LECTURE NOTES: CONFORMAL NETS AND TENSOR CATEGORIES

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Abstract. ATTENTION: not proof read lecture notes in progress.....

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Part 1. Subfactors and Tensor Categories

Goal: Understanding the basic structure of subfactors and tensor categories in the type III setting.

Disclaimer: Many things can be abstractly and independently studied. We will go a pedestrian way.

1.1. Finite Index Subfactors

Let $M$ be a factor, i.e. $M' \cap M = \mathbb{C} \cdot 1$, and of type III. Fact: If $p \neq 0$ is an orthogonal projection in $M$, then there is a isometry $u \in M$ (isometry means $u^*u = 1$) such that $uu^* = p$. In other words, $p$ is Murray–von Neumann equivalent to the identity 1, denoted $p \sim 1$.

Let $N \subset M$ be a subfactor, i.e. $N \subset M$ is a unital von Neumann subalgebra, which is itself a factor, i.e. $N' \cap N = \mathbb{C}$.

Consider the canonical inclusion map:

$$\iota: N \to M$$

(1.1)

We now write $\iota(N) \subset M$.

For $\alpha, \beta: P \to Q$ unital *-morphisms, we define the space of intertwiners

$$\text{Hom}(\alpha, \beta) = \{ t \in Q : \alpha(p) = \beta(p)t \text{ for all } p \in P \}.$$  

(1.2)

We write $\alpha \prec \beta$ if there is an isometry $u \in \text{Hom}(\alpha, \beta)$, i.e. $u^*u = 1$.

Date: today.
We call
\[ [\alpha] = \{ \text{Ad} u \circ \alpha : u \in Q \text{ unitary} \} \quad (1.3) \]
the **sector** of \( \alpha \).

**Definition 1.1.1.** We call \( \iota(N) \subset M \) finite index if there is a unital \( * \)-morphism
\[ \iota: N \to M \quad (1.4) \]
such that \( \text{id}_N \prec \bar{\iota} \) and \( \text{id}_M \prec \iota \circ \bar{\iota} \) with \( v, w \) multiple of isometries
\[ v \in \text{Hom}(\text{id}_M, \iota \circ \bar{\iota}) \quad w \in \text{Hom}(\text{id}_N, \bar{\iota} \circ \iota) \quad (1.5) \]
fulfilling the **conjugate equations** ("zig-zag equations")
\[ \iota(w^*)v = 1_M \quad \bar{\iota}(v^*)w = 1_N, \quad (1.7) \]
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v
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1_N
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\end{array}.
\end{array} \quad (1.8) \]

Because \( M \) and \( N \) are factors, we get
\[ v^*v \in \text{Hom}(\text{id}_M, \text{id}_M) = M' \cap M = \mathbb{C} \cdot 1_M \quad (1.9) \]
\[ w^*w \in \text{Hom}(\text{id}_N, \text{id}_N) = N' \cap N = \mathbb{C} \cdot 1_N \quad (1.10) \]
and we denote the **minimal index** \([M : N]\) to be the positive number
\[ [M : N] = \inf_{(v,w)} v^*v \cdot w^*w \quad (1.11) \]
where the infimum runs over all solutions \((v, w)\) of (1.7).

If \( \|v\| = \|w\| \) and \((v, w)\) saturate the the bound \((1.11)\), then we call the pair \((v, w)\) a **standard solution of the conjugate equation**. Then there is a \( \delta > 0 \) such that
\[ v^*v = \delta \cdot 1_M \quad w^*w = \delta \cdot 1_N \quad (1.12) \]
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\circ
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\end{array} \quad (1.13) \]
and \( \delta \) is called the **dimension** of \( \iota \) denoted \( d \iota \) and we have
\[ [M : N] = \delta^2. \quad (1.14) \]

Further, we denote
\[ \theta := \bar{\iota} \circ \iota \in \text{End}(N) \quad \text{dual canonical endomorphism}, \quad (1.15) \]
\[ \gamma := \iota \circ \bar{\iota} \in \text{End}(M) \quad \text{canonical endomorphism}. \quad (1.16) \]
1.2. Q-systems

From \( \iota(N) \subset M \) and \( \bar{i}: M \to N \) together with a standard solution \((v, w)\) of the conjugate equation, we get a triple \((\theta, w, x := \iota(v))\), graphically displayed as:

\[
\begin{align*}
\theta &= \bar{i} \iota, & 
\bar{i} \iota &= \theta; \quad w &= \bar{i} \iota, & w &= \iota; \\
\bar{i} \iota &= \theta, & x &= \iota. \\
\end{align*}
\]

It is a categorical version of an algebra, namely an algebra object in \( \text{End}(N) \). As comparison: a unital associative algebra is an algebra object in the category of vector spaces. Namely, \( w \) is the unit and \( x^* \) is the multiplication. From the conjugate equation it easily follows (written this way it is actually a co-algebra):

\[
\begin{align*}
xx &= \theta(x)x & (x \otimes 1_\theta)x &= (1_\theta \otimes x)x & \text{(associativity)} \\
w^*x &= \theta(w^*)x &= 1_\theta & (w^* \otimes 1_\theta)x &= (1_\theta \otimes w^*)x &= 1_\theta & \text{(unit law)}.
\end{align*}
\]

In graphical notation this reads:

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\theta \quad \theta \quad \theta \\
\theta
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\begin{array}{c}
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\begin{array}{c}
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\theta \quad \theta \quad \theta \\
\theta \quad \theta \quad \theta \\
\theta
\end{array}
\end{array}
\]

and by taking \( * \) corresponding to reflect the pictures vertical on gets a algebra.

**Definition 1.2.1.** A triple \( \Theta = (\theta, w, x) \) with \( \theta \in \text{End}(N) \) and isometries \( \sqrt{d\theta}^{-1}w: \text{id}_N \to \theta \) and \( \sqrt{d\theta}^{-1}x: \theta \to \theta^2 \), which we will graphically display as

\[
\begin{align*}
\sqrt{d\theta} w &= \theta; & \sqrt{d\theta} x &= \theta \theta \\
\end{align*}
\]

is called a **Q-system** (cf. [Lon94], [LR97])

Two Q-systems \( \Theta = (\theta, w, x) \) and \( \tilde{\Theta} = (\tilde{\theta}, \tilde{w}, \tilde{x}) \) in \( \text{End}(N) \) are called equivalent, if there is a unitary \( u \in \text{Hom}(\theta, \tilde{\theta}) \), such that

\[
\tilde{x}u = (u \otimes u)x \equiv u\theta(u)x; \quad u\tilde{w} = w
\]

hold, or graphically:

\[
\begin{array}{c}
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\tilde{\theta} \quad \tilde{\theta}
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\end{array}
\begin{array}{c}
\begin{array}{c}
\tilde{x} \quad \tilde{x} \\
\tilde{x} \quad \tilde{x}
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\begin{array}{c}
\begin{array}{c}
\tilde{\theta} \quad \tilde{\theta} \\
\tilde{\theta} \quad \tilde{\theta}
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\begin{array}{c}
\begin{array}{c}
\tilde{x} \quad \tilde{x} \\
\tilde{x} \quad \tilde{x}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\tilde{w}^* = w^* \\
\tilde{w}^* = w^*
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\begin{array}{c}
\begin{array}{c}
\tilde{w}^* = w^* \\
\tilde{w}^* = w^*
\end{array}
\end{array}
\]
A Q-system in a C*-tensor category automatically \cite{LR97} fulfills the “Frobenius law” which is easy to check for our example from a subfactor:

\[(x^* \otimes 1_\theta)(1_\theta \otimes x) \equiv x^*\theta(x) = xx^* = (1_\theta \otimes x^*)(x \otimes 1_\theta) \equiv \theta(x^*)x\]

or graphically:

\[
\begin{array}{ccc}
\theta & \theta & \theta \\
\theta & \theta & \theta \\
\theta & \theta & \theta
\end{array}
\]

\[
\begin{array}{ccc}
\theta & \theta & \theta \\
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\theta & \theta & \theta
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\]

\[
\begin{array}{ccc}
\theta & \theta & \theta \\
\theta & \theta & \theta \\
\theta & \theta & \theta
\end{array}
\]

Definition 1.2.2. Let \(\Theta = (\theta, w, x)\) be a Q-system. It is called irreducible if \(\dim\text{Hom}(\text{id}, \theta) = 1\) and standard if \((\theta, r, \bar{r})\) with \(\bar{\theta} = \theta\), \(r = \bar{r} = xw\) is a standard solution for the conjugate equation for \(\theta\).

So an irreducible finite index subfactor \(N \subset M\) of type III gives an irreducible standard Q-system in \(\text{End}(N)\).

Conversely, given an abstract Q-system in \(\text{End}(N)\) there is a von Neumann algebra \(M \supset N\) by the Longo–Rehren construction.

Proposition 1.2.3 (Longo–Rehren construction). Let \(\Theta = (\theta, w, x)\) be an irreducible standard Q-system in \(\text{End}(N)\) for a type III factor \(N\). Then there is a von Neumann algebra \(M\) with canonical inclusion \(\iota: N \to M\) with conjugate \(\bar{\iota}: M \to N\) and \(v \in \text{Hom}(\text{id}_M, \bar{\iota})\), such that \((\iota, v, w)\) fulfills the conjugate equation for \(v\) and \(N \subset M\) is irreducible.

Proof. The short way is to define \(M\) to be the Jones basic construction \(E_0(N) \subset N \subset M\) where \(E_0 = \sqrt{d\theta^{-1}} x^*\theta(\cdot)x\) is a conditional expectation by the Q-system properties. Namely, it is normal and completely positive. Associativity of the Q-system gives \(E_0 \circ E_0 = E_0\) and the Frobenius law gives \(E_0(xE_0(y)) = E_0(x)E_0(y)\) which shows that \(E_0(M)\) is a subalgebra.

But we can build \(M\) directly from \(N\) by adjoining one element \(v\) fulfilling:

\[v\iota(n) = \iota\theta(n)v\]

with \(n \in N\) and \(\iota: N \to M\) the embedding of \(N\) into the still to be constructed algebra \(M\). This means \(v \in \text{Hom}(\iota, \theta)\) and its square and adjoint are defined to be

\[v^2 = \iota(x)v \quad v^* = \iota(wx^*)v\]  

(1.17)

A general element of \(M\) can be written as \(\iota(n)v\) for some \(n \in n\). The unit is \(1_M = \iota(w^*)v\) and \((\iota(n)v)^* = v^*\iota(n^*) = \iota(w^*x^*\theta(n^*))v\). By the Frobenius property this turns the set \(M = \{\iota(n)v : n \in A\}\) into a \(*\)-algebra. We induce the weak topology from \(N\) to \(M\) with help of the conditional expectation \(E: M \to N\) given by

\[E(m) = \sqrt{d\theta^{-1}} w^*\iota(m)w \quad E(vv^*) = \sqrt{d\theta^{-1}} 1_N\]  

(1.18)

where \(\iota: M \to N: \iota(n)v \mapsto \theta(n)x\) and \((w, v)\) is a standard solution to the conjugate equation. \(\square\)

We call the Q-system \((\theta, w, x)\) factorial if the constructed \(M \supset N\) is a factor. If \((\theta, w, x)\) is irreducible, i.e. \(\dim\text{Hom}(\text{id}_N, \theta) = 1\), then \(M\) is always a factor.
1.3. Subobjects and semi-simplicity

Consider \( \alpha: P \to Q \) for \( P, Q \) type III factors with finite dimension. \( p \in \text{Hom}(\alpha, \alpha) \) is a non-trivial \( (0 \neq p \neq 1) \) orthogonal projection \( p = p^2 = p^* \), then by the type III properties there are isometries \( u_i \) with \( u_1 u_1^* = p \) and \( u_2 u_2^* = 1 - p \). Note that \( u_i \) are the generators of the Cuntz algebra \( \mathcal{O}_2 \). We can write

\[
\alpha = \text{Ad}(u_1) \circ \alpha_1 + \text{Ad}(u_2) \circ \alpha_2
\]

where \( \alpha_i: P \to Q \) are unital \(*\)-morphisms with finite dimension. In other words, \( \alpha \) is a direct sum of \( \alpha_1 \) and \( \alpha_2 \). We use the notation \( \mathcal{O}_2 \) and we use the notation \( \mathcal{O}_2 \setminus \mathcal{O}_2 \).

It follows that \( \alpha \) is a finite direct sum of inequivalent irreducibles \( \alpha_i \), i.e. \( \text{Hom}(\alpha_i, \alpha_j) = \delta_{i,j} \).}

1.4. Even part, finite depth and fusion categories

Consider the categories

\[
\mathcal{C}_N = \langle \rho \prec (\bar{\iota} \circ \iota)^n \rangle \subset \text{End}_0(N)
\]

\[
\mathcal{C}_M = \langle \beta \prec (\bar{\iota} \circ \iota)^n \rangle \subset \text{End}_0(M)
\]

where \( \langle \cdot \rangle \) is the full subcategory with subobjects and finite direct sums generated by the listed sectors. A subcategory \( \mathcal{D} \subset \mathcal{C} \) is full if for any \( \rho, \sigma \) objects in \( \mathcal{D} \), then \( \text{Hom}_\mathcal{D}(\rho, \sigma) = \text{Hom}_\mathcal{C}(\rho, \sigma) \).

Let us denote \( \text{Irr}(\mathcal{C}) \) denotes the collection of irreducible sectors of \( \mathcal{C} \). We say \( \iota(N) \subset M \) has finite depth if and only if \( | \text{Irr}(\mathcal{C}_N) | < \infty \) if and only if \( | \text{Irr}(\mathcal{C}_M) | < \infty \).

**Exercise 1.4.1.** Let \( P = M, N \) and \( \Delta \) a set of endomorphisms of a choice of an endomorphism for sector in \( \text{Irr}(P, \mathcal{C}_P) \). Then we have

1. \( \text{id} \in \Delta \) (we choose \( \text{id} = \text{id}_P \) as representant of \( \text{id}_P \) in \( \text{Irr}(P, \mathcal{C}_P) \)).
2. \( \Delta \) is closed under conjugates: If \( \rho \in \Delta \) then \( \bar{\rho} \in \Delta \).
3. \( \Delta \) is closed under fusion: If \( \rho, \sigma \in \Delta \), then

\[
[\rho] \otimes [\sigma] = [\rho \circ \sigma] = \bigoplus_{\tau \in \Delta} N^{\tau}_{\rho, \sigma}[\tau]
\]
If is a finite system $\Delta \subset \text{End}_0(P)$ fulfilling the properties (1)-(3) then we call $\mathcal{F} = \langle \Delta \rangle \subset \text{End}_0(P)$ a (concrete type III) unitary fusion category.

**Proposition 1.4.2.** Let $\mathcal{C} \subset \text{End}(N)$ be a fusion category, then there is a subfactor $N \subset M$, such that $\mathcal{C} = {}_N \mathcal{C}_N$.

*Proof.* Choose a $\rho \in \mathcal{C}$ with $[\rho] = \bigoplus_{\rho_i \in \Delta} [\rho_i]$ and a $\bar{\rho}$ and $(v, w)$ a standard solution of the conjugate equation. Then $\theta = \bar{\rho} \circ \rho, w, \rho(v)$ is a Q-system which gives the Jones basic extension $\rho(N) \subset N \subset M$.

It follows immediately that $\mathcal{C} = {}_N \mathcal{C}_N$. $\blacksquare$

**Fact 1.4.3.** It is a deep result, that an abstract unitary fusion category $\mathcal{C}$ can be realized as $\mathcal{C} \subset \text{End}(N)$ for a type III factor $N$. It follows from Popa’s theorem [Pop95], that there is a (essentially) unique realization if $N$ is the hyperfinite type III$_1$ factor.

**Example 1.4.4.** Let $G$ be a finite group and $\alpha: G \to \text{Aut}(M)$ be an outer action. Then $M^G = \{ m \in M : \alpha_g(m) = m \text{ for all } g \in G \}$ is a factor. Consider the subfactor:

$$N = M^G \subset M$$ (1.26)

It turns out that

$$[\gamma] = [\iota \circ \iota] = \bigoplus_{g \in G} [\alpha_g]$$

$$\mathcal{M} \cong \text{Vect}_G$$

$$\text{Hom}(\gamma, \gamma) \cong \ell^\infty(G)$$ (1.27)

$$[\gamma] = [\iota \circ \iota] \cong \bigoplus_{\pi \in \text{Rep}(G)} \dim(\pi)[\pi]$$

$$\mathcal{N} \cong \text{Rep}(G)$$

$$\text{Hom}(\theta, \theta) \cong \mathbb{C}[G]$$ (1.28)

(1.29)

where $\text{Vect}_G$ is the category of finite-dimensional $G$-graded Hilbert spaces and $\text{Rep}(G)$ is the category of finite-dimensional unitary representations of $G$ and we mean an equivalence of tensor categories. We note that for any $\alpha$ with $d\alpha < \infty$ the space of intertwiners $\text{Hom}(\alpha, \alpha)$ is a finite-dimensional $\text{C}^*$ (actually von Neumann) algebra and the corresponding isomorphism is an isomorphism of $*$-algebras.

**Example 1.4.5.** A unitary fusion category $\mathcal{C} \subset \text{End}_0(N)$ is called pointed if for any $[\rho] \in \text{Irr}(\mathcal{C})$ we have $d\rho = 1$. Then $G = \text{Irr}(\mathcal{C})$ is a finite group and $\mathcal{C}$ is classified by this group $G$ and an element in $\omega \in H^3(G, \mathbb{T})$. Note that we can see $\text{Irr}(\mathcal{C}) \subset \text{Out}(N) = \text{Aut}(N)/\text{Inn}(N)$ and this is also called a $G$-kernel.

We get a true action by automorphisms (then $M^G$ is a factor) if only if $\omega = 0$. If $\omega \neq 0$ we can do the construction from Proposition 1.4.2, this is called the diagonal subfactor associated with the $G$-kernel.

### 1.5. Subfactors in Unitary Fusion Categories

Let $\mathcal{N} \mathcal{F}_N \subset \text{End}(N)$ be a unitary fusion category.

**Problem 1.5.1.** Find all overfactors $M \supset \iota(N)$ such that the associated Q-system lies in $\mathcal{N} \mathcal{F}_N$. 


Let us denote from now on
\[ N \mathcal{F}^{N \subset M} = \langle \rho \prec (\bar{\iota} \circ \iota)^n \rangle, \quad M \mathcal{F}^{N \subset M} = \langle \beta \prec (\iota \circ \bar{\iota})^n \rangle. \quad (1.30) \]

In other words, we ask \( N \mathcal{F}^{N \subset M} \subset M \mathcal{F}^{N \subset M} \).

In this case, we define the dual category \( M \mathcal{C} = \langle \rho \circ (\bar{\iota} \circ \iota) \rangle_{\rho \in N \mathcal{F}^{N \subset M}} \) which is a unitary fusion category. Note that Problem 1.5.1 is equivalent with asking \( M \mathcal{F}^{N \subset M} \subset M \mathcal{F}^{M \subset M} \).

Example 1.5.2. Let \( N \subset M \) be a finite index finite depth subfactor, then \( N \mathcal{F}^{N \subset M} \) and \( M \mathcal{F}^{M \subset M} \) are Morita equivalent. The Morita equivalence is exactly given by the subfactor. Again Morita equivalence can be defined abstractly, see [Müg03].

Fact 1.5.3. Let \( N \mathcal{F}^{N \subset M} \) and \( M \mathcal{F}^{M \subset M} \) be Morita equivalent, then
\[ \text{GDim}(N \mathcal{F}^{N \subset M}) = \text{GDim}(M \mathcal{F}^{M \subset M}) \]

Since \( d\pi = \dim(\pi) \) for \( \pi \in \text{Rep}(G) \), considering \( M^G \subset M \) we get this classical result from group theory.

Corollary 1.5.4. We have
\[ \sum_{\pi \in \text{Irr}(\text{Rep}(G))} \dim(\pi)^2 = |G|. \]

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<thead>
<tr>
<th>Group</th>
<th>Quantum symmetry</th>
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<tbody>
<tr>
<td>( G )</td>
<td>( N \subset M )</td>
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<tr>
<td>( \text{Rep}(G) )</td>
<td>( N \mathcal{F}^{N \subset M} )</td>
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<tr>
<td>( \text{Vect}_G )</td>
<td>( M \mathcal{F}^{M \subset M} )</td>
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<td>G</td>
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1.6. Quantum Double Subfactors

Consider \( \mathcal{F} \subset \text{End}(N) \) be a unitary fusion category. Then with \( A = N \otimes N^{\text{op}} \) there is a Q-system:
\[ [\theta] = \bigoplus_{[\sigma] \in \text{Irr}(\mathcal{F})} [\sigma \otimes \sigma^{\text{op}}] \]

in \( A \mathcal{C}_A = \langle \rho \otimes \sigma^{\text{op}} : \rho, \sigma \in \mathcal{F} \rangle \), which gives the so-called Longo–Rehren subfactor \( A \subset B \).

We have \( B \mathcal{C}_B \cong Z(\mathcal{F}) \), where \( Z(\mathcal{F}) \) is the Drinfeld center.

The category \( A \mathcal{C}_A \) is equivalent to the Drinfeld tensor product \( \mathcal{F} \boxtimes \mathcal{F}^{\text{op}} \), which is therefore Morita equivalent to \( Z(\mathcal{F}) \).
References


