LECTURE NOTES: CONFORMAL NETS AND TENSOR CATEGORIES

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Abstract. ATTENTION: not proof read lecture notes in progress.....

Contents

Part 1. Subfactors and Tensor Categories

Goal: Understanding the basic structure of subfactors and tensor categories in the type III setting.

Disclaimer: Many things can be abstractly and independently studied. We will go a pedestrian way.

1.1. Finite Index Subfactors

Let M be a factor, i.e. $M' \cap M = \mathbb{C} \cdot 1$, and of type III. Fact: If $p \neq 0$ is an orthogonal projection in M, then there is a isometry $u \in M$ (isometry means $u^*u = 1$) such that $uu^* = p$. In other words, p is Murray–von Neumann equivalent to the identity 1, denoted $p \sim 1$.

Let $N \subset M$ be a subfactor, i.e. $N \subset M$ is a unital von Neumann subalgebra, which is itself a factor, i.e. $N' \cap N = \mathbb{C}$.

Consider the canonical inclusion map:

$$\iota \colon N \to M \tag{1.1}$$

We now write $\iota(N) \subset M$.

For $\alpha, \beta \colon P \to Q$ unital $*$-morphisms, we define the space of **intertwiners**

$$\mathrm{Hom}(\alpha, \beta) = \{t \in Q : t\alpha(p) = \beta(p)t \text{ for all } p \in P\}\,. \tag{1.2}$$

We write $\alpha \prec \beta$ if there is an isometry $u \in \mathrm{Hom}(\alpha, \beta)$, i.e. $u^*u = 1$.

Date: today.

We call

$$[\alpha] = \{\mathrm{Ad}\, u \circ \alpha : u \in Q \text{ unitary}\} \tag{1.3}$$

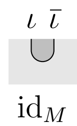
the **sector** of α.

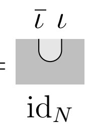
Definition 1.1.1. We call $\iota(N) \subset M$ **finite index** if there is a unital $*$-morphism

$$\iota \colon N \to M \tag{1.4}$$

such that $\mathrm{id}_N \prec \bar\iota$ and $\mathrm{id}_M \prec \iota \circ \bar\iota$ with v, w multiple of isometries

$$v \in \mathrm{Hom}(\mathrm{id}_M, \iota \circ \bar\iota) \qquad\qquad w \in \mathrm{Hom}(\mathrm{id}_N, \bar\iota \circ \iota) \tag{1.5}$$

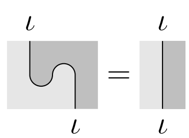
$v =$ 

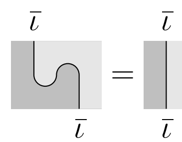
$w =$ 

(1.6)

fulfilling the **conjugate equations** ("zig-zag equations")

$$\iota(w^*)v = 1_M \qquad\qquad \bar\iota(v^*)w = 1_N\,, \tag{1.7}$$





. (1.8)

Because M and N are factors, we get

$$v^*v \in \mathrm{Hom}(\mathrm{id}_M, \mathrm{id}_M) = M' \cap M = \mathbb{C} \cdot 1_M \tag{1.9}$$

$$w^*w \in \mathrm{Hom}(\mathrm{id}_N, \mathrm{id}_N) = N' \cap N = \mathbb{C} \cdot 1_N \tag{1.10}$$

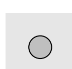
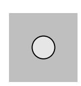
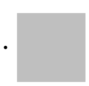
and we denote the **minimal index** $[M : N]$ to be the positive number

$$[M : N] = \inf_{(v,w)} v^*v \cdot w^*w \tag{1.11}$$

where the infimum runs over all solutions (v, w) of (1.7).

If $\|v\| = \|w\|$ and (v, w) saturate the the bound (1.11), then we call the pair (v, w) a **standard solution of the conjugate equation**. Then there is a $\delta > 0$ such that

$$v^*v = \delta \cdot 1_M \qquad\qquad w^*w = \delta \cdot 1_N \tag{1.12}$$

 $= \delta \cdot$  $= \delta \cdot$    (1.13)

and δ is called the **dimension** of ι denoted $d\iota$ and we have

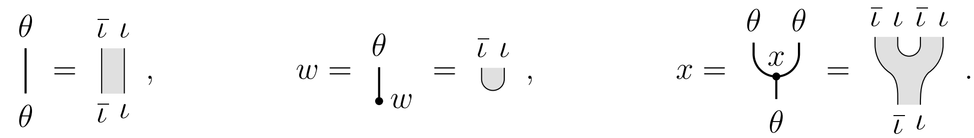
$$[M : N] = \delta^2\,. \tag{1.14}$$

Further, we denote

$$\theta := \bar\iota \circ \iota \in \mathrm{End}(N) \qquad \textbf{dual canonical endomorphism}\,, \tag{1.15}$$

$$\gamma := \iota \circ \bar\iota \in \mathrm{End}(M) \qquad \textbf{canonical endomorphism}\,. \tag{1.16}$$

From $\iota(N) \subset M$ and $\bar\iota\colon M \to N$ together with a standard solution (v, w) of the conjugate equation, we get a triple $(\theta, w, x := \iota(v))$, graphically displayed as:



It is a categorical version of an algebra, namely an algebra object in $\mathrm{End}(N)$. As comparison: a unital associative algebra is an algebra object in the category of vector spaces. Namely, w is the unit and x^* is the multiplication. From the conjugate equation it easily follows (written this way it is actually a co-algebra):

$$xx = \theta(x)x \qquad (x \otimes 1_\theta)x = (1_\theta \otimes x)x \qquad \text{(associativity)}$$
$$w^*x = \theta(w^*)x = 1_\theta \qquad (w^* \otimes 1_\theta)x = (1_\theta \otimes w^*)x = 1_\theta \qquad \text{(unit law)}\,.$$

In graphical notation this reads:

and by taking $*$ corresponding to reflect the pictures vertical on gets a algebra.

Definition 1.2.1. A triple $\Theta = (\theta, w, x)$ with $\theta \in \mathrm{End}(N)$ and isometries $\sqrt{d\theta}^{-1} w\colon \mathrm{id}_N \to \theta$ and $\sqrt{d\theta}^{-1} x\colon \theta \to \theta^2$, which we will graphically display as

is called a **Q-system** (cf. [Lon94, LR97])

Two Q-systems $\Theta = (\theta, w, x)$ and $\tilde\Theta = (\tilde\theta, \tilde w, \tilde x)$ in $\mathrm{End}(N)$ are called equivalent, if there is a unitary $u \in \mathrm{Hom}(\theta, \tilde\theta)$, such that

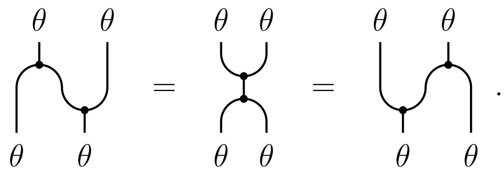
$$\tilde x u = (u \otimes u)x \equiv u\theta(u)x\,; \qquad u\tilde w = w$$

hold, or graphically:

A Q-system in a C*-tensor category automatically [LR97] fulfills the "Frobenius law" which is easy to check for our example from a subfactor:

$$(x^* \otimes 1_\theta)(1_\theta \otimes x) \equiv x^*\theta(x) \quad = \quad xx^* \quad = \quad (1_\theta \otimes x^*)(x \otimes 1_\theta) \equiv \theta(x^*)x$$

or graphically:



Definition 1.2.2. Let $\Theta = (\theta, w, x)$ be a Q-system. It is called **irreducible** if $\dim \mathrm{Hom}(\mathrm{id}, \theta) = 1$ and standard if $(\bar\theta, r, \bar r)$ with $\bar\theta = \theta$, $r = \bar r = xw$ is a standard solution for the conjugate equation for θ.

So an irreducible finite index subfactor $N \subset M$ of type III gives an irreducible standard Q-system in $\mathrm{End}(N)$.

Conversely, given an abstract Q-system in $\mathrm{End}(N)$ there is a von Neumann algebra $M \supset N$ by the Longo–Rehren construction.

Proposition 1.2.3 (Longo–Rehren construction)**.** *Let $\Theta = (\theta, w, x)$ be an irreducible standard Q-system in* $\mathrm{End}(N)$ *for a type III factor N. Then there is a von Neumann algebra M with canonical inclusion $\iota \colon N \to M$ with conjugate $\bar\iota \colon M \to N$ and $v \in \mathrm{Hom}(\mathrm{id}_M, \iota\bar\iota)$, such that $(\bar\iota, v, w)$ fulfills the conjugate equation for ι and $N \subset M$ is irreducible.*

Proof. The short way is to define M to be the Jones basic construction $E_0(N) \subset N \subset M$ where $E_0 = \sqrt{d\theta}^{-1} x^*\theta(\,\cdot\,)x$ is a conditional expectation by the Q-system properties. Namely, it is normal and completely positive. Associativity of the Q-system gives $E_0 \circ E_0 = E_0$ and the Frobenius law gives $E_0(xE_0(y)) = E_0(x)E_0(y)$ which shows that $E_0(M)$ is a subalgebra.

But we can build M directly from N by adjoining one element v fulfilling:

$$v\iota(n) = \iota\theta(n)v$$

with $n \in N$ and $\iota \colon N \to M$ the embedding of N into the still to be constructed algebra M. This means $v \in \mathrm{Hom}(\iota, \iota\theta)$ and its square and adjoint are defined to be

$$v^2 = \iota(x)v \qquad\qquad v^* = \iota(wx^*)v \tag{1.17}$$

A general element of M can be written as $\iota(n)v$ for some $n \in n$. The unit is $1_M = \iota(w^*)v$ and $(\iota(n)v)^* = v^*\iota(n^*) = \iota(w^*x^*\theta(n^*))v$. By the Frobenius property this turns the set $M = \{\iota(n)v : n \in A\}$ into a $*$-algebra. We induce the weak topology from N to M with help of the conditional expectation $E \colon M \to N$ given by

$$E(m) = \sqrt{d\theta}^{-1} w^*\iota(m)w \qquad\qquad E(vv^*) = \sqrt{d\theta}^{-1} 1_N \tag{1.18}$$

where $\bar\iota \colon M \to N : \iota(n)v \mapsto \theta(n)x$ and (w, v) is a standard solution to the conjugate equation. □

We call the Q-system (θ, w, x) **factorial** if the constructed $M \supset N$ is a factor. If (θ, w, x) is irreducible, i.e. $\dim \mathrm{Hom}(\mathrm{id}_N, \theta) = 1$, then M is always a factor.

1.3. Subobjects and semi-simplicity

Consider $\alpha\colon P \to Q$ for P, Q type III factors with finite dimension. $p \in \mathrm{Hom}(\alpha, \alpha)$ is a non-trivial ($0 \neq p \neq 1$) orthogonal projection $p = p^2 = p^*$, then by the type III properties there are isometries u_i with $u_1 u_1^* = p$ and $u_2 u_2 * = 1 - p$. Note that u_i are the generators of the Cuntz algebra $\mathcal{O}_2$. We can write

$$\alpha = \mathrm{Ad}(u_1) \circ \alpha_1 + \mathrm{Ad}(u_2) \circ \alpha_2 \qquad \text{where } \alpha_i = \mathrm{Ad}(u_i^*)\rho \tag{1.19}$$

where $\alpha_i\colon P \to Q$ are unital $*$-morphisms with finite dimension. In other words, α is a **direct sum** of α_1 and α_2. We use the notation $[\alpha] = [\alpha_1] \oplus [\alpha_2]$ since direct sums are well-defined on sectors. In general, using u_i generators of $\mathcal{O}_n$ in Q and $\alpha_i\colon P \to Q$ with finite dimension, we can define direct sums

$$\alpha = \sum_{i=1} n \,\mathrm{Ad}(u_i) \circ \alpha_i \qquad [\alpha] = \bigoplus_{i=1}^{n} [\alpha_i] \tag{1.20}$$

and we use the notation $N \in \mathbb{N}$

$$N[\alpha] = \bigoplus_{i=1}^{N} [\alpha] \tag{1.21}$$

It follows that α is a finite direct sum of inequivalent irreducibles α_i, i.e. $\mathrm{Hom}(\alpha_i, \alpha_j) = \delta_{i,j}\mathbb{C} \cdot 1_Q$

$$[\alpha] = \bigoplus_i n_i[\alpha_i] \qquad \Longrightarrow \qquad \mathrm{Hom}(\alpha, \alpha) \cong \bigoplus M_{n_i}(\mathbb{C})\,. \tag{1.22}$$

In other words, the category $\mathrm{End}_0(M)$ of unital $*$-morphisms with finite dimensions of a type III factor M has **direct sums and subobjects** and is **semisimple**.

1.4. Even part, finite depth and fusion categories

Consider the categories

$$ {}_N\mathcal{C}_N = \langle \rho \prec (\bar{\iota} \circ \iota)^n \rangle \subset \mathrm{End}_0(N) \tag{1.23}$$
$$ {}_M\mathcal{C}_M = \langle \beta \prec (\iota \circ \bar{\iota})^n \rangle \subset \mathrm{End}_0(M) \tag{1.24}$$

where $\langle \,\cdot\, \rangle$ is the full subcategory with subobjects and finite direct sums generated by the listed sectors. A subcategory $\mathcal{D} \subset \mathcal{C}$ is **full** if for any ρ, σ objects in $\mathcal{D}$, then $\mathrm{Hom}_{\mathcal{D}}(\rho, \sigma) = \mathrm{Hom}_{\mathcal{C}}(\rho, \sigma)$.

Let us denote $\mathrm{Irr}(\mathcal{C})$ denotes the collection of irreducible sectors of $\mathcal{C}$. We say $\iota(N) \subset M$ has finite depth if and only if $|\,\mathrm{Irr}({}_N\mathcal{C}_N)| < \infty$ if and only if $|\,\mathrm{Irr}({}_M\mathcal{C}_M)| < \infty$.

Exercise 1.4.1. Let $P = M, N$ and Δ a set of endomorphisms of a choice of an endomorphism for sector in $\mathrm{Irr}({}_P\mathcal{C}_P)$. Then we have

(1) $\mathrm{id} \in \Delta$ (we choose $\mathrm{id} = \mathrm{id}_P$ as representant of $[\mathrm{id}_P] \in \mathrm{Irr}({}_P\mathcal{C}_P)$).
(2) Δ is closed under conjugates: If $[\rho] \in \Delta$ then $[\bar{\rho}] \in \Delta$.
(3) Δ is closed under fusion: If $\rho, \sigma \in \Delta$, then

$$[\rho] \otimes [\sigma] = [\rho \circ \sigma] = \bigoplus_{\tau \in \Delta} N_{\sigma,\rho}^{\tau}[\tau] \tag{1.25}$$

If is a finite system $\Delta \subset \mathrm{End}_0(P)$ fulfilling the properties (1)-(3) then we call $\mathcal{F} = \langle \Delta \rangle \subset \mathrm{End}_0(P)$ a (concrete type III) **unitary fusion category**.

Proposition 1.4.2. *Let $\mathcal{C} \subset \mathrm{End}(N)$ be a fusion category, then there is a subfactor $N \subset M$, such that $\mathcal{C} = {}_N\mathcal{C}_N$.*

Proof. Choose a $\rho \in \mathcal{C}$ with $[\rho] = \bigoplus_{\rho_i \in \Delta}[\rho_i]$ and a $\bar{\rho}$ and (v, w) a standard solution of the conjugate equation. Then $\theta = \bar{\rho} \circ \rho, w, \rho(v)$ is a Q-system which gives the Jones basic extension

$$\rho(N) \subset N \subset M\,.$$

It follows immediately that ${}_N\mathcal{C}_N = \mathcal{C}$. □

Fact 1.4.3. It is a deep result, that an abstract unitary fusion category $\mathcal{C}$ can be realized as $\mathcal{C} \subset \mathrm{End}(N)$ for a type III factor N. It follows from Popa's theorem [Pop95], that there is a (essentially) unique realization if N is the hyperfinite type III_1 factor.

Example 1.4.4. Let G be a finite group and $\alpha\colon G \to \mathrm{Aut}(M)$ be an outer action. Then $M^G = \{m \in M : \alpha_g(m) = m \text{ for all } g \in G\}$ is a factor. Consider the subfactor:

$$N = M^G \subset M \tag{1.26}$$

It turns out that

$$[\gamma] = [\iota \circ \bar{\iota}] = \bigoplus_{g\in G}[\alpha_g] \qquad {}_M\mathcal{C}_M \cong \mathrm{Vect}_G \qquad \mathrm{Hom}(\gamma,\gamma) \cong \ell^\infty(G) \tag{1.27}$$

$$[\gamma] = [\iota \circ \bar{\iota}] \cong \bigoplus_{\pi\in\mathrm{Rep}(G)} \dim(\pi)[\pi] \qquad {}_N\mathcal{C}_N \cong \mathrm{Rep}(G) \qquad \mathrm{Hom}(\theta,\theta) \cong \mathbb{C}[G] \tag{1.28}$$

(1.29)

where Vect_G is the category of finite-dimensional G-graded Hilbert spaces and $\mathrm{Rep}(G)$ is the category of finite-dimensional unitary representations of G and we mean an equivalence of tensor categories. We note that for any α with $d\alpha < \infty$ the space of intertwiners $\mathrm{Hom}(\alpha, \alpha)$ is a finite-dimensional C* (actually von Neumann) algebra and the corresponding isomorphism is an isomorphism of $*$-algebras.

Example 1.4.5. A unitary fusion category $\mathcal{C} \subset \mathrm{End}_0(N)$ is called pointed if for any $[\rho] \in \mathrm{Irr}(\mathcal{C})$ we have $d\rho = 1$. Then $G = \mathrm{Irr}(\mathcal{C})$ is a finite group and $\mathcal{C}$ is classified by this group G and an element in $\omega \in H^3(G, \mathbb{T})$. Note that we can see $\mathrm{Irr}(\mathcal{C}) \subset \mathrm{Out}(N) = \mathrm{Aut}(N)/\mathrm{Inn}(N)$ and this is also called a ***G*-kernel**.

We get a true action by automorphisms (then M^G is a factor) if only if $\omega = 0$. If $\omega \neq 0$ we can do the construction from Proposition 1.4.2, this is called the **diagonal subfactor** associated with the G-kernel.

1.5. Subfactors in unitary fusion categories

Let ${}_N\mathcal{F}_N \subset \mathrm{End}(N)$ be a unitary fusion category.

Problem 1.5.1. Find all overfactors $M \supset \iota(N)$ such that the associated Q-system lies in ${}_N\mathcal{F}_N$.

Let us denote from now on

$$ {}_N\mathcal{F}_N^{N\subset M} = \langle \rho \prec (\bar\iota \circ \iota)^n \rangle \,, \qquad {}_M\mathcal{F}_M^{N\subset M} = \langle \beta \prec (\iota \circ \bar\iota)^n \rangle \,. \tag{1.30} $$

In other words, we ask ${}_N\mathcal{F}_N^{N\subset M} \subset {}_N\mathcal{F}_N$.

In this case, we define the **dual category** ${}_M\mathcal{C}_M = \langle \iota\circ\rho\circ\bar\iota : \rho \in {}_N\mathcal{F}_N \rangle$ which is a unitary fusion category. Note that Problem 1.5.1 is equivalent with asking ${}_M\mathcal{F}_M^{N\subset M} \subset {}_M\mathcal{F}_M$.

In this case, we say ${}_N\mathcal{F}_N$ and ${}_M\mathcal{F}_M$ are **Morita equivalent**. The Morita equivalence is exactly given by the subfactor. Again Morita equivalence can be define abstractly, see [Müg03].

Example 1.5.2. Let $N \subset M$ be a finite index finite depth subfactor, then ${}_N\mathcal{F}_N^{N\subset M}$ and ${}_M\mathcal{F}_M^{N\subset M}$ are Morita equivalent.

Rep(G) is Morita equivalen to Vect$_G$ for every finite group G.

We define the **global dimension** of a unitary fusion category $\mathcal{F}$ to be

$$ \mathrm{GDim}(\mathcal{F}) = \sum_{[\rho]\in\mathrm{Irr}(\mathcal{F})} (d\rho)^2 \,. $$

Similarly, we define the global dimension for a subfactor $N \subset M$ to be $\mathrm{GDim}(N\subset M) = \mathrm{GDim}({}_N\mathcal{F}_N^{N\subset M})$. Note that in general $\mathrm{GDim}(N\subset M) \neq [M:N]$, while $\mathrm{GDim}(M^G \subset M) = [M:N^G] = |G|$, while

Fact 1.5.3. Let ${}_N\mathcal{F}_N$ and ${}_M\mathcal{F}_M$ be Morita equivalent, then

$$ \mathrm{GDim}({}_N\mathcal{F}_N) = \mathrm{GDim}({}_M\mathcal{F}_M) $$

Since $d\pi = \dim(\pi)$ for $\pi \in \mathrm{Rep}(G)$, considering $M^G \subset M$ we get this classical result from group theory.

Corollary 1.5.4. *We have*

$$ \sum_{\pi\in\mathrm{Irr}(\mathrm{Rep}(G))} \dim(\pi)^2 = |G| \,. $$

Group	Quantum symmetry
G	$N \subset M$
Rep(G)	${}_N\mathcal{F}_N^{N\subset M}$
Vect$_G$	${}_M\mathcal{F}_M^{N\subset M}$
$\lvert G\rvert$	$[M:N]$

1.6. Quantum Double Subfactors

Consider $\mathcal{F} \subset \mathrm{End}(N)$ be a unitary fusion category. Then with $A = N \otimes N^{\mathrm{op}}$ there is a Q-system:

$$ [\theta] = \bigoplus_{[\sigma]\in\mathrm{Irr}(\mathcal{F})} [\sigma \otimes \sigma^{\mathrm{op}}] $$

in ${}_A\mathcal{C}_A = \langle \rho\otimes\sigma^{\mathrm{op}} : \rho,\sigma \in \mathcal{F}\rangle$. which gives the so-called **Longo–Rehren subfactor** $A \subset B$.

We have ${}_B\mathcal{C}_B \cong Z(\mathcal{F})$, where $Z(\mathcal{F})$ is the **Drinfeld center**.

The category ${}_A\mathcal{C}_A$ is equivalent to the Deligne tensor product $\mathcal{F}\boxtimes\mathcal{F}^{\mathrm{op}}$, which is therefore Morita equivalent to $Z(\mathcal{F})$.

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