epsilon_M(\rho \eta) \equiv T^*_L \rho(T_L T^*_R) T_R = 1 \) iff \( \rho(T_L T^*_R) = T_L T^*_R \). The statement follows, realizing \( T_L T^*_R \) equals \( T^* \) up to some constant:
Criterion for triviality of a representation: If $\rho$ act trivially on $\hat{\mathcal{A}}(E)$ then $\rho \simeq N \cdot \text{id}$, thus trivial.

**Theorem 3.1.** $Z_2(\text{Rep}_f \mathcal{A})$ is trivial thus $\text{Rep}_f \mathcal{A}$ is modular.

*Proof.* $\pi \in Z_2(\text{Rep}_f \mathcal{A})$ and $\rho \in [\pi]$ localized as above and $E$ the union of intervalls left and right from the localization intervall of $\rho$. $\rho \in Z_2$ implies $\rho(T) = 1$ for all possible charge transporters $T$ from left to the right using the first criterion.

We have seen that the big factor $\hat{\mathcal{A}}(E)$ is generated by the small $\mathcal{A}(E)$ and the intertwiner $R_i$, this turns out to be equivalent with $\hat{\mathcal{A}}(E)$ generated by $\mathcal{A}(E)$ and interwiner $T_i$ which transport $\eta = \rho_i$ from left to right, i.e.

$$\hat{\mathcal{A}}(E) = \mathcal{A}(E) \vee \{R_i\} = \mathcal{A}(E) \vee \{T_i\}$$

By definition $\rho$ acts trivially on $\mathcal{A}(E)$, but also on all charge transporters $T_i$ thus on $\hat{\mathcal{A}}(E)$. But this is the second criteria which implies triviality of $\rho$ thus $\pi$. Thus the center is trivial. □