Kasparov's operator K-theory and applications
3. Baum-Connes conjecture

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Nashville - May 2008

Classifying spaces

Classifying space BG of a discrete group G:

- Principal G-bundle $\xi_0 : EG \to BG$.
- Universal: Every G-bundle $\xi : \tilde{X} \to X$ is a pull back $\xi \simeq f^*(\xi_0)$ where $f : X \to BG$ continuous, unique up to homotopy.
- These properties determine BG up to homotopy equivalence.

Principal G-bundle $\xi : \tilde{X} \to X$ then G acts *freely* and *properly* on $\tilde{X}$ and $\tilde{X}/G = X$ ($(x, g) \mapsto (x, gx)$ injective and proper).
EG *final object* in category of free and proper G-spaces.

Baum-Connes conjecture uses *final object in category of proper G-spaces.*
Exists and unique up to G-equivariant homotopy. Notation $\underline{E}G$.
Construction simpler than that of EG

Topological K-theory group (Baum-Connes, BC+ Higson, Tu)

G locally compact group.

$$K_{0,top}(G) = \varinjlim KK_G(C_0(Y), \mathbb{C}),$$

inductive limit taken over closed, G-invariant $Y \subset \underline{E}G$ with Y/G compact.

Remarks

- *G is discrete,* $K_{0,top}(G) = \varinjlim KK(C_0(Y) \rtimes G, \mathbb{C})$.
- *G discrete torsion-free,* $C_0(Y) \rtimes G$ *isomorphic to* $C(Y/G) \otimes \mathcal{K}$.
  $K_{0,top}(G) = \varinjlim KK(C(X), \mathbb{C})$ *limit over* $X \subset BG = \underline{E}G/G$, X *compact.*
  $K_{0,top}(G) = K_{0,c}(BG)$, *($K$-homology with compact supports).*

The Baum-Connes map

Y locally compact space *proper* action of G and Y/G *compact*.

If action free, Morita equivalence $\to$ canonical Hilbert $C_0(Y) \rtimes G$-module E, such that $\mathcal{K}(E)$ isomorphic to $C(Y/G)$.

Same construction in non free case! (E no longer *full*). Since $\mathcal{K}(E)$ is unital, $(E, \pi, 0)$ gives $\Lambda_Y \in KK(\mathbb{C}, C_0(Y) \rtimes G)$.

Equivalently, Λ_Y class of idempotent of $C_0(Y) \rtimes G$ constructed using a *'cut-off' function* $c \in C_c(G)$ non negative with integral 1.

- $x \in K_{0,top}(G)$represented by a closed G-invariant$Y \subset \underline{E}G$ with Y/G compact and $y \in KK_G(C_0(Y), \mathbb{C})$.
- $\mu_G(x)$: Kasparov product of $\Lambda_Y \in KK(\mathbb{C}, C_0(Y) \rtimes G)$ with $j_G(y) \in KK(C_0(Y) \rtimes G, C^*(G))$.
- Well defined homomorphism $\mu_G : K_{0,top}(G) \to K_0(C^*(G))$ at inductive limit level.

Baum-Connes' conjecture

Definition

Baum-Connes homomorphism $\mu_{G,r} = \lambda_* \circ \mu_G : K_{0,top}(G) \to K_0(C^*_r(G))$ where $\lambda : C^*(G) \to C^*_r(G)$.

Baum-Connes Conjecture

Homomorphism $\mu_{G,r}$ is an isomorphism.

Baum-Connes Conjecture with coefficients

A a G-algebra.

$K_{0,top}(G; A) = \varinjlim KK_G(C_0(Y), A)$

Baum-Connes homomorphisms $\mu_G^A : K_{0,top}(G; A) \to K_0(A \rtimes G)$ *and* $\mu_{G,r}^A : K_{0,top}(G; A) \to K_0(A \rtimes_r G)$.

$\mu_{G,r}^A$ *isomorphism?*

Generalizations and related conjectures

- Baum-Connes' conjecture for foliations
- Groupoids (Tu)
- 'Coarse' Baum-Connes' Conjecture (Roe, Higson Roe, Yu)
- Baum-Connes' conjecture with values in L^1 ("Bost conjecture")

The Novikov conjecture

M, N homotopy equivalent smooth oriented manifolds same signature.
Atiyah-Hirzebruch theorem, $\langle L(M), [M] \rangle = \langle L(N), [N] \rangle$.
Novikov theorem: when M and N are simply connected *only oriented homotopy invariant characteristic number.*

Novikov Conjecture (cohomological formulation)

$f : N \to M$ homotopy equivalence of smooth oriented manifolds
Γ discrete group and $g : M \to B\Gamma$ continuous map.
For every $\xi \in H^*(\Gamma; \mathbb{Q}) = H^*(B\Gamma; \mathbb{Q})$ we have

$$\langle (g \circ f)^*(\xi) \cup L(N), [N] \rangle = \langle g^*(\xi) \cup L(M), [M] \rangle.$$

Novikov Conjecture (homological formulation)

M, N, f, Γ, G as above
$(g \circ f)_*(L(N) \cap [N]) = g_*(L(M) \cap [M]) \in H_*(B\Gamma; \mathbb{Q})$.

- *Chern character.* Isomorphism $\mathrm{ch} : K^*(X) \otimes \mathbb{Q} \to H^*(X;\mathbb{Q})$ (X nice).
- Dually $\mathrm{ch} : K_*(X) \otimes \mathbb{Q} \to H_*(X;\mathbb{Q})$ given by equality:
  $\langle \mathrm{ch}(x), \mathrm{ch}(y)\rangle = \langle x, y\rangle \ \ (\in \mathbb{Q} \ - \ \ x \in K_*(X) \otimes \mathbb{Q}, \ y \in K^*(X) \otimes \mathbb{Q})$.
  True for general CW complexes and in particular for classsifying spaces of discrete groups.
- M oriented compact manifold, D_M signature operator of M defines an element $[D_M] \in K_*(M)$. $\quad \mathrm{ch}[D_M] = L(M) \cap [M]$.

Novikov Conjecture (K-homological formulation)

$f : N \to M$ homotopy equivalence between smooth oriented manifolds.
Γ discrete group and $g : M \to B\Gamma$ continuous map.
Then $(g \circ f)_*([D_N]) - g_*([D_M])$ is a torsion element of $K_{*,c}(B\Gamma)$.

Baum-Connes $\Rightarrow$ Novikov

Γ discrete group.
Natural homomorphism $\Phi : K_{*,c}(B\Gamma) \to K_{*,top}(\Gamma)$ *rationally injective.*

Theorem (Miščenko, Kasparov)

$\mu \circ \Phi(g_*([D_M])) \in K_*(C^*(\Gamma))$ *only depends on homotopy type of* (M, f).

In other words, if f is a homotopy equivalence,
$\mu(\Phi((g \circ f)_*([D_N]))) = \mu(\Phi(g_*([D_M])))$.

Injectivity of Baum-Connes map μ (and even injectivity up to torsion) implies *Novikov's conjecture.*

Remark

Rational injectivity of μ *also implies Lichnerowicz conjecture.*

What is known about Baum-Connes conjecture?

Positive results *Baum-Connes' conjecture with coefficients*:

- Higson-Kasparov: All *amenable* locally compact groups and, more generally, those *acting properly by affine isometries on a Hilbert space* (*cf.* Tu groupoids, Yu coarse Baum-Connes).
  Contains all previously known cases:
  - Amenable real Lie groups (Kasparov)
  - Lorenz groups $SO(n, 1)$ (Kasparov),
  - $SU(n, 1)$ (Julg-Kasparov).
- Stability results of conjecture with coefficients by operations on groups (subgroups, group extensions, amalgamated free products *etc.*) have been established (Chabert, Echterhoff, Oyono...).

What is known about Baum-Connes conjecture? (2)

Injectivity of Baum-Connes' map (with coefficients)

- Groups acting properly by isometries on complete, connected, simply connected riemannian manifolds with *nonpositive sectional curvature;* closed subgroups of *real Lie groups* (Miščenko, Kasparov);
- groups acting properly by isometries on *affine buildings*, closed subgroups of *p-adic Lie groups* (Solov'ev, Kasparov-Sk);
- Groups acting properly by isometries on discrete metric spaces with "nice" behaviour at infinity: weakly geodesic, with bounded coarse geometry and *"bolic"* (Kasparov-Sk);
- groups admitting *amenable actions* on compact space (Higson),
- more generally for all groups admitting *uniform embedding* in a Hilbert space (*cf.* Yu, Sk-Tu-Yu);
- even more generally, groups admitting a *uniform embedding in 'nice Banach spaces'* (uniformly strictly convex... Kasparov-Yu).

What is known about Baum-Connes conjecture? (3) Lafforgue's method

Baum-Connes' conjecture holds:

- Real or p-adic *reductive Lie groups.* Result previously proved in real case (Wassermann) and p-adic GL_n groups (Baum-Higson-Plymen);
- Chabert-Echterhoff-Nest include all *almost connected* locally-compact groups.
- *Word hyperbolic groups* (in sense of Gromov); this contains all discrete cocompact subgroups of Lie groups with real rank 1; (use Mineyev-Yu)
- For all discrete cocompact subgroups of $SL_3(\mathbb{K})$ - ($\mathbb{K} = \mathbb{R},\ \mathbb{C},\ \mathbb{Q}_p$ or quaternions) - and products of those (use Ramagge-Robertson-Steger, Chatterji).
- Some coefficients allowed.

What is known about Baum-Connes conjecture? (4)

Using Baum-Connes map together with tools of *cyclic cohomology,* Connes-Gromov-Moscovici prove that Novikov conjecture holds for any group Γ and a class in *"HyperEuclidean" cohomology* of Γ, *e.g.* :

- *"secondary" cohomology classes* - like Godbillon-Vey class
- Chern classes of *"almost flat bundles".*

Counterexamples

Higson-Lafforgue-Sk: Counterexamples to *almost all generalizations* of Baum-Connes conjecture. Idea: observation of Gromov putting together sequences of 'expanders', space which does not embed in a Hilbert space.

Example

Non Hausdorff groupoid $G = \Gamma \times [0,1]/\sim$, where Γ free group, $\Gamma \times [0,1]$ bundle of groups $(g,s) \sim (h,t) \iff s = t \neq 0$ or $s = t$ and $g = h$.
$C^*(G) = \{(x,f) \in C^*(\Gamma) \times C([0,1]);\ \varepsilon(x) = f(0)\}$, where ε trivial representation.
Since Γ is non amenable, $C^*_r(G) = C^*_r(\Gamma) \oplus C([0,1])$.
$K_0(C^*(G)) = \mathbb{Z}$ and $K_0(C^*_r(G)) = \mathbb{Z} \oplus \mathbb{Z}$; therefore, λ_* not onto, whence Baum-Connes map not onto.

Counterexamples (2)

Building out of this example, one may also give examples of

- (non Hausdorff) foliation groupoids for which Baum-Connes map is neither surjective nor injective,
- Hausdorff groupoids, not surjective Baum-Connes map.
- Hausdorff groupoids, not injective Baum-Connes map.
- Counterexample to surjectivity of coarse Baum-Connes (first counterexample - due to Higson).
- Gromov: group whose Cayley graph 'almost contains' sequence of expanders. Admits action on a separable abelian C^*-algebra for which Baum-Connes map with coefficients not onto. N. Osawa: similar construction $\rightarrow$ nonabelian algebra with trivial action of Γ Baum-Connes map with coefficients is not onto.
- In these counterexamples, one can prove that the nonreduced morphism μ is injective. *Not counterexamples to Novikov conjecture.*
- No known counterexamples to the Bost conjecture.

Kasparov's method: proper algebras

Definition

G locally compact group.

1. X locally compact space with continuous action of G.
   $G, C_0(X)$-algebra: G-algebra A with G-equivariant $*$-homomorphism $\pi : C_0(X) \to Z(M(A))$ (center of the multiplier algebra) such that $\pi(C_0(X))A = A$.
2. *Proper G-algebra* $G, C_0(\underline{E}G)$-algebra.

Theorem (Chabert-Echterhoff-Mayer)

Baum-Connes conjecture with coefficients on proper algebras holds.

Kasparov's method: the 'γ' element

Definition

G locally compact group. γ element: $\gamma \in KK_G(\mathbb{C}, \mathbb{C})$ such that:

1. There exists a proper G-algebra A and elements $\alpha \in KK_G(A, \mathbb{C})$ and $\beta \in KK_G(\mathbb{C}, A)$ such that $\gamma = \beta \otimes_A \alpha$.
2. For every proper G-space X, we have $p^*(\gamma) = 1 \in KK_{X \rtimes G}(\mathbb{C}, \mathbb{C})$, where $p : X \to \mathrm{pt}$.

Proposition (Tu)

If γ exists, it is unique and an idempotent of $KK_G(\mathbb{C}, \mathbb{C})$.

Theorem

1. *If G admits γ element, Baum-Connes map with coefficients injective.*
2. *If moreover $\gamma = 1$ Baum-Connes conjecture with coefficients holds.*

Actually, in case 2 both μ_G^B and $\mu_{G,r}^B$ are isomorphisms.

The obstruction of Kazhdan's property T

G has property T if trivial representation is isolated in its dual.
This means that $\exists p \in C^*(G)$ such that, $p^2 = p$, and $\pi(p)$ orthogonal projection on space of G-invariant vectors, for every representation π of G.
G be a locally compact non compact group with property T.

- $\varepsilon(p) = 1$, hence $[p] \neq 0 \in K_0(C^*(G)$ (ε trivial representation;
- $\lambda(p) = 0$ where $\lambda : C^*(G) \to C_r^*(G)$, hence λ_* not injective.

μ_G and $\mu_{G,r}$ cannot be both isomorphisms, hence we cannot have $\gamma = 1$.

Worse: G closed discrete subgroup with finite covolume in $Sp(n, 1)$, $n \geq 2$, then $j_{G,r}(\gamma_G) \neq 1_{C_r^*(G)}$. Therefore, if true, this conjecture cannot be proved only by means of KK-theory.

γ element for $\widetilde{A_2}$ buildings

Building: higher dimensional generalization of trees. Here two dimensional.

$\widetilde{A_2}$ building: pair (X, B) where X is a simplicial complex of dimension 2 and B geometric realization of X, endowed with metric.

Apartments: subcomplexes of X isometric to euclidean $\mathbb{R}^2$. Isomorphism: affine on every simplex, equilateral triangles.

1. Apartments: homogeneous
2. Every geodesic segment contained in an apartment.
3. Any pair of simplices of X contained in an apartment.
4. The intersection of two apartments is convex and there exists a simplicial isometry $g : S \to S'$ which is the identity on $S \cap S'$.

Julg-Valette γ for buildings

G acts properly, isometrically on $\widetilde{A_2}$ building X.

$X^{(i)}$ $(0 \leq i \leq 2)$ set of faces of dimension i in X.

$(e_x)_{x \in X^{(0)}}$ canonical Hilbert basis of $H_0 = \ell^2(X^{(0)})$.

$H_1 \subset \Lambda^2(H_0)$ vector span of $e_\sigma = e_x \wedge e_y$, $\sigma = (x, y) \in X^{(1)}$
$H_2 \subset \Lambda^3(H_0)$ vector span of $e_\sigma = e_x \wedge e_y \wedge e_z$, $\sigma = (x, y, z) \in X^{(2)}$.

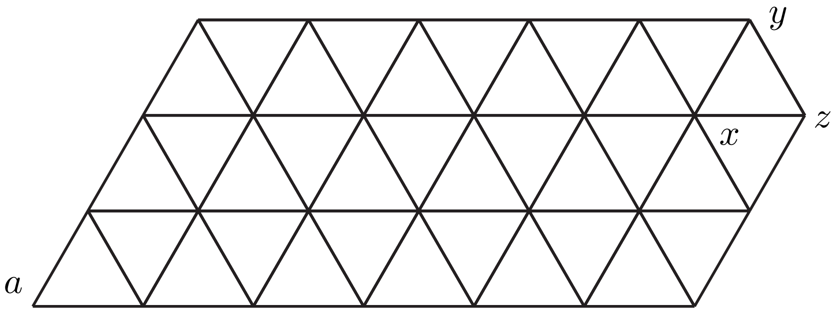
- $H = (H_0 \oplus H_2) \oplus H_1$.
- $F = F_a = T_a + T_a^*$ depends on an origin $a \in X^{(0)}$.

The operator T_a is an exterior product:
$T_a(e_\sigma) = v_{a,\sigma} \wedge e_\sigma$.

Construction of T_a: first case

Let $(x, y, z) \in X^{(2)}$.
Barycentric coordinates $a = \lambda x + \mu y + \nu z$ in any apartment containing $\{x_0, x, y, z\}$. We may assume that $\lambda \geq \mu \geq \nu$.

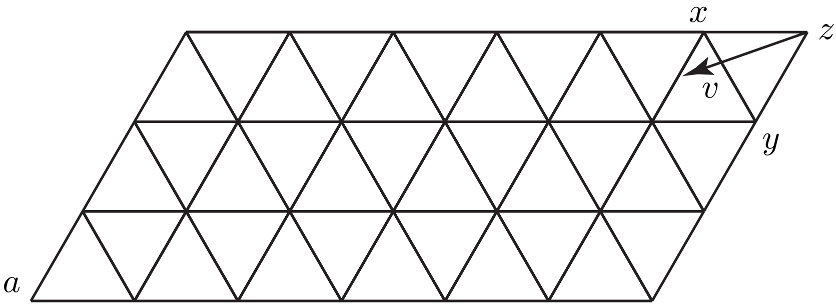
First case. $\lambda > 0 > \mu \geq \nu$.



Then $v = e_x$ and $T_a(e_y \wedge e_z) = e_x \wedge e_y \wedge e_z$.

Construction of T_a: second case

Assume that $\lambda \geq \mu \geq 0 \geq \nu$.



Set $\lambda' = (\lambda^2 + \mu^2)^{-1/2}\lambda$ and $\mu' = (\lambda^2 + \mu^2)^{-1/2}\mu$
Put $v = \lambda' e_x + \mu' e_y$.

- $T_a(e_z) = v \wedge e_z$,
- $T_a(e_y \wedge e_z) = v \wedge (e_y \wedge e_z) = \lambda'(e_x \wedge e_y \wedge e_z)$
- $T_a(e_x \wedge e_z) = v \wedge (e_x \wedge e_z) = -\mu'(e_x \wedge e_y \wedge e_z)$.

Julg-Valette γ for buildings

- Easy check $1 - F_a^2$ orthogonal projection on e_a.
- Action of G changes origin a.
- If $a, b \in X^{(0)}$, difference between the barycentric coordonnates of a and b bounded on the whole building.
- $F_a - F_b$ is compact

Consequence: (H, F) is an element of $KK_G(\mathbb{C}, \mathbb{C})$.

Theorem (Kasparov-Sk)

The Julg-Valette element is a γ element.

In the above construction, we used T_a which satisfied $T_a^2 = 0$ to construct KK-elements. This idea will be very useful in the homotopy.