Kasparov's operator K-theory and applications
4. Lafforgue's approach

Georges Skandalis

Université Paris-Diderot Paris 7
Institut de Mathématiques de Jussieu

NCGOA Vanderbilt University
Nashville - May 2008

Lafforgue's approach

3 steps

1. Construction of a KK-theory for Banach algebras, and main properties.
2. Equality $\gamma = 1$ in $KK_G^{ban}(\mathbb{C}, \mathbb{C})$ in many cases. *Almost all.*

   The first two steps establish the Bost conjecture and some variants.
3. Construction of a suitable spectral dense subalgebra.

B-pairs

B Bancah algebra.

Right (*resp.* left) *Banach B-module:* Banach space E endowed with a right (*resp.* left) action of B such that, for all $x \in E$ and $a \in B$, we have $\|xa\| \leq \|x\|\|a\|$ (*resp.* $\|ax\| \leq \|a\|\|x\|$).

B-pair:

- left Banach B-module $E^<$,
- right Banach B-module $E^>$,
- bilinear map $\langle\ ,\ \rangle : E^< \times E^> \to B$

satisfying: $\forall x \in E^>,\ \xi \in E^<$, the map $\eta \mapsto \langle \eta, x \rangle$ (*resp.* $y \mapsto \langle \xi, y \rangle$) left (*resp.* right) B-linear and $\|\langle \xi, x \rangle\| \leq \|\xi\|\|x\|$. Often $E^<$ not specified: take $E^< = (E^>)^* = \mathcal{L}(E, B)$.

Morphisms of B-pairs E

- *Morphism* from $E = (E^<, E^>)$ to $F = (F^<, F^>)$ couple $f = (f^<, f^>)$ where $f^< : F^< \to E^<$ and $f^> : E^> \to F^>$ $\mathbb{C}$-linear, left (*resp.* right) B-linear, continuous; $f^>$ and $f^<$ ajoint: $\langle \eta, f^>(x)\rangle = \langle f^<(\eta), x\rangle$.
- $\mathcal{L}(E, F)$ the Banach space of morphisms from E to F (norm $(f^<, f^>) \mapsto \sup(\|f^<\|, \|f^>\|)$).
- $f \in \mathcal{L}(E, F)$ and $g \in \mathcal{L}(F, G)$ define morphism $gf = (f^< \circ g^<, g^> \circ f^>) \in \mathcal{L}(E, F)$.
- $\mathcal{L}(E) = \mathcal{L}(E, E)$ Banach algebra.
- Let $y \in F^>$ and $\xi \in E^<$. We note $\theta_{y,\xi} \in \mathcal{L}(E, F)$ (or $|y\rangle\langle\xi|$) the morphism given by
  - $F^< \ni \eta \mapsto \langle \eta, y\rangle\xi \in E^<$
  - $E^> \ni x \mapsto y\langle \xi, x\rangle \in F^>$.
- $\mathcal{K}(E, F)$ closed vector span in $\mathcal{L}(E, F)$ of morphisms $\theta_{y,\xi}$.

Definition of the Banach KK-theory

A, B be Banach algebras. Cycle for $KK^{\mathrm{ban}}(A, B)$: triple (E, π, f) where

- E is a $\mathbb{Z}/2\mathbb{Z}$-graded B-pair,
- $\pi : A \to \mathcal{L}(E)^{(0)}$ is a homomorphism
- $f \in \mathcal{L}(E)^{(1)}$

such that $f\pi(a) - \pi(a)f \in \mathcal{K}(E), \quad (f^2 - 1)\pi(a) \in \mathcal{K}(E)$ $(a \in A)$.

Sum of cycles and *homotopy* defined exactly as for Hilbert modules.
$KK^{\mathrm{ban}}(A, B)$ set of homotopy classes of cycles. Abelian group.

Bifunctor: contravariant in A, covariant in B.

Action on K-theory

- $KK^{\mathrm{ban}}(M_n(A), B)$ isomorphic $KK^{\mathrm{ban}}(A, B)$.
- Idempotent $p \in A$, homomorphism $i_p : \mathbb{C} \to A$ setting $i_p(\lambda) = \lambda p$.
- bilinear map $\varphi : K_0(A) \times KK^{\mathrm{ban}}(A, B) \to KK^{\mathrm{ban}}(\mathbb{C}, B)$
  $(p, x) \mapsto (i_p)^*(x)$. Can be constructed when A not unital.
- $K_0(A) \simeq KK^{\mathrm{ban}}(\mathbb{C}, A) \quad (x \mapsto \varphi(x, 1_A))$.
- *Banach-KK-theory acts on the K-theory.*

The equivariant case

G locally compact group acting on A and B. Define $KK_G^{\mathrm{ban}}(A, B)$ same way as the corresponding Kasparov group.

A, B C^*-algebras, natural homomorphism $KK_G(A, B) \to KK_G^{\mathrm{ban}}(A, B)$.

Morphism j_G

Morphism $j_G^r : KK_G^{\mathrm{ban}}(A, B) \to KK^{\mathrm{ban}}(A \rtimes_r G, B \rtimes_r G)$ *impossible.* Need Banach algebras crossed product, extending reduced crossed product.

Natural crossed product for Banach algebras: $L^1(G; A)$. Morphism $j_G^{L^1} : KK_G^{\mathrm{ban}}(A, B) \to KK^{\mathrm{ban}}(L^1(G; A), L^1(G; B))$.

Using $j_G^{L^1}$ + equality $\gamma = 1$ in $KK_G^{\mathrm{ban}}(\mathbb{C}, \mathbb{C}) \Rightarrow$ BC conjecture in $L^1(G)$.

$L^1(G) \to C_r^*(G)$ isomorphism in K-theory? No general result (Bost).

Lafforgue: may perform other Banach crossed product.

Definition

Algebra norm N on $C_c(G)$ is *unconditional* if $N(f) = N(|f|)$ $(f \in C_c(G))$.

Unconditional norms and crossed products

N *unconditional norm* $\rightarrow$ natural norm on $C_c(G;A)$: $f \mapsto N(\| \, \|_A \circ f)$.
Completion: Banach algebra $A \rtimes_N G$.

Same construction for B-pairs: $E \rtimes_N G$.

Morphism $j_G^N : KK_G^{\mathrm{ban}}(A,B) \rightarrow KK^{\mathrm{ban}}(A \rtimes_N G, B \rtimes_N G)$.

Conclusion of first step

Baum-Connes' conjecture for G with element $\gamma \in KK_G(\mathbb{C},\mathbb{C})$:

1. Prove that $\gamma = 1$ in $KK_G^{\mathrm{ban}}(\mathbb{C},\mathbb{C})$.
2. Construct unconditional completion of $C_c(G)$ with same K-theory as $C_r^*(G)$.

Non isometric representations of G

Equality $\gamma = 1$ proved in slight generalization $KK_G^{\mathrm{ban}}(\mathbb{C}, \mathbb{C})$.

ℓ *length function* on G: $\ell : G \to \mathbb{R}_+$ (continuous) such that $\ell(1) = 0$ and $\ell(xy) \leq \ell(x) + \ell(y)$ for all $x, y \in G$.

$KK_{G,\ell}^{\mathrm{ban}}(A, B)$ same way as $KK_G^{\mathrm{ban}}(A, B)$ action of G in the B-pairs not isometric but is *controled by ℓ* *i.e.* continuous and $\|x \cdot \xi\| \leq \exp(\ell(x))\|\xi\|$.

ℓ length function, N unconditional norm.

- Put $N' : f \mapsto N(e^\ell f)$: unconditional norm.
- $j_G^{N,\ell} : KK_{G,\ell}^{\mathrm{ban}}(A, B) \to KK^{\mathrm{ban}}(A \rtimes_{N'} G, B \rtimes_N G)$.

ℓ length function. For $s \in \mathbb{R}_+^*$, define N_s unconditional norm $f \mapsto N(e^{s\ell} f)$.
Subalgebra $\displaystyle\bigcup_{s \in \mathbb{R}_+^*} A \rtimes_{N_s} G$ of $A \rtimes_N G$ *same K-theory* as $A \rtimes_N G$.

$\gamma = 1$ in $KK_{G,s\ell}^{\mathrm{ban}}(\mathbb{C}, \mathbb{C})$ *for all s*, implies γ identity in $K_0(A \rtimes_N G)$.

Homotopy between γ and 1

Two different cases:

1. *"geometric":* complete riemannian manifold with nonpositive sectional curvature; real Lie groups and closed subgroups.
2. *"Combinatoric":* "strongly bolic" metric space; p-adic Lie groups and closed subgroups.

Concentrate here to the combinatoric case: case of *buildings of type $\widetilde{A_2}$.*

Recall: Julg Vallette γ element for $\widetilde{A_2}$ buildings

G acts properly, isometrically on $\widetilde{A_2}$ building X.

$X^{(i)}$ $(0 \leq i \leq 2)$ set of faces of dimension i in X.

$(e_x)_{x \in X^{(0)}}$ canonical Hilbert basis of $H_0 = \ell^2(X^{(0)})$.

$H_1 \subset \Lambda^2(H_0)$ vector span of $e_\sigma = e_x \wedge e_y$, $\sigma = (x, y) \in X^{(1)}$
$H_2 \subset \Lambda^3(H_0)$ vector span of $e_\sigma = e_x \wedge e_y \wedge e_z$, $\sigma = (x, y, z) \in X^{(2)}$.

- $H = (H_0 \oplus H_2) \oplus H_1$.
- $F = F_a = T_a + T_a^*$ depends on an origin $a \in X^{(0)}$.

Where $T_a(e_\sigma) = v_{a,\sigma} \wedge e_\sigma$
v_a: unit vector which points from σ to a.

Abstract results: elliptic complexes

G locally compact group, length function ℓ, A, B Banach algebras, E a $\mathbb{Z}/2\mathbb{Z}$-graded B-pair with actions of A and G-action "controled by ℓ".

$D = \{S \in \mathcal{L}(E);\ [S, a] \in \mathcal{K}(E),\ g.S - S \in \mathcal{K}(E);\ g \mapsto g.S$ continuous$\}$.
$F \in D^{(1)}$ with $\mathrm{id}_E - F^2 \in \mathcal{K}(E)$: $(E, F) \in KK^{\mathrm{ban}}_{G,\ell}(A, B)$.

Lemma

Let $S \in D^{(1)}$ such that $S^2 \in \mathcal{K}(E)$ and $\exists T \in D^{(1)}$ with $\mathrm{id}_E - (TS + ST) \in \mathcal{K}(E)$ ((E, S) - or S elliptic complex).

1. *There exists such a T with $T^2 \in \mathcal{K}(E)$. Then $(E, S + T) \in KK^{\mathrm{ban}}_{G,\ell}(A, B)$.*
2. *The class of $(E, S + T)$ in $KK^{\mathrm{ban}}_{G,\ell}(A, B)$ does not depend on T.*

Elliptic complexes define KK-elements.

1. TST is OK.
2. $\{T \in D^{(1)};\ ST + TS - \mathrm{id}_E \in \mathcal{K}(E)\}$ is affine.

Abstract results: Elliptic complexes (2)

Lemma

$S, T \in D^{(1)}$ such that $S^2 \in \mathcal{K}(E)$ and $T^2 \in \mathcal{K}(E)$. Assume that the spectrum of $S + T$ in $D/\mathcal{K}(E)$ is disjoint from $\mathbb{R}_-$.

1. *S and T are elliptic complexes.*
2. *S and T define the same element of $KK^{\mathrm{ban}}_{G,\ell}(A, B)$.*

1. In $D/\mathcal{K}(E)$, $(ST + TS)$ commutes with S and T. $T(ST + TS)^{-1}$ and $S(ST + TS)^{-1}$ 'quasi-inverses'.
2. We may define a logarithm of $ST + TS$. The desired homotopy is $S(ST + TS)^{-t} + T(ST + TS)^{t-1}$.

Abstract results: Elliptic complexes (3)

Lemma

$S, T \in D^{(1)}$. Assume that S commutes exactly to A and to G, that $S^2 = 0$ and $T^2 \in \mathcal{K}(E)$, $ST + TS = \mathrm{id}_E$. Then the class of $(E, S + T)$ in $KK^{\mathrm{ban}}_{G,\ell}(A, B)$ is zero.

May assume $T^2 = 0$ (replace T by TST). $S(E) \subset E$ invariant by A and G.

Decomposition $E = S(E) \oplus T(E)$, matrix of these elements is of the form $\begin{pmatrix} c_{1,1} & c_{1,2} \\ 0 & c_{2,2} \end{pmatrix}$. Note that $c_{1,2} \in \mathcal{K}(E)$ since $T \in D$.

Change these actions through a homotopy $\begin{pmatrix} c_{1,1} & tc_{1,2} \\ 0 & c_{2,2} \end{pmatrix}$ $(t \in [0, 1])$. At $t = 0$, $S + T$ is degenerate.

Homotopy

$\varphi = \varphi_a : X \to \mathbb{R}_+$: $\varphi(f)$ distance to a of most remote point of f.

Put $\ell(g) = \varphi(g(a))$.

It is a length function: indeed
$\ell(gh) = d(gh(a), a) \leq d(gh(a), g(a)) + d(g(a), a) = d(h(a), a) + d(g(a), a)$
(g isometry).

Theorem

For all $s > 0$, the images of γ and 1 in $KK^{\mathrm{ban}}_{G,s\ell}(\mathbb{C}, \mathbb{C})$ coincide.

Homotopy (2)

E_p Banach space, graded by $0, 1, 2$: replace ℓ^2 norm by ℓ^p norm in construction of H: ℓ^p basis e_σ.

Consider $\partial : E_p \to E_p$ given by $\partial(e_x) = 0$, $\partial(e_x \wedge e_y) = e_y - e_x$ and $\partial(e_x \wedge e_y \wedge e_z) = (e_y \wedge e_z) - (e_x \wedge e_z) + (e_x \wedge e_y)$ (for all $(x, y, z) \in X^{(2)}$).

For $t > 0$, let A_t multiplication by $e^{t\varphi}$ (unbounded) $\partial_t = A_t \circ \partial \circ A_t^{-1}$ (bounded).

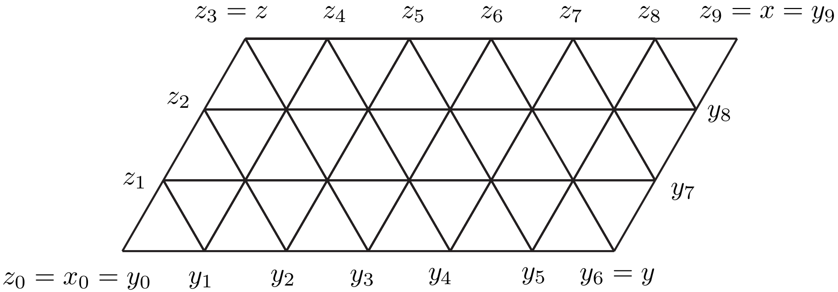
$\partial_t - g.\partial_t$ is compact in every E_p ($\varphi_a - \varphi_{ga}$ *almost constant at infinity*).

Proposition

1. *For all $s > 0$, $\partial_s : E_1 \to E_1$ elliptic complex.*
2. *There exists $s > 0$ such that for all $p \in [1, 2]$, $\partial_s : E_p \to E_p$ elliptic complex.*

Construction of a quasi-inverse

Let $x \in X^{(0)}$. Points x and $a = x_0$ determine a parallelogram x_0, y, x, z in the building X.



We set $T_0(e_x) = (1 - \frac{j}{n}) \sum_{k=1}^{n} e_{z_{k-1}} \wedge e_{z_k} + \frac{j}{n} \sum_{k=1}^{n} e_{y_{k-1}} \wedge e_{y_k}$ where $d(x, y) = j(= 6)$ and $d(x, z) = n - j(= 3)$.

Clearly $\partial \circ T_0(e_x) = e_x - e_0$.

Construction of a quasi-inverse (2)

In order to define $T_0(e_x \wedge e_y)$, one uses the following lemma:

Lemma

$e_x \wedge e_y - T_0\partial(e_x \wedge e_y)$ is in the image of ∂.

Restrict to the parallelogram containing $\{x_0, x, y\}$. ∂ restricted to vertices, edges and faces of this parallelogram is exact in dimensions 1 and 2.

$T_0(e_x \wedge e_y)$ is the element ξ such that $\partial(\xi) = e_x \wedge e_y - T_0\partial(e_x \wedge e_y)$ decribed above.

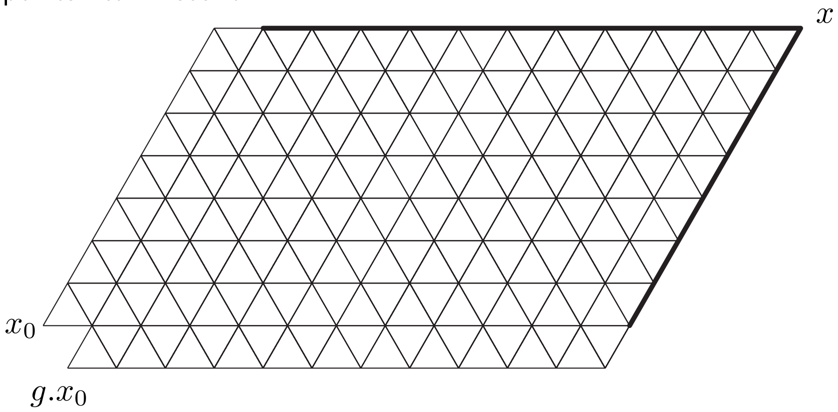
The desired quasi-inverse of ∂_s is $T_s = A_s \circ T_0 \circ A_s^{-1}$.

1. For all $s > 0$, T_s is continuous from E_1 into E_1.
2. There exists $t > 0$ such that, for all $p \in [1, 2]$, T_t is continuous from E_p in E_p and, for all $g \in G$, $T_t - g.T_t \in \mathcal{K}(E_p)$.
3. For all $s > 0$ and $g \in G$, $T_s - g.T_s \in \mathcal{K}(E_1)$.

1. Check $\|T_s(e_x)\|$ and $\|T_s(e_x \wedge e_y)\|$ bounded on X. Faces appearing located in parallelogram above; their number grows polynomially with the distance from x to x_0; coefficients appearing bounded. Conjugation by A_s multiplies by function with exponential decay.
2. In $T_s^*(e_x \wedge e_y)$ appear all points z such that x, y is on the path from z to x_0. The coefficients appearing have exponential decay $\exp(-s\varphi(z))$; their number z increases exponentially.
   Taking s large enough, one may control the ℓ^1 norm of $T_s^*(e_x \wedge e_y)$ and that of $T_s^*(e_x \wedge e_y \wedge e_z)$.
   It follows that T_s^* is continuous from E_1 into E_1, whence T_s is continuous from E_∞ into E_∞. As it is continuous from E_1 into E_1, it is continuous from E_p into E_p for all p (by interpolation).

Similar arguments show that $T_s - g.T_s \in \mathcal{K}(E_p)$ for all $g \in G$.

3 Check that when $\{x, y\}$ goes to infinity, $\|(T_s - g.T_s)(e_x)\|$ and $\|(T_s - g.T_s)(e_x \wedge e_y)\|$ go to 0. $g.T_0$ is T_0 with x_0 replaced by $g.x_0$. For x far from x_0, from x to x_0 and from x to $g.x_0$ used in construction of T_0 coincide near x. Since we conjugate by A_s only points near x count.



Homotopy: end

For s small, ∂_s almost invariant and we can use abstract results above to show that (E_1, ∂_s) defines element 1. (Use E_1^s: where norm is changed - with ∂).

For s large, ∂_s almost quasi-inverse of T_a appearing in γ and we can use abstract results above to show that (E_2, ∂_s) defines element γ.

These elements are homotopic!!

The last step... Case of Lie groups

G (real or p-adic) reductive Lie group: slight modification of Schwarz space (Harish-Chandra algebra) *spectral unconditional completion* of $C_c(G)$.

In this way, we get more direct proof of results of Wassermann (real case) and a generalization of Baum, Higson and Plymen (p-adic case).

Property (RD) Haagerup-Jolissaint

G discrete group ℓ length function on G. If G is finitely generated, may take word length.

$H^\infty(G,\ell)$ vector space functions $f : G \to \mathbb{C}$ such that, for all $p \in \mathbb{R}_+$, $\sum_{x\in G} \ell(x)^p |f(x)|^2 < \infty$. *Unconditional.*

Theorem (Haagerup)

G finitely generated free group. $H^\infty(G)$ subalgebra of $C^*_r(G)$ stable under holomorphic functional calculus (spectral). In particular, inclusion $H^\infty(G) \to C^*_r(G)$ induces K-theory isomorphism.

Jolissaint:

Definition

Fnitely generated group G property (RD) if $H^\infty(G) \subset C^*_r(G)$.

Then $H^\infty(G)$ spectral subalgebra of $C^*_r(G)$.

Groups with property (RD)

Jolissaint: many groups behaving like free groups (*e.g.* cocompact subgroups of simple real Lie groups of rank 1) have property (RD).

De la Harpe: Gromov's hyperbolic groups.

Ramagge, Robertson and Steger: discrete groups acting properly with compact quotient on $\widetilde{A}_2$ buildings (*e.g.* discrete subgroups of $SL_3(\mathbb{Q}_p)$).

Lafforgue adapts proof of Ramagge, Robertson and Steger to the case of cocompact subgroups of $SL_3(\mathbb{R})$ and $SL_3(\mathbb{C})$.

Finally, these results extended by M. Talbi and I. Chaterji so to contain also the quaternionic case and products of above groups.