Planar algebras

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Further: Isotopy invariance, \(P_0 = \mathbb{C}\), \(*\)-structure, positivity, spherical invariance.

Jones '83: There is \(\delta \in \{2 \cos(\pi/n) : n \geq 3\} \cup [2, \infty)\) s.t. \(Z_T' = \delta \cdot Z_T\).
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\[ T \circ_2 S = * \]

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* \\
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The Temperley-Lieb planar algebra

$TL_n(\delta)$ has basis given by planar tangles with $2n$ marked points on the outer disc and no input discs or interior loops (up to isotopy):

For positivity require $\delta \geq 2$. 

Stephen Curran (UCLA)
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The polynomial planar algebra

\[ P_k \subset \mathbb{C}\langle X_1, X_1^*, \ldots, X_n, X_n^* \rangle \]

\[ P_k = \text{span}\{ X_{i_1} X_{j_1}^* \cdots X_{i_k} X_{j_k}^* \} \]
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\[ Z_T(X_{i_1}X_{j_1}^*X_{i_2}X_{j_2}^*X_{i_3}X_{j_3}^* \otimes X_{k_1}X_{l_1}^*X_{k_2}X_{l_2}^*) = \]
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Planar algebra of modulus \( \delta = n \).
Subfactor $N \subset M$ with $[M : N] < \infty$,

$$P_{2k} = \text{Hom}_{N,N} L^2 M_k, \quad P_{2k+1} = \text{Hom}_{N,M} L^2(M_k).$$

where $N = M_0 \subset M_1 = M \subset M_2 \subset \cdots$ is the Jones tower.
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**Theorem (Jones ’99)**

\[ \mathcal{P} = (P_n)_{n \geq 0} \text{ has a natural planar algebra structure with } \delta^2 = [M : N]. \]
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**Theorem (Jones ’99)**

\( \mathcal{P} = (P_n)_{n \geq 0} \) has a natural planar algebra structure with \( \delta^2 = [M : N] \).

**Theorem (Popa ’95)**

Any planar algebra (\( \lambda \)-lattice) is isomorphic to the planar algebra of some finite-index subfactor \( N \subset M \).
Planar algebra subfactors

Let $\mathcal{P} = (P_n)_{n \geq 0}$ be a planar algebra.

Guionnet-Jones-Shlyakhtenko ’08,’09:

- Tower of graded algebras $\text{Gr}_0(\mathcal{P}) \subset \text{Gr}_1(\mathcal{P}) \subset \cdots$.
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- If $\mathcal{P}$ is finite-depth then $M_k \cong L(\mathbb{F}_{r_k})$,

$$r_k = 1 + 2I\delta^{-2k}(\delta - 1),$$

where $\delta^2 = [M_1 : M_0]$ and $I$ is the global index.

Kodiyalam-Sunder ’09: Independent proof that $M_k$ are interpolated free group factors in finite-depth case.
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- If $\mathcal{P}$ is finite-depth then $M_k \cong L(\mathbb{F}_{r_k})$,
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Graded algebras associated to planar algebras

Let $\mathcal{P} = (P_n)_{n \geq 0}$ be a planar algebra.

For $n, k \geq 0$ let $P_{n,k}$ be a copy of $P_{n+k}$, represented by:

![Diagram](attachment:diagram.png)
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- Multiplication: $\wedge_k : P_{n,k} \times P_{m,k} \to P_{n+m,k}$

\[ x \wedge_k y = \]

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\begin{array}{c}
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   \kappa \\
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- Voiculescu trace: $\tau_k : P_{n,k} \to P_0 \simeq \mathbb{C}$

  \[ \tau_k(x) = \delta^{-k}. \]

Loopless diagrams with $2n$ boundary points.
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- Graded algebras: $(Gr_k, \wedge_k, \tau_k),$

  $Gr_k = \bigoplus_{n \geq 0} P_{n,k}.$
Key example: Polynomial planar algebra

- \( A = \mathbb{C}\langle X_1, X_1^*, \ldots, X_n, X_n^* \rangle. \)

\[
P_m = \text{span}\{X_{i_1}X_{i_2}^* \cdots X_{i_{2m-1}}X_{i_{2m}}^*: 1 \leq i_1, \ldots, i_{2m} \leq n\}
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Planar tangles act by “contracting indices”.

\[\top\]
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Planar tangles act by “contracting indices”.

- \( \wedge_0 : P_m \times P_k \to P_{m+k} : \)

\[
\begin{array}{cccc}
X_{i_1} & X_{i_2}^* & \cdots & X_{i_{2m}}^* \\
X_{j_1} & X_{j_2}^* & \cdots & X_{j_{2k}}^*
\end{array}
\]

= \[
\begin{array}{cccc}
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Inclusion \( Gr_0 \hookrightarrow A \).
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\hline
\ldots & \ldots & \ldots & \ldots \\
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\end{array}
\]

\[
\begin{array}{ccc}
\ldots & \ldots & \ldots \\
\hline
X_{i_1} & X_{i_2}^* & \ldots & X_{i_{2m}}^* & X_{j_1}X_{j_2}^* & \ldots & X_{j_{2k}}^*
\end{array}
\]

Inclusion \( Gr_0 \hookrightarrow A. \)

- \( \tau_0 : \) Restriction to \( Gr_0 \) of the Voiculescu trace on \( A \), i.e. \( X_1, \ldots, X_n \) freely independent circular random variables. Large \( N \)-limit of random matrices \( X_1(N), \ldots, X_n(N) \) where

\[
\{X_l(N; i, j) : 1 \leq l \leq n, 1 \leq i, j \leq N\}
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are i.i.d. complex Gaussian random variables.
Key example: Polynomial planar algebra

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- \( P_m = \text{span}\{X_{i_1}X_{i_2}^* \cdots X_{i_{2m-1}}X_{i_{2m}}^*: 1 \leq i_1, \ldots, i_{2m} \leq n\} \)

Planar tangles act by “contracting indices”.

- \( \wedge_0: P_m \times P_k \to P_{m+k} \):

Inclusion \( G_{r_0} \hookrightarrow A \).

- \( \tau_0 \): Restriction to \( G_{r_0} \) of the Voiculescu trace on \( A \), i.e. \( X_1, \ldots, X_n \) freely independent circular random variables. Large \( N \)-limit of random matrices \( X_1(N), \ldots, X_n(N) \) where

\[ \{X_l(N; i, j): 1 \leq l \leq n, 1 \leq i, j \leq N\} \]

are i.i.d. complex Gaussian random variables.

- \( (G_{r_k}, \tau_k) \simeq (G_{r_0} \otimes M_n(\mathbb{C}) \otimes^k, \tau_0 \otimes \text{tr} \otimes^k) \).
Inclusions: $Gr_k \subset Gr_{k+1}$:

\[ x \rightarrow x \]

Theorem (Guionnet-Jones-Shlyakhtenko '08, Popa '95)
For $k \geq 0$ the Voiculescu trace $\tau_k$ is positive and faithful, and the GNS completion $M_k$ is a $\text{II}_1$ factor. Moreover, $M_0 \subset M_1 \subset M_2 \subset \cdots$ is the Jones tower for the subfactor $M_0 \subset M_1$, and its planar algebra is $P$. 
Inclusions: $Gr_k \subset Gr_{k+1}$:

Jones projections: $e_k \in Gr_{k+2}$,

$$e_k = \delta^{-1}.$$
Subfactors associated to planar algebras

- Inclusions: \( Gr_k \subset Gr_{k+1} \):

- Jones projections: \( e_k \in Gr_{k+2} \),

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Theorem (Guionnet-Jones-Shlyakhtenko ’08, Popa ’95)

For \( k \geq 0 \) the Voiculescu trace \( \tau_k \) is positive and faithful, and the GNS completion \( M_k \) is a \( II_1 \) factor. Moreover, \( M_0 \subset M_1 \subset M_2 \subset \cdots \) is the Jones tower for the subfactor \( M_0 \subset M_1 \), and its planar algebra is \( \mathcal{P} \).
Theorem (Popa ’94)

Let $N \subset M$ be an inclusion of II$_1$ factors with $[M : N] < \infty$. Then there is a (unique) II$_1$ factor $M \boxtimes_{e_N} M^{op}$ such that:

1. There is an anti-automorphism $x \mapsto x^{op}$ of $M \boxtimes_{e_N} M^{op}$ such that $(x^{op})^{op} = x$.
2. There is an inclusion $M \otimes M^{op} \hookrightarrow M \boxtimes_{e_N} M^{op}$ such that $(x \otimes y^{op})^{op} = y \otimes x^{op}$.
3. There is a projection $e_N \in M \boxtimes_{e_N} M^{op}$ such that $e^{op} = e_N$ and $e_N$ is the Jones projection for the inclusions $N \subset M$ and $N^{op} \subset M^{op}$.
4. $M \boxtimes_{e_N} M^{op}$ is generated by $M \otimes M^{op}$ and $e_N$.

$M$ amenable, $N \subset M$ finite-depth $\Rightarrow M \otimes M^{op} \subset M \boxtimes_{e_N} M^{op} \cong$ Ocneanu’s asymptotic inclusion. Related to Drinfeld double. Popa ’94, ’99: $M \otimes M^{op} \subset M \boxtimes_{e_N} M^{op}$ encodes a number of important analytic properties of $N \subset M$.
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Popa’s symmetric enveloping algebra

**Theorem (Popa ’94)**

Let $N \subset M$ be an inclusion of $II_1$ factors with $[M : N] < \infty$. Then there is a (unique) $II_1$ factor $M \boxtimes e_N M^{op}$ such that:

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Theorem (Popa ’94)

Let \( N \subseteq M \) be an inclusion of II\(_1\) factors with \([M : N] < \infty\). Then there is a (unique) II\(_1\) factor \( M \boxtimes_{e_N} M^{\text{op}} \) such that:

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1. There is an anti-automorphism $x \mapsto x^{\text{op}}$ of $M \boxtimes_{e_N} M^{\text{op}}$ such that $(x^{\text{op}})^{\text{op}} = x$.

2. There is an inclusion $M \otimes M^{\text{op}} \hookrightarrow M \boxtimes_{e_N} M^{\text{op}}$ such that $(x \otimes y^{\text{op}})^{\text{op}} = y \otimes x^{\text{op}}$.

3. There is a projection $e_N \in M \boxtimes_{e_N} M^{\text{op}}$ such that $e_N^{\text{op}} = e_N$, and $e_N$ is the Jones projection for the inclusions $N \subset M$ and $N^{\text{op}} \subset M^{\text{op}}$.

4. $M \boxtimes_{e_N} M$ is generated by $M \otimes M^{\text{op}}$ and $e_N$.

- $M$ amenable, $N \subset M$ finite-depth $\Rightarrow M \otimes M^{\text{op}} \subset M \boxtimes_{e_N} M^{\text{op}} \simeq$ Ocneanu’s asymptotic inclusion. Related to Drinfeld double.

- Popa ’94, ’99: $M \otimes M^{\text{op}} \subset M \boxtimes_{e_N} M^{\text{op}}$ encodes a number of important analytic properties of $N \subset M$. 

- Stephen Curran (UCLA)

- PA of asymptotic inclusion

- October 30, 2011
Symmetric enveloping graded algebra

For $k, s, t \geq 0$ let $V_{k,s,t}$ be a copy of $P_{2k+s+t}$, represented by:

![Diagram of tensor product]
For $k, s, t \geq 0$ let $V_{k,s,t}$ be a copy of $P_{2k+s+t}$, represented by:

$$
\begin{array}{c}
\times \\
2k \\
\ast \\
2s \\
\ast \\
2t \\
2k
\end{array}
$$

**Multiplication:** $\land_k : V_{k,s,t} \times V_{k,s',t'} \to V_{k,s+s',t+t'}$

$$
\begin{array}{c}
\ast \\
x
\ast \\
y
\end{array}
$$
Symmetric enveloping graded algebra

For $k, s, t \geq 0$ let $V_{k,s,t}$ be a copy of $P_{2k+s+t}$, represented by:

- **Multiplication:** $\wedge_k : V_{k,s,t} \times V_{k,s',t'} \to V_{k,s+s',t+t'}$
  
  $$x \wedge_k y = \begin{array}{c}
  x \\
  \wedge_k \\
  y
  \end{array}$$

- **Trace:** $\tau_k \boxtimes \tau_k : V_{k,s,t} \to P_0 \simeq \mathbb{C}$.
  
  $$\tau_k \boxtimes \tau_k(x) = \delta^{-2k}.$$
For $k, s, t \geq 0$ let $V_{k,s,t}$ be a copy of $P_{2k+s+t}$, represented by:

- **Multiplication**: $\wedge_k : V_{k,s,t} \times V_{k,s',t'} \to V_{k,s+s',t+t'}$

$$x \wedge_k y = *x*y$$

- **Trace**: $\tau_k \boxtimes \tau_k : V_{k,s,t} \to P_0 \cong \mathbb{C}$.

$$\tau_k \boxtimes \tau_k (x) = \delta^{-2k}.$$
Symmetric enveloping inclusion

- Anti-automorphism: $y \in V_{k,s,t} \mapsto y^{op} \in V_{k,t,s}$

\[ y^{op} = \lambda \]
Symmetric enveloping inclusion

- **Anti-automorphism:** \( y \in V_{k,s,t} \mapsto y^{op} \in V_{k,t,s} \)

- **Inclusion:** \( Gr_k \otimes Gr_k^{op} \hookrightarrow Gr_k \boxtimes Gr_k^{op} \), \( (x \otimes y^{op})^{op} = y \otimes x^{op} \).
Symmetric enveloping inclusion

- Anti-automorphism: $y \in V_{k,s,t} \mapsto y^{op} \in V_{k,t,s}$

- Inclusion: $Gr_k \otimes Gr_k^{op} \hookrightarrow Gr_k \boxtimes Gr_k^{op}$, $(x \otimes y^{op})^{op} = y \otimes x^{op}$.

- Jones projection: $e_{k-1} \in V_{k,0,0}$.

$$e_{k-1} = \delta^{-1}.$$
Symmetric enveloping algebra

Theorem (C.-Jones-Shlyakhtenko ’11)

For $k \geq 0$, $\tau_k \boxtimes \tau_k$ is a faithful, positive state on $Gr_k \boxtimes Gr^\text{op}_k$. Moreover, for $k \geq 1$ the GNS completion $M_k \boxtimes M^\text{op}_k$ is isomorphic to Popa’s symmetric enveloping algebra for $M_{k-1} \subset M_k$.
Theorem (C.-Jones-Shlyakhtenko ’11)

For \( k \geq 0 \), \( \tau_k \boxtimes \tau_k \) is a faithful, positive state on \( \text{Gr}_k \boxtimes \text{Gr}_k^{\text{op}} \). Moreover, for \( k \geq 1 \) the GNS completion \( M_k \boxtimes M_k^{\text{op}} \) is isomorphic to Popa’s symmetric enveloping algebra for \( M_{k-1} \subset M_k \).

- Proof of positivity uses diagrammatic orthogonalization procedure from Jones-Shlyakhtenko-Walker ’09.
Theorem (C.-Jones-Shlyakhtenko '11)

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- Proof of positivity uses diagrammatic orthogonalization procedure from Jones-Shlyakhtenko-Walker '09.
- Popa’s characterization of symmetric enveloping algebra implies

$$W^*(M_k \otimes M_k^{op}, e_{k-1}) \simeq M_k \boxtimes e_{k-1} M_k^{op}.$$
**Theorem (C.-Jones-Shlyakhtenko ’11)**

For $k \geq 0$, $\tau_k \boxtimes \tau_k$ is a faithful, positive state on $Gr_k \boxtimes Gr_k^{op}$. Moreover, for $k \geq 1$ the GNS completion $M_k \boxtimes M_k^{op}$ is isomorphic to Popa’s symmetric enveloping algebra for $M_{k-1} \subset M_k$.

- Proof of positivity uses diagrammatic orthogonalization procedure from Jones-Shlyakhtenko-Walker ’09.
- Popa’s characterization of symmetric enveloping algebra implies

$$W^*((M_k \otimes M_k^{op}, e_{k-1}) \simeq M_k \boxtimes_{e_{k-1}} M_k^{op}.$$

- Main difficulty in proof is showing:

$$M_k \boxtimes M_k^{op} = W^*(M_k \otimes M_k^{op}, e_{k-1}).$$
Derivations: $\delta_Q : Gr_0 \to Gr_0 \otimes Gr_0^{op}$,

$$\delta_Q(x) = \sum_{0 \leq k < 2n} \sum_{k+t \text{ even}} E_{M_0 \otimes M_0^{op}}$$

Related to free difference quotient, Schwinger-Dyson equation.
Application to derivations

- Derivations: $\delta_Q : Gr_0 \to Gr_0 \otimes Gr_0^{op}$,

$$\delta_Q(x) = \sum_{0 \leq k < 2n \atop k + t \text{ even}} E_{M_0 \otimes M_0^{op}} \{\delta_Q\}$$

Related to free difference quotient, Schwinger-Dyson equation.

- Murray von Neumann dimension of space of such derivations:

$$\dim_{M_0 \otimes M_0^{op}} \{\delta_Q\} \sim 1 + 2I(\delta - 1),$$

where $I$ is the global index.
Derivations: $\delta_Q : Gr_0 \to Gr_0 \otimes Gr_0^{op}$,

$$\delta_Q(x) = \sum_{0 \leq k < 2n \atop k+t \text{ even}} E_{M_0 \otimes M_0^{op}} \left[ \begin{array}{cc} x & k \\ Q & t \end{array} \right]$$

Related to free difference quotient, Schwinger-Dyson equation.

Murray von Neumann dimension of space of such derivations:

$$\dim_{M_0 \otimes M_0^{op}} \{\delta_Q\} \sim 1 + 2I(\delta - 1),$$

where $I$ is the *global index*.

Dimensions of these spaces of derivations are related to free entropy dimension, so this provides some intuition for the formula

$$M_0 \simeq L^F 1 + 2I(\delta - 1).$$
This description can be used to compute a number of invariants of the asymptotic inclusion (for finite depth $\mathcal{P}$).
This description can be used to compute a number of invariants of the asymptotic inclusion (for finite depth $\mathcal{P}$).

Previous computations of fusion rules:

- Ocneanu and Evans-Kawahigashi in the language of TQFT.
- Longo-Rehren, Izumi in the language of sectors (type III setting).
- Müger in the language of tensor categories.
This description can be used to compute a number of invariants of the asymptotic inclusion (for finite depth $\mathcal{P}$).

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In this talk we’ll describe the planar algebra of the asymptotic inclusion in terms of the original planar algebra $\mathcal{P}$.
This description can be used to compute a number of invariants of the asymptotic inclusion (for finite depth $\mathcal{P}$).

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- Ocneanu and Evans-Kawahigashi in the language of TQFT.
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In this talk we’ll describe the planar algebra of the asymptotic inclusion in terms of the original planar algebra $\mathcal{P}$.

We will then try to explain the following theorem.

**Theorem (Ocneanu, Evans-Kawahigashi ’94)**

The (reduced) fusion algebra of $M_0 \boxtimes M_0^{op}$-bimodules is commutative.
Fix $2k \geq \text{depth} (\mathcal{P})$. We have

$$P_{2k} \simeq \bigoplus_{v \in \Gamma_+} M_{n_v} (\mathbb{C}),$$

$\Gamma = \text{principal graph of } \mathcal{P}$, Perron-Frobenius eigenvector $\mu$. 
Fix $2k \geq \text{depth}(\mathcal{P})$. We have

$$P_{2k} \cong \bigoplus_{v \in \Gamma_+} M_{n_v}(\mathbb{C}),$$

where $\Gamma = \text{principal graph of } \mathcal{P}$, Perron-Frobenius eigenvector $\mu$. Define

$$p = \sum_{v \in \Gamma_+} n_v^{-1} \sum_{1 \leq i, j \leq n_v} e_{ij}(v) \otimes e_{ji}(v)^{op} \in P_{2k} \otimes P_{2k}^{op}.$$
Fix $2k \geq \text{depth}(\mathcal{P})$. We have

$$P_{2k} \simeq \bigoplus_{v \in \Gamma_+} M_{n_v}(\mathbb{C}),$$

where $\Gamma = \text{principal graph}$ of $\mathcal{P}$, Perron-Frobenius eigenvector $\mu$. Define

$$p = \sum_{v \in \Gamma_+} n_v^{-1} \sum_{1 \leq i,j \leq n_v} e_{ij}(v) \otimes e_{ji}(v)^{op} \in P_{2k} \otimes P_{2k}^{op}.$$ 

Have $\text{tr}(p) = \delta^{-2k} \cdot I$, where $I = \sum_{v \in \Gamma_+} \mu(v)^2$ is the global index.
Key relation: \((\text{Sweedler-notation } p = p^{(1)} \otimes p^{(2)})\)
“Splitting” the symmetric enveloping algebra

Key relation: \( p = p^{(1)} \otimes p^{(2)} \)

\[
\sum_{v \in \Gamma_+} \frac{\mu_v}{n_v}.
\]

Follows from

\[
\text{Tr}(xy) = \sum_{1 \leq i, j \leq n} x_{ji}y_{ij} = \sum_{1 \leq i, j \leq n} \text{Tr}(e_{ij} \cdot x)\text{Tr}(y \cdot e_{ji}).
\]
“Splitting” the symmetric enveloping algebra

Natural $M_0 \otimes M_0^{\text{op}}$-bimodule map $\Psi : M_k \otimes M_k^{\text{op}} \to M_0 \boxtimes M_0$. 

Theorem (C.-Jones-Shlyakhtenko '11, Popa '94)

$\Psi$ extends to an isomorphism of $\mathbb{L}^2(M_k \otimes M_k^{\text{op}}) \to \mathbb{L}^2(M_0 \boxtimes M_0)$. 

Corollary (Ocneanu, Popa '94)

As a $M_0 \otimes M_0^{\text{op}}$-bimodule, we have $\mathbb{L}^2(M_0 \boxtimes M_0) \cong \bigoplus_{v \in \Gamma^+} X_v \otimes X_v$. 

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“Splitting” the symmetric enveloping algebra

Natural $M_0 \otimes M_0^{op}$-bimodule map $\Psi : M_k \otimes M_k^{op} \to M_0 \boxtimes M_0$.

\[
\Psi(x, y) \mapsto I^{-1/2} \sum_{v \in \Gamma_+} \left( \frac{\mu_v}{n_v} \right)^{1/2}.
\]
“Splitting” the symmetric enveloping algebra

Natural $M_0 \otimes M_0^{op}$-bimodule map $\Psi : M_k \otimes M_k^{op} \to M_0 \boxtimes M_0$.

$\langle \Psi(x_1 \otimes y_1), \Psi(x_2 \otimes y_2) \rangle = I^{-1} \sum_{v, w \in \Gamma_+} \sqrt{\frac{\mu_v \mu_w}{n_v n_w}}$.
“Splitting” the symmetric enveloping algebra

Natural $M_0 \otimes M_0^{op}$-bimodule map $\Psi : M_k \otimes M_k^{op} \to M_0 \boxtimes M_0$.

\[ \langle \Psi(x_1 \otimes y_1), \Psi(x_2 \otimes y_2) \rangle = l^{-1} \sum_{v \in \Gamma_+} \frac{\mu_v}{n_v} \sum TL \]

\[ \rightarrow l^{-1/2} \sum_{v \in \Gamma_+} \left( \frac{\mu_v}{n_v} \right)^{1/2} \]
“Splitting” the symmetric enveloping algebra

Natural $M_0 \otimes M_0^{op}$-bimodule map $\Psi : M_k \otimes M_k^{op} \rightarrow M_0 \boxtimes M_0$.

\[ \langle \Psi(x_1 \otimes y_1), \Psi(x_2 \otimes y_2) \rangle = l^{-1} \langle p(x_1 \otimes y_1), x_2 \otimes y_2 \rangle. \]
“Splitting” the symmetric enveloping algebra

Natural $M_0 \otimes M_0^{op}$-bimodule map $\Psi : M_k \otimes M_k^{op} \to M_0 \boxtimes M_0$.

$$\Psi : M_k \otimes M_k^{op} \to M_0 \boxtimes M_0.$$  

Theorem (C.-Jones-Shlyakhtenko ’11, Popa ’94)

$\Psi$ extends to an isomorphism of $p \cdot L^2(M_k \otimes M_k^{op}) \to L^2(M_0 \boxtimes M_0^{op})$. 

Corollary (Ocneanu, Popa ’94)

As a $M_0 \otimes M_0^{op}$-bimodule, we have

$$L^2(M_0 \boxtimes M_0^{op}) \cong \bigoplus_{v \in \Gamma_+} X_v \otimes X_v.$$
“Splitting” the symmetric enveloping algebra

Natural \( M_0 \otimes M_0^{op} \)-bimodule map \( \Psi : M_k \otimes M_k^{op} \to M_0 \boxtimes M_0 \).

\[
\Psi : M_k \otimes M_k^{op} \to M_0 \boxtimes M_0.
\]

Theorem (C.-Jones-Shlyakhtenko ’11, Popa ’94)

\( \Psi \) extends to an isomorphism of \( p \cdot L^2(M_k \otimes M_k^{op}) \to L^2(M_0 \boxtimes M_0^{op}) \).

Corollary (Ocneanu, Popa ’94)

As a \( M_0 \otimes M_0^{op} \)-bimodule, we have

\[
L^2(M_0 \boxtimes M_0^{op}) \simeq \bigoplus_{v \in \Gamma_+} X_v \otimes \overline{X_v}
\]
“Splitting” the symmetric enveloping algebra

We want to “lift” the multiplication on $Gr_0 \boxtimes Gr_0^{op}$ back to $p(Gr_k \otimes Gr_k)^{op}$.

Define $q = q(1) \otimes q(2) \in P_3 k \otimes P_{op} 3 k$ by the relation:

$q(2) q(1) y x y x v w z v w z = I - \frac{1}{2} \sum v, w, z \in \Gamma_+(\mu v \mu w \mu z n v n w n z) \frac{1}{2} \cdot x_2 x_1 y_2 y_1 z w = I - \frac{1}{2} \sum w, z \in \Gamma_+(\mu w \mu z n w n z) \frac{1}{2} \cdot x_1 x_2 y_1 y_2 z w$
We want to “lift” the multiplication on $Gr_0 \boxtimes Gr_0^{op}$ back to $p(Gr_k \otimes Gr_k)^{op}$. Define $q = q^{(1)} \otimes q^{(2)} \in P_{3k} \otimes P_{3k}^{op}$ by the relation:

$$q = q^{(1)} \otimes q^{(2)} \in P_{3k} \otimes P_{3k}^{op}$$

$$= l^{-1/2} \sum_{v, w, z \in \Gamma_+} \left( \frac{\mu_v \mu_w \mu_z}{n_v n_w n_z} \right)^{1/2} \cdot \begin{array}{c}
\text{(diagram)}
\end{array}$$
“Splitting” the symmetric enveloping algebra

We want to “lift” the multiplication on $\text{Gr}_0 \boxtimes \text{Gr}_0^{\text{op}}$ back to $p(\text{Gr}_k \otimes \text{Gr}_k)^{\text{op}}$. Define $q = q^{(1)} \otimes q^{(2)} \in P_{3k} \otimes P_{3k}^{\text{op}}$ by the relation:

$$q^{(2)} q^{(1)} = l^{-1/2} \sum_{v,w,z \in \Gamma_+} \left( \frac{\mu_v \mu_w \mu_z}{n_v n_w n_z} \right)^{1/2} \cdot$$

If $2k \geq \text{depth}(P)$ then we have:

$$l^{-1/2} \sum_{v \in \Gamma_+} \left( \frac{\mu_v}{n_v} \right)^{1/2} \cdot \text{Diagram} = l^{-1} \sum_{w,z \in \Gamma_+} \left( \frac{\mu_w \mu_z}{n_w n_z} \right)^{1/2} \cdot \text{Diagram}$$
“Splitting” the symmetric enveloping algebra

We want to “lift” the multiplication on $Gr_0 \boxtimes Gr_0^{op}$ back to $p(Gr_k \otimes Gr_k)^{op}$.

Define $q = q^{(1)} \otimes q^{(2)} \in P_{3k} \otimes P_{3k}^{op}$ by the relation:

$$I^{-1/2} \sum_{v,w,z \in \Gamma_+} \left( \frac{\mu_v \mu_w \mu_z}{n_v n_w n_z} \right)^{1/2} \cdot$$

If $2k \geq \text{depth}(P)$ then we have:

$$I^{-1/2} \sum_{v \in \Gamma_+} \left( \frac{\mu_v}{n_v} \right)^{1/2} \cdot$$
Skein relations for $q$

$$\begin{align*}
q^{(1)} & \ast p^{(1)} & = & \ast q^{(1)} \\
q^{(2)} & \ast p^{(2)} & = & \ast q^{(2)}
\end{align*}$$

and

$$\begin{align*}
p^{(1)} & \ast q^{(1)} & = & \ast q^{(1)} \\
p^{(2)} & \ast q^{(2)} & = & \ast q^{(2)}
\end{align*}$$
Skein relations for $q$

\[
\begin{align*}
q^{(1)} & \quad p^{(1)} \\
q^{(2)} & \quad p^{(2)} \\
\end{align*}
\]

\[=\]

\[
\begin{align*}
q^{(1)} & \quad \text{and} \quad p^{(1)} \\
q^{(2)} & \quad q^{(2)} \\
\end{align*}
\]

\[=
\]

\[
\begin{align*}
q^{(2)} & \\
p^{(2)} & \quad q^{(2)} \\
\end{align*}
\]

\[=\]

\[
\begin{align*}
q^{(1)} & \\
q^{(2)} & \\
\end{align*}
\]
The basic construction

\[ \Psi(x \otimes y) = I^{-1/2} \sum_{v \in \Gamma_+} \left( \frac{\mu_v}{n_v} \right)^{1/2}. \]
The basic construction

\[ \Psi(x \otimes y) = I^{-1/2} \sum_{v \in \Gamma_+} \left( \frac{\mu_v}{n_v} \right)^{1/2}. \]

Inclusion \( M_0 \boxtimes M_0^{op} \hookrightarrow p(M_{2k} \otimes M_{2k}^{op})p, \)

\[ \Psi(x \otimes y) \mapsto \]

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The basic construction

\[ \psi(x \otimes y) = I^{-1/2} \sum_{v \in \Gamma_+} \left( \frac{\mu_v}{n_v} \right)^{1/2}. \]

Inclusion \( M_0 \boxtimes M_0^{op} \hookrightarrow p(M_{2k} \otimes M_{2k}^{op})p \), Jones projection \( e_0 \),

\[ e_0 = \frac{1^{1/2} n_*}{\delta^{2k}}. \]
The Jones tower

Define \( p_n = p_n^{(1)} \otimes p_n^{(2)} \in P_{2nk} \otimes P_{2nk}^{\text{op}} \) by:

\[
p_n = \begin{array}{c}
\begin{array}{c}
\ast \\
\ast
\end{array}
\end{array}
\begin{array}{c}
p_n^{(1)}\begin{array}{c}
\ast
\end{array}
\end{array}
\begin{array}{c}
p_{n-1}^{(1)}\begin{array}{c}
\ast
\end{array}
\end{array}
\begin{array}{c}
p_n^{(2)}\begin{array}{c}
\ast
\end{array}
\end{array}
\begin{array}{c}
p_{n-1}^{(2)}\begin{array}{c}
\ast
\end{array}
\end{array}
\end{array}
\]

where \( p_0 \) is the empty diagram, i.e. \( p_n \sim p^{\otimes n} \).
The Jones tower

Define $p_n = p_n^{(1)} \otimes p_n^{(2)} \in P_{2nk} \otimes P_{2nk}^{\text{op}}$ by:

\[
p_n = p_n^{(1)} \otimes p_n^{(2)}
\]

where $p_0$ is the empty diagram, i.e. $p_n \sim p^\otimes n$. Define

\[
\mathcal{M}_{2n} = p_n(M_{2nk} \otimes M_{2nk}^{\text{op}})p_n, \quad \mathcal{M}_{2n+1} = p_n(M_{2nk} \boxtimes M_{2nk}^{\text{op}})p_n.
\]

In particular, $\mathcal{M}_0 = M_0 \otimes M_0^{\text{op}}$, $\mathcal{M}_1 = M_0 \boxtimes M_0^{\text{op}}$. 
The Jones tower

Define $p_n = p_n^{(1)} \otimes p_n^{(2)} \in P_{2nk} \otimes P_{2nk}^{op}$ by:

\[
p_n = \begin{array}{c}
p_n^{(1)} \bigstar p_{n-1}^{(1)} \\
p_n^{(2)} \bigstar p_{n-1}^{(2)}
\end{array}
\]

where $p_0$ is the empty diagram, i.e. $p_n \sim p^\otimes n$. Define

\[
\mathcal{M}_{2n} = p_n(M_{2nk} \otimes M_{2nk}^{op})p_n, \quad \mathcal{M}_{2n+1} = p_n(M_{2nk} \boxtimes M_{2nk}^{op})p_n.
\]

In particular, $\mathcal{M}_0 = M_0 \otimes M_0^{op}$, $\mathcal{M}_1 = M_0 \boxtimes M_0^{op}$.

Theorem (C. ’11)

$(\mathcal{M}_n)_{n \geq 0}$ is the Jones tower for the subfactor $M_0 \otimes M_0^{op} \subset M_0 \boxtimes M_0^{op}$.
The planar algebra

The PA of $M_0 \otimes M_0^{op} \subset M_k \otimes M_k^{op}$ is the $k$-cabling $C_k(P \otimes P^{rev})$, the $n$-th box space is $P_{nk} \otimes P_{nk}^{op}$. 

Skein relations for $p$ and $q$:

1. $p p = q q$
2. $q q p q = 3$
The planar algebra

The planar algebra of $M_0 \otimes M_0^{op} \subset M_k \otimes M_k^{op}$ is the $k$-cabling $\mathcal{C}_k(\mathcal{P} \otimes \mathcal{P}^{rev})$, the $n$-th box space is $P_{nk} \otimes P_{nk}^{op}$.

Skein relations for $p$ and $q$:

1. $p^* \ast p = p$ and $q^* \ast q = q^*$

2. $q \ast q = p$

3. $q \ast q = q \ast q$
The planar algebra

Let $\text{Sym}(\mathcal{P})_n \subset C_k(\mathcal{P} \otimes \mathcal{P}^{\text{rev}})_n$ be the range of:

\[
\begin{align*}
\mathcal{F}_n = & \quad * \quad p \\ \mathcal{C}_n = & \quad * \quad q
\end{align*}
\]

If $T$ has $2^n$ marked points on the outer disc, and $2^n_i$ marked points on the $i$-th input disc for $1 \leq i \leq m$, define $Z_{\text{Sym}}(T)(x) = \sigma(T) \cdot Z_{\mathcal{C}_n}(Z_{\tilde{T}}(Z_{\mathcal{F}_n}^1 \otimes \cdots \otimes Z_{\mathcal{F}_n}^m))$.

where $\tilde{T}$ is given by doubling the strings of $T$, and $\sigma(T)$ is a "spin" factor.
Let $\text{Sym}(\mathcal{P})_n \subset C_k(\mathcal{P} \otimes \mathcal{P}^{\text{rev}})_n$ be the range of:

\begin{align*}
F_n &= q \cdots q \\
C_n &= q \cdots q
\end{align*}

Define tangles $F_n, C_n$ by

\begin{align*}
F_n &= q \cdots q \\
C_n &= q \cdots q
\end{align*}
The planar algebra

Let $\text{Sym}(\mathcal{P})_n \subset C_k(\mathcal{P} \otimes \mathcal{P}^{\text{rev}})_n$ be the range of:

Define tangles $F_n$, $C_n$ by

If $T$ has $2n$ marked points on the outer disc, and $2n_i$ marked points on the $i$-th input disc for $1 \leq i \leq m$, define

$$Z_T^{\text{Sym}}(x) = \sigma(T) \cdot Z_{C_n}(Z_{\tilde{T}}(Z_{F_{n_1}} \otimes \cdots \otimes Z_{F_{n_m}}(x))).$$

where $\tilde{T}$ is given by doubling the strings of $T$, and $\sigma(T)$ is a “spin” factor.
Identity: For $n$ even,

\[
\begin{array}{c}
\text{Identity: For } n \text{ even,} \\
\begin{array}{c}
\ast \\
\hat{n} \\
\end{array} = & \begin{array}{c}
q \\
\vdots \\
q \\
\end{array} = \begin{array}{c}
p \\
\vdots \\
p \\
\end{array}
\end{array}
\]
The planar algebra: examples

1. Identity: For $n$ even,

\[ \ast \quad \begin{array}{c} n \end{array} = \begin{array}{c} q \quad q \quad \cdots \quad q \end{array} = \begin{array}{c} p \quad \cdots \quad p \end{array} \]

2. Inclusion: $x \in \text{Sym}(\mathcal{P})_n$, $n$ odd,

\[ \ast \quad \begin{array}{c} x \end{array} = \begin{array}{c} p \quad p \quad \cdots \quad p \end{array} = \begin{array}{c} x \end{array} \]
1. Identity: For $n$ even,

\[ n^* = q \cdot q \cdot \ldots = p \cdot \ldots \]

2. Inclusion: $x \in \text{Sym}(\mathcal{P})_n$, $n$ odd,

\[ x^* = p \cdots p = q \]

3. Jones projections: For $n$ even,

\[ n^* = l^{1/2} \cdot p \cdot \ldots = l^{1/2} \cdot e_n \]
The planar algebra

Theorem (C. ’11)

\[(\text{Sym}(\mathcal{P})_n)_{n \geq 0} \text{ is the planar algebra of the asymptotic inclusion}\]
\[M_0 \otimes M_0^{\text{op}} \subset M_0 \boxdot M_0^{\text{op}}.\]
Theorem (C. ’11)

\((\text{Sym}(\mathcal{P})_n)_{n \geq 0}\) is the planar algebra of the asymptotic inclusion 
\(M_0 \otimes M_0^{\text{op}} \subset M_0 \boxtimes M_0^{\text{op}}\).

Idea of proof: After showing \(\text{Sym}(\mathcal{P})\) is a spherical C*-planar algebra, just need to check generating tangles.

To show \(\text{Sym}(\mathcal{P})\) is a planar algebra, we need the skein relation:

\[
\text{\begin{array}{c}
\text{\begin{array}{c}
\ast \\
\begin{array}{c}
\begin{array}{c}
\cdots \\
\begin{array}{c}
\begin{array}{c}
q \\
q \\
q \\
\…”
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\text{\begin{array}{c}
\text{\begin{array}{c}
\ast \\
\begin{array}{c}
\begin{array}{c}
\cdots \\
\begin{array}{c}
\begin{array}{c}
q \\
q \\
q \\
\…”
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}} =
\text{\begin{array}{c}
\text{\begin{array}{c}
\begin{array}{c}
\cdots \\
\begin{array}{c}
\begin{array}{c}
q \\
q \\
q \\
\…”
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}}
\end{array}
\end{array}
\
\text{\begin{array}{c}
\text{\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\cdots \\
\begin{array}{c}
\begin{array}{c}
q \\
q \\
q \\
\…”
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}}
\end{array}}
\]

where there are \(n q\)’s appearing on the left hand side of the equation, and \(n - 2\) on the right hand side \((n \geq 3)\).
Compatibility with gluing:

\[ T = \quad \rightarrow \quad \text{Sym}(T) = \]

\[ \quad \Rightarrow \quad \text{Sym}(T) = \quad \]

\[ \quad \Rightarrow \quad \text{Sym}(T) = \quad \]
Hom_{M_1, M_1}(L^2(M_n)) \simeq M'_1 \cap M_{2n+1} is isomorphic to:

\begin{align*}
q_1^{(1)} & \quad x \quad q_2^{(1)} \\
q_2^{(2)} & \quad y \quad q_1^{(2)}
\end{align*}

for x, y \in P_{2nk+2k}.
\( \text{Hom}_{\mathcal{M}_1, \mathcal{M}_1}(L^2(\mathcal{M}_n)) \cong \mathcal{M}'_1 \cap \mathcal{M}_{2n+1} \) is isomorphic to:

\[
\begin{align*}
q_1^{(1)} & \quad \quad X \quad \quad q_2^{(1)} \\
q_2^{(2)} & \quad \quad q_2^{(2)} \\
y & \quad \quad y \\
q_1^{(2)} & \quad \quad q_1^{(2)}
\end{align*}
\]

\( \cong \)

for \( x, y \in P_{2nk+2k} \).
\[
\text{Hom}_{\mathcal{M}_1, \mathcal{M}_1}(L^2(\mathcal{M}_n)) \cong \mathcal{M}_1' \cap \mathcal{M}_{2n+1} \text{ is isomorphic to:}
\]

\[
q^{(1)}_1 x q^{(2)}_1 q^{(1)}_2 y q^{(2)}_2
\]

\[
\cong
\]

\[
q^{(1)}_1 x q^{(2)}_1 y
\]

\[
\cong
\]

\[
q^{(1)}_1 q^{(2)}_1 y
\]

for \(x, y \in P_{2nk+2k}\).
Affine tangles

$\mathcal{P}$-labelled tangle $T$: all input discs labelled by elements of $\mathcal{P}$ ($\Rightarrow Z_T \in \mathcal{P}$).
**Affine tangles**

$\mathcal{P}$-labelled tangle $T$: all input discs labelled by elements of $\mathcal{P}$ ($\Rightarrow Z_T \in \mathcal{P}$).

- $\mathcal{P}$-labelled affine $n$-tangles:
  - 2$n$ marked points on output disc ($|z| = 2$).
  - Preferred input disc ($|z| = 1$) with 2$n$ marked pts. on boundary.
  - All other input discs labelled by elements of $\mathcal{P}$.
  - Up to isotopies preserving boundaries of outer and preferred discs.
Affine tangles

\( \mathcal{P} \)-labelled tangle \( T \): all input discs labelled by elements of \( \mathcal{P} \) (\( \Rightarrow Z_T \in \mathcal{P} \)).

\( \mathcal{P} \)-labelled affine \( n \)-tangles:

- 2\( n \) marked points on output disc (\(|z| = 2\)).
- Preferred input disc (\(|z| = 1\)) with 2\( n \) marked pts. on boundary.
- All other input discs labelled by elements of \( \mathcal{P} \).
- Up to isotopies preserving boundaries of outer and preferred discs.

Every affine \( n \)-tangles is of the form

\[
\Psi_{n,m}(T) = \quad * \quad \text{for some } \mathcal{P} \text{-labelled } n + m \text{-tangle } T.
\]
Affine tangles: Multiplication

\[ \Psi_{n,m}(T) = \]

Multiplication:
Affine tangles: Multiplication

\[ \Psi_{n,m}(T) = \]

Multiplication:

\[ \Psi_{n,m}(T_1) \cdot \Psi_{n,m'}(T_2) = \]
Affine tangles: Affine category

\[ \Psi_{n,m}(T) = \]

\[ \mathcal{P}\text{-labelled } n + m \text{ tangle } T \Rightarrow Z_T \in \mathcal{P}_{n+m}. \]
Affine tangles: Affine category

$$\Psi_{n,m}(T) =$$

$\mathcal{P}$-labelled $n + m$ tangle $T \Rightarrow Z_T \in P_{n+m}$.

**Definition**

- $F\text{Aff}(\mathcal{P})_n$: basis given by isotopy classes of $\mathcal{P}$-labelled affine $n$-tangles.
- $\text{Aff}(\mathcal{P})_n$: quotient of $F\text{Aff}(\mathcal{P})_n$ by $\bigcup_{m \geq 0} \{\Psi_{n,m}(T) : Z_T = 0\}$.
Affine tangles: Affine category

\[ \Psi_{n,m}(T) = \]

\[ \mathcal{P}\text{-labelled } n + m \text{ tangle } T \Rightarrow Z_T \in P_{n+m}. \]

**Definition**

- \( F_{\text{Aff}}(\mathcal{P})_n \): basis given by isotopy classes of \( \mathcal{P} \)-labelled affine \( n \)-tangles.
- \( \text{Aff}(\mathcal{P})_n \): quotient of \( F_{\text{Aff}}(\mathcal{P})_n \) by \( \bigcup_{m \geq 0} \{ \Psi_{n,m}(T) : Z_T = 0 \} \).

Note \( \Psi_{n,m} : P_{n+m} \to \text{Aff}(\mathcal{P})_n \) is well-defined, surjective if \( m \geq \text{depth}(\mathcal{P}) \).
“Splitting” affine tangles

For $x \in P_{n+2k}$ define $x_p$ by
“Splitting” affine tangles

For \( x \in P_{n+2k} \) define \( x_p \) by

\[
\begin{array}{c}
p^{(1)} \\
\ast \\
x \\
p^{(2)}
\end{array}
\]

Note that \( \Psi_{n,2k}(x_p) = \Psi_{n,2k}(x) \).
For \( x \in P_{n+2k} \) define \( x_p \) by

\[
\begin{align*}
\xrightarrow{p(1)} x \xrightarrow{p(2)} p(1) & \\
p(2) & \\
\end{align*}
\]

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For $x \in P_{n+2k}$ define $x_p$ by

Note that $\Psi_{n,2k}(x_p) = \Psi_{n,2k}(x)$.

Proposition

The restriction of $\Psi_{n,2k}$ to $\{x_p : x \in P_{n+2k}\}$ is a bijection onto $\text{Aff}(P)_n$. 
For $x \in P_{n+2k}$ define $x_p$ by

Note that $\Psi_{n,2k}(x_p) = \Psi_{n,2k}(x)$.

**Proposition**

The restriction of $\Psi_{n,2k}$ to \{ $x_p : x \in P_{n+2k}$ \} is a bijection onto $\text{Aff}(\mathcal{P})_n$.

“Lift” of multiplication on $\text{Aff}(\mathcal{P})$ can be expressed using $q$. 
The affine category and the asymptotic inclusion

Theorem (C. ’11)

There is an isomorphism of $\text{Hom}_{\mathcal{M}_1, \mathcal{M}_1}(L^2(\mathcal{M}_n))$ with the compression

$$p_n(\text{Aff}(\mathcal{P})_{4nk})p_n.$$
The affine category and the asymptotic inclusion

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Corollary (Ocneanu, Evans-Kawahigashi ’94)

The irreducible $\mathcal{M}_1, \mathcal{M}_1$-bimodules appearing in the Jones tower are indexed by central projections in $\text{Aff}(\mathcal{P})_{2k}$. 

The affine category and the asymptotic inclusion

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- Their result was in terms of Ocneanu’s (reduced) tube algebra - this can be identified with a subalgebra of \( \text{Aff}(\mathcal{P})_{2k} \) which intersects all central components, so our results are equivalent.
The affine category and the asymptotic inclusion

**Theorem (C. ’11)**

*There is an isomorphism of* $\text{Hom}_{\mathcal{M}_1,\mathcal{M}_1}(L^2(\mathcal{M}_n))$ *with the compression*

$$p_n(\text{Aff}(\mathcal{P})_{4nk})p_n.$$  

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*The irreducible* $\mathcal{M}_1,\mathcal{M}_1$*-bimodules appearing in the Jones tower are indexed by central projections in* $\text{Aff}(\mathcal{P})_{2k}$.

- Their result was in terms of Ocneanu’s (reduced) *tube algebra* - this can be identified with a subalgebra of $\text{Aff}(\mathcal{P})_{2k}$ which intersects all central components, so our results are equivalent.
- Ghosh ’05: For group planar algebra, representations of affine category $\sim$ representations of quantum double.
The half-twist

Rotation $v \in Aff_{4nk}$:

$$v_n = \star$$

Note that $v_n$ exchanges $x_1$ and $x_2$. 

Stephen Curran (UCLA)
PA of asymptotic inclusion
October 30, 2011 35 / 37
The half-twist

Rotation $v \in \text{Aff}_{4nk}$:

$$v_n = \begin{array}{c}
\ast \\
\ast
\end{array}$$

Every element of $\text{Aff}_{4nk}$ is of the form:

$$x = \begin{array}{c}
\ast \\
\ast \\
\ast \hspace{1cm} \ast
\end{array}$$

Note that $v_n^\ast n x v_n$ exchanges $x_1$ and $x_2$. 
The half-twist

Rotation $v \in \text{Aff}_{4nk}$:

$\nu_n =$

Every element of $\text{Aff}_{4nk}$ is of the form:

$x =$

Note that $\nu_n^* x \nu_n$ exchanges $x_1$ and $x_2$. 
The half-twist

Rotation $\nu \in \text{Aff}_{4nk}$:

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Rotation $v \in \text{Aff}_{4nk}$:

Every element of $\text{Aff}_{4nk}$ is of the form:

Note that $v_n^*x v_n$ exchanges $x_1$ and $x_2$. 
The braiding

\( x, y \in \text{Hom}_{\mathcal{M}_1, \mathcal{M}_1} L^2(\mathcal{M}_n) \cong p_n(\text{Aff}_{4nk}(P))p_n. \)

\[ x = \begin{array}{c}
\ast \\
\ast \\
(2k) \\
\ast \\
\ast \\
\ast
\end{array} \]

\[ y = \begin{array}{c}
\ast \\
\ast \\
(2k) \\
\ast \\
\ast \\
\ast
\end{array} \]

\[ x \otimes y \in \text{Hom}_{\mathcal{M}_1, \mathcal{M}_1} L^2(\mathcal{M}_{2n}): \]
The braiding

\[ x \otimes y \in \text{Hom}_{\mathcal{M}_1, \mathcal{M}_1} L^2(\mathcal{M}_{2n}) : \]

\[ x \otimes y = \]

\[ x_1 \quad q_1^{(1)} \quad y_1 \quad y_2 \quad q_2^{(2)} \quad q_1^{(2)} \quad x_2 \]

Conclusion: \( x \otimes y \) is unitarily conjugate to \( y \otimes x \).
The braiding

\[ x \otimes y \in \text{Hom}_{\mathcal{M}_1, \mathcal{M}_1} L^2(\mathcal{M}_{2n}) : \]

\[ \nu_{2n+1}^*(x \otimes y) \nu_{2n+1} = \]

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The braiding

\( x \otimes y \in \text{Hom}_{\mathcal{M}_1, \mathcal{M}_1} L^2(\mathcal{M}_{2n}) \):

\[
v_{2n+1}^* \left( v_n^* x v_n \otimes v_n^* y v_n \right) v_{2n+1} =
\]

Conclusion: \( x \otimes y \) is unitarily conjugate to \( y \otimes x \).
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Thank you!