Local Lagrange Interpolation with C^2 Splines of Degree Seven on Triangulations

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Abstract. We describe a method based on C^2 polynomial bivariate splines of degree seven which can be used to interpolate function values at a set of arbitrarily scattered points in a planar domain. The method starts with an arbitrary triangulation of the data points, and involves refining some of the triangles with Clough-Tocher splits. Some additional function values are required at selected points in the domain. The method is local, which ensures that the process of constructing the Lagrange interpolant is of linear complexity while providing optimal order approximation of smooth functions.

§1. Introduction

Given a set of points $\mathcal{V} := \{\eta_i\}_{i=1}^n$ in the plane, our aim in this paper is to provide a constructive method for solving the following problem.

Problem 1. *Find a triangulation $\triangle^*$ whose set of vertices includes $\mathcal{V}$, a space $\mathcal{S}$ of C^2 splines defined on $\triangle^*$, and a set of additional points $\{\eta_i\}_{i=n+1}^N$ such that for every choice of the data $\{z_i\}_{i=1}^N$, there is a unique spline $s \in \mathcal{S}$ satisfying*

$$s(\eta_i) = z_i, \qquad i = 1, \ldots, N. \tag{1}$$

We call $P := \{\eta_i\}_{i=1}^N$ and $\mathcal{S}$ a Lagrange interpolation pair.

Although constructing Lagrange interpolation pairs sounds simple at first glance, it is in fact a complex problem, especially since we want a local and stable method which has linear complexity and provides optimal order approximation. To achieve this, both $\mathcal{S}$ and P must be carefully chosen.

Advances in Constructive Approximation: Vanderbilt 2003
Mike Neamtu and Ed Saff (eds.), pp. 1–27.

ISBN 0-9728482-2-3

In particular, to construct $\triangle^*$ we will choose additional vertices beyond $\mathcal{V}$, and to construct $\mathcal{S}$ we will begin with a classical superspline space and enforce additional special supersmoothness conditions.

The analog of Problem 1 for C^1 splines has recently been treated in [11–13,15–18]; see also the survey paper [14] for further references. In this paper, we present what we believe is the first C^2 method. Our construction makes use of splines of degree seven, and is based on the following steps:

1) Choose a triangulation $\triangle$ with vertices at the points of $\mathcal{V}$.

2) Separate $\triangle$ into black and white triangles such that each triangle has at most one neighbor of the same color.

3) Define the triangulation $\triangle^*$ by subdividing some of the white triangles into subtriangles.

4) Define the space $\mathcal{S}$ as a subspace of the classical superspline space $\mathcal{S}_7^{2,3}(\triangle^*)$ (see (11)) by requiring certain additional smoothness conditions at the inserted vertices or across inserted edges.

5) Insert additional interpolation points into the black triangles so as to uniquely and locally define a spline $s \in \mathcal{S}$ on the black triangles.

6) Show that the smoothness conditions uniquely determine s on all of the refined triangulation $\triangle^*$, *i.e.* s is smoothly extended to the white triangles.

The paper is organized as follows. In Section 2 we introduce some notation and describe the Bernstein-Bézier representation of splines, while in Section 3 we present a useful result on interpolation by polynomials in Bernstein-Bézier form. Our coloring algorithm is given in Section 4, while in Section 5 we discuss a number of three and four-sided macro-elements needed for the construction. Sections 6 and 7 contain our main results, namely the construction of the Lagrange interpolating pair $P, \mathcal{S}$, and error bounds for the interpolating splines, while Section 8 includes several remarks.

§2. Preliminaries

Given a triangulation $\triangle$ and integers $0 \le r < d$, we write

$$\mathcal{S}_d^r(\triangle) := \{s \in C^r(\Omega) : \ s|_T \in \mathcal{P}_d, \text{ all } T \in \triangle\}$$

for the usual space of splines of degree d and smoothness r, where $\mathcal{P}_d$ is the $\binom{d+2}{2}$ dimensional space of bivariate polynomials of degree d. Throughout the paper we shall make extensive use of the well-known Bernstein-Bézier representation of splines. We recall that in this representation of a spline, for each triangle $T = \langle v_1, v_2, v_3 \rangle$ in $\triangle$ with vertices v_1, v_2, v_3, the corresponding polynomial piece $s|_T$ is written in the form

$$s|_T = \sum_{i+j+k=d} c_{ijk}^T B_{ijk}^d, \tag{2}$$

where B_{ijk}^d are the Bernstein basis polynomials of degree d associated with T. As usual, we identify the Bernstein-Bézier coefficients $\{c_{ijk}^T\}_{i+j+k=d}$ with the set of domain points $\mathcal{D}_T := \{\xi_{ijk}^T := (iv_1 + jv_2 + kv_3)/d\}_{i+j+k=d}$. We write $\mathcal{D}_{d,\triangle}$ for the union of the sets of domain points associated with the triangles of $\triangle$.

Given $T := \langle v_1, v_2, v_3 \rangle$ and an integer $0 \le m < d$, we set $R_m^T(v_1) := \{\xi_{ijk}^T : i = d-m\}$, $D_m^T(v_1) := \{\xi_{ijk}^T : i \ge d-m\}$, and associated with the edge $e := \langle v_2, v_3 \rangle$, we let $E_m^T(e) := \{\xi_{ijk}^T : i \le m\}$. We recall that the ring of radius m around v_1 is the set $R_m(v_1) := \bigcup\{R_m^T(v_1) : T$ has a vertex at $v_1\}$, and the disk of radius m around v_1 is the set $D_m(v_1) := \bigcup\{D_m^T(v_1) : T$ has a vertex at $v_1\}$. The rings and disks around v_2 and v_3 are defined similarly.

It is well known that a spline s in $\mathcal{S}_d^0(\triangle)$ is uniquely determined by its set $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$ of B-coefficients. To describe higher order smoothness, we recall some notation introduced in [2]. Suppose that $T := \langle v_1, v_2, v_3 \rangle$ and $\tilde{T} := \langle v_4, v_3, v_2 \rangle$ are two adjoining triangles from $\triangle$ which share the oriented edge $e := \langle v_2, v_3 \rangle$, and let

$$\begin{aligned} s|_T &:= \sum_{i+j+k=d} c_{ijk} B_{ijk}^d, \\ s|_{\tilde{T}} &:= \sum_{i+j+k=d} \tilde{c}_{ijk} \widetilde{B}_{ijk}^d, \end{aligned} \tag{3}$$

where B_{ijk}^d and $\widetilde{B}_{ijk}^d$ are the Bernstein polynomials of degree d on the triangles T and $\tilde{T}$, respectively. Given integers $0 \le n \le j \le d$, let $\tau_{j,e}^n$ be the linear functional defined on $\mathcal{S}_d^0(\triangle)$ by

$$\tau_{j,e}^n s := c_{n,d-j,j-n} - \sum_{\nu+\mu+\kappa=n} \tilde{c}_{\nu,\mu+j-n,\kappa+d-j} \widetilde{B}_{\nu\mu\kappa}^n(v_1). \tag{4}$$

These are called smoothness functionals of order n. A spline $s \in \mathcal{S}_d^0(\triangle)$ belongs to $C^r(\Omega)$ for some $r > 0$ if and only if

$$\tau_{m,e}^n s = 0, \qquad n \le m \le d, \quad 0 \le n \le r. \tag{5}$$

Below we shall often make use of smoothness conditions to calculate one coefficient of a spline in terms of others. One way to do this is to simply use $\tau_{j,e}^n s = 0$ to solve for the coefficient $c_{n,d-j,j-n}$ in terms of the coefficients in the sum. We shall also make use of the following well-known lemma.

Lemma 2. [1] *Suppose T and $\tilde{T}$ are as above, where v_1, v_2, v_4 are not collinear. Suppose that all coefficients c_{ijk} and $\tilde{c}_{ijk}$ on the disk $D_m(v_2)$ of the spline s are known except for*

$$\begin{aligned} c_\nu &:= c_{\nu,d-m,m-\nu}, & \nu &= \ell+1,\ldots,q, \\ \tilde{c}_\nu &:= \tilde{c}_{\nu,m-\nu,d-m}, & \nu &= \ell+1,\ldots,\tilde{q}, \end{aligned} \tag{6}$$

for some $\ell, m, q, \tilde{q}$ with $0 \le q, \tilde{q}$, $-1 \le \ell \le q, \tilde{q}$, and $q+\tilde{q}-\ell \le m \le d$. Then the coefficients (6) are uniquely determined by the smoothness conditions

$$\tau^n_{m,e} s = 0, \qquad \ell+1 \le n \le q+\tilde{q}-\ell. \tag{7}$$

As is well known, Lemma 2 should not be used when the edge e is near degenerate, *i.e.*, when v_1, v_2, v_4 are nearly collinear.

The smoothness functionals in (4) can be used to define very general superspline spaces

$$\mathcal{S}^{\mathcal{T}}_d(\triangle) := \{s \in \mathcal{S}^0_d(\triangle) : \ \tau s = 0, \text{ all } \tau \in \mathcal{T}\}$$

associated with an arbitrary set $\mathcal{T}$ of smoothness functionals. These spaces encompass all the classical bivariate spline spaces and superspline subspaces.

Recall that a determining set for a spline space $\mathcal{S} \subseteq \mathcal{S}^0_d(\triangle)$ is a subset $\mathcal{M}$ of the set of domain points $\mathcal{D}_{d,\triangle}$ such that if $s \in \mathcal{S}$ and $c_\xi = 0$ for all $\xi \in \mathcal{M}$, then $c_\xi = 0$ for all $\xi \in \mathcal{D}_{d,\triangle}$, *i.e.*, $s \equiv 0$. The set $\mathcal{M}$ is called a minimal determining set (MDS) for $\mathcal{S}$ if there is no smaller determining set. It is known that $\mathcal{M}$ is a MDS for $\mathcal{S}$ if and only if every spline $s \in \mathcal{S}$ is uniquely determined by its set of B-coefficients $\{c_\xi\}_{\xi \in \mathcal{M}}$. We recall that a MDS $\mathcal{M}$ is said to be local provided there is an integer ℓ such that for every triangle $T \in \triangle$, the coefficients of s associated with domain points in T depend only on the values of the coefficients associated with domain points in $\mathcal{M} \cap \operatorname{star}^\ell(T)$, where $\operatorname{star}^0(T) := T$ and $\operatorname{star}^i(T)$ is the union of the set of all triangles which touch a triangle in $\operatorname{star}^{i-1}(T)$. Finally, we recall that a MDS $\mathcal{M}$ for a spline space $\mathcal{S}$ of degree d defined on a triangulation $\triangle$ is said to be stable provided there exists a constant K depending only on the smallest angle in $\triangle$ such that

$$\max_{\eta \in \mathcal{D}_{d,\triangle}} |c_\eta| \le K \max_{\xi \in \mathcal{M}} |c_\xi|. \tag{8}$$

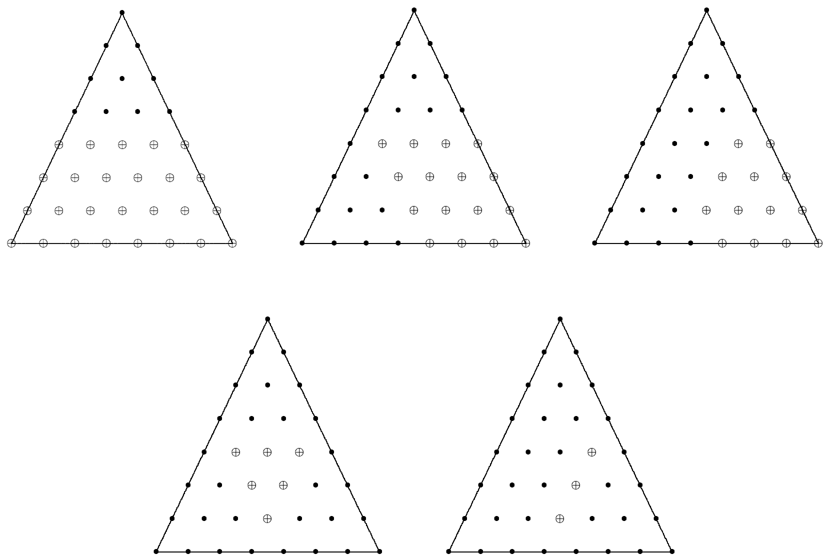

Fig. 1. The interpolation sets $\mathcal{A}_1^T, \mathcal{A}_{2a}^T, \mathcal{A}_{2b}^T, \mathcal{A}_{3a}^T, \mathcal{A}_{3b}^T$.

§3. Polynomial Interpolation

For later use, we now state a useful property of the Bernstein basis polynomials of degree d associated with a given triangle $T := \langle u, v, w \rangle$.

Theorem 3. *Let $\mathcal{A}$ be an arbitrary subset of the set $\mathcal{D}_T$ of domain points associated with T, and let $0 \le d \le 7$. Then for any $\{z_\eta\}_{\eta \in \mathcal{A}}$, there is a unique polynomial of the form*

$$p := \sum_{\xi \in \mathcal{A}} c_\xi B_\xi^d$$

such that

$$p(\eta) = z_\eta, \qquad \eta \in \mathcal{A}. \tag{9}$$

Proof: The result is easy to prove for $d = 0, 1, 2$, but so far remains a conjecture for arbitrary degree d, see [21]. However, we have verified the result for all $d \le 7$ and all choices of $\mathcal{A}$ by explicitly computing the determinants of the matrices $M_\mathcal{A} := [B_\xi^d(\eta)]_{\xi, \eta \in \mathcal{A}}$ using Mathematica. Note that since the entries depend only on barycentric coordinates of domain points in T, these matrices do not depend on the size, shape, or orientation of the triangle T. $\square$

The following cases of Theorem 3 are of particular interest here (see Algorithm 12):

$$\begin{aligned}
\mathcal{A}_0^T &:= \mathcal{D}_T, \\
\mathcal{A}_1^T &:= \mathcal{D}_T \setminus D_3^T(u), \\
\mathcal{A}_{2a}^T &:= \mathcal{A}_1^T \setminus D_3^T(v), \\
\mathcal{A}_{2b}^T &:= \mathcal{A}_{2a}^T \setminus E_2^T(e), \\
\mathcal{A}_{3a}^T &:= \mathcal{A}_{2a}^T \setminus D_3^T(w), \\
\mathcal{A}_{3b}^T &:= \mathcal{A}_{3a}^T \setminus E_2^T(e),
\end{aligned} \tag{10}$$

where $e = \langle u, v \rangle$ and $d = 7$. These configurations are shown in Fig. 1, where the points in the sets $\mathcal{A}_i^T$ are marked with $\oplus$, and where the vertices are labelled in counterclockwise order with u at the top. Using Mathematica, we found the following values for $\|M_i^{-1}\| := \|M_{\mathcal{A}_i^T}^{-1}\|$:

$\|M_0^{-1}\|$	$\|M_1^{-1}\|$	$\|M_{2a}^{-1}\|$	$\|M_{2b}^{-1}\|$	$\|M_{3a}^{-1}\|$	$\|M_{3b}^{-1}\|$
412.020	350.077	293.792	104.931	86.024	46.902

§4. Coloring a Triangulation

Given a planar triangulation $\triangle$ of a domain Ω, we say that two triangles in $\triangle$ are neighbors provided they have a common edge. We say that they touch provided they have at least one vertex in common. We now present an algorithm for creating a black & white coloring of $\triangle$. This coloring will be of importance later as a means for organizing the triangles of $\triangle$ so that a Lagrange interpolation pair can be constructed.

Algorithm 4. *Start with any black and white coloring of $\triangle$.*

1) *Repeat until every triangle of $\triangle$ has at most one neighbor of the same color:*
   a) *Choose a triangle T with at least two neighbors of the same color.*
   b) *Switch the color of T.*
2) *Switch the color of every white triangle having a white neighbor and two edges on the boundary.*

Discussion: It is easy to see that the number of edges shared by two triangles with the same color decreases at each step. There are two possible cases, where this number decreases either by 3 or 1:

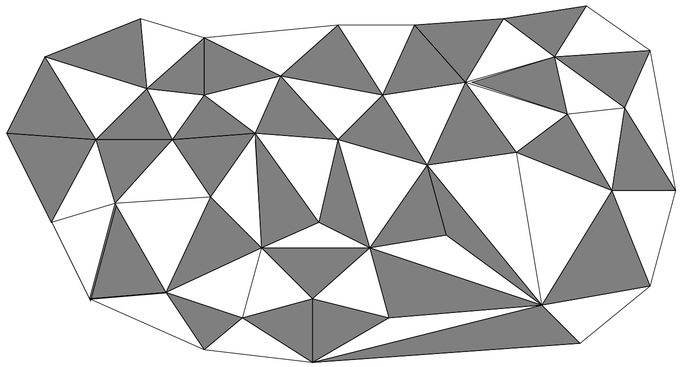

Fig. 2. A triangulation colored by Algorithm 4.

It follows that the essential part of the algorithm, step 1), terminates after at most E_I steps, where E_I is the number of interior edges in $\triangle$. $\square$

We note that since the dual graph of a triangulation is a planar graph of maximal degree three, the above coloring algorithm can be considered as a variant of the coloring method in [10] for bounded degree graphs. In graph coloring theory, such colorings where neighboring knots are allowed to have the same color are called improper or defective colorings, see [4,6].

Fig. 2 shows an example of a triangulation that has been colored by the above algorithm. We note that the algorithm is fast, namely linear in the number of triangles, which follows from the above discussion and Euler's formulae (see, for instance [14]).

After applying Algorithm 4 to a triangulation, it is clear that all black triangles appear singly or in pairs sharing a common edge. Moreover, any black triangle or pair of black triangles can touch other such clusters only at a vertex. The analogous statement holds for the white triangles. We also note that no white triangle with two edges on the boundary of Ω has a white neighbor.

§5. Macro-elements

In this section we describe several C^2 macro-element methods which will be useful in solving Problem 1. In particular, given a black & white coloring as constructed in the previous section, and given a spline s defined on the black triangles, we will use the macro-elements described here to extend s onto the white triangles in such a way as to define a C^2 spline of degree 7. In view of the nature of the coloring, we have to deal with extensions onto either a single triangle or a pair of adjoining triangles. We can think of macro-elements as schemes for filling three or four-sided holes in a triangulation. For a construction of n-sided C^r macro elements, see [20].

5.1. Three-sided macro-elements

Let $T := \langle v_1, v_2, v_3\rangle$ be a triangle in a triangulation $\triangle$ of a domain Ω. We consider three cases depending on how many edges of T lie on the boundary $\partial\Omega$ of Ω. We shall make use of the superspline spaces

$$\mathcal{S}_7^{2,3}(\triangle) := \{s \in \mathcal{S}_7^2(\triangle) : \ s \in C^3(v), \text{ all vertices } v \text{ of } \triangle\}. \qquad (11)$$

As usual, $s \in C^3(v)$ means that all polynomial pieces of s associated with triangles sharing the vertex v have common derivatives up to order 3 at v.

First, we assume that no edges of T lie on $\partial\Omega$. In this case, we consider the Clough-Tocher split $\triangle_T$ of T consisting of the three triangles $T_i := \langle v_T, v_i, v_{i+1}\rangle$ for $i = 1, 2, 3$, where we identify $v_4 = v_1$, and where v_T is the barycenter of T, see Fig. 3. The following lemma is a special case of results in [1].

Lemma 5. *Let $\mathcal{S}_T$ be the subspace of all splines s in $\mathcal{S}_7^{2,3}(\triangle_T)$ satisfying the following additional smoothness conditions:*

a) $s \in C^6(v_T)$,

b) $\tau^5_{5,\langle v_1, v_T\rangle} s = 0$.

Then dim $\mathcal{S}_T = 39$, *and the set $\mathcal{M}$ containing the domain points*

1) $\{D_3^{T_i}(v_i)\}_{i=1}^3$,

2) $\{\xi_{133}^{T_i}, \xi_{232}^{T_i}, \xi_{223}^{T_i}\}_{i=1}^3$

is a stable MDS for $\mathcal{S}_T$.

The points listed in 1) of Lemma 5 are marked with $\odot$ in Fig. 3, while the points in 2) are marked with $\otimes$. The tip of the special smoothness condition $\tau^5_{5,\langle v_1, v_T\rangle}$ is marked with an arrowhead.

Remark. Clearly, if s is a spline belonging to $C^2(\Omega \setminus T)$, where T is as in Lemma 5, then s can be uniquely extended to $\triangle_T$ to produce a spline in $C^2(\Omega)$ using only C^2 smoothness across the edges and C^3 smoothness at the vertices. In particular, referring to Fig. 3, the coefficients corresponding to domain points marked with $\odot$ can be computed from C^3 smoothness at the vertices, while those corresponding to points marked with $\otimes$ can be computed from the C^2 smoothness across the edges.

We now assume that exactly one edge of $T := \langle v_1, v_2, v_3\rangle$ lies on the boundary $\partial\Omega$ of Ω. We may suppose that the vertices of T are numbered such that $\langle v_3, v_1\rangle$ is the edge on $\partial\Omega$. As above, let $\triangle_T$ be the Clough-Tocher split of T consisting of the three triangles $T_i := \langle v_T, v_i, v_{i+1}\rangle$ for $i = 1, 2, 3$, where v_T is the barycenter of T.

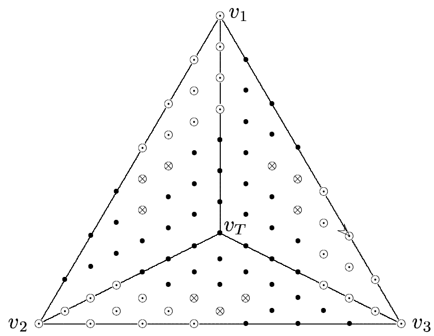

Fig. 3. The split and MDS of Lemma 5.

Lemma 6. *Let $\mathcal{S}_T$ be the subspace of all splines $s \in \mathcal{S}_7^{2,3}(\triangle_T)$ satisfying the following additional smoothness conditions:*

a) $s \in C^6(v_T)$,

b) $\tau^j_{j,\langle v_2,v_T\rangle} s = 0$ *for* $j = 5,6,7$.

Then dim $\mathcal{S}_T = 37$, *and the set* $\mathcal{M}$ *containing the domain points*

1) $\{D_3^{T_i}(v_i)\}_{i=1}^3$,

2) $\{\xi_{133}^{T_i}, \xi_{232}^{T_i}, \xi_{223}^{T_i}\}_{i=1}^2$,

3) $\xi_{331}^{T_1}$ *or* $\xi_{313}^{T_2}$

is a stable MDS for $\mathcal{S}_T$.

Proof: First we show that $\mathcal{M}$ is a determining set. Suppose that we set the coefficients c_ξ of $s \in \mathcal{S}_T$ to zero for all $\xi \in \mathcal{M}$. Then we claim that all other coefficients must be zero. By the C^3 supersmoothness at the vertices, all coefficients corresponding to domain points in the disks $D_3(v_i)$ must be zero for $i = 1,2,3$. Now we solve for the unset coefficients corresponding to domain points on the rings $R_i(v_2)$ for $i = 4,\ldots,7$, using conditions a) and b). Each step involves solving a nonsingular homogeneous system of equations, see Lemma 2. The remaining coefficients in T_3 are zero by the C^6 smoothness at vertex v_T. This shows that all coefficients of s corresponding to domain points in $T_1 \cup T_2 \cup T_3$ must be zero, and thus $\mathcal{M}$ is a determining set.

To show that $\mathcal{M}$ is a minimal determining set, we now show that its cardinality is equal to the dimension of $\mathcal{S}_T$. It is easy to see that $\#\mathcal{M}$ is equal to 37, and it is known (cf. [8]) that the dimension of the superspline space $\mathcal{S}_7^{2,3}(\triangle) \cap C^6(v_T)$ is 40. Our space $\mathcal{S}$ is the subspace of $\mathcal{S}_7^{2,3}(\triangle_T) \cap C^6(v_T)$ that satisfies the 3 additional special conditions b).

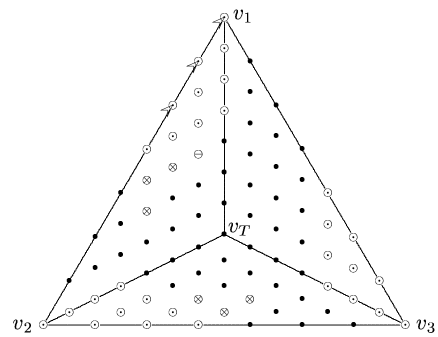

Fig. 4. The split and MDS of Lemma 6.

Thus, $40 - 3 \le \dim \mathcal{S} \le \#\mathcal{M} = 37$, and we conclude that the dimension of $\mathcal{S}$ is 37 and $\mathcal{M}$ is a MDS for $\mathcal{S}$.

To prove that $\mathcal{M}$ is a stable MDS, we must check (8). First, it is well known that the process of computing a coefficient directly from a smoothness condition is stable, *i.e.*, if we solve (4) for a coefficient of the form $c_{n,d-j,j-n}$, its absolute value is bounded by a constant times the maximum absolute value of the coefficients of the form $\tilde{c}_{\nu,\mu+j-n,\kappa+d-j}$, $\nu + \mu + \kappa = n$, where the constant depends only on the smallest angle in the two triangles containing the edge e. The computation of coefficients by Lemma 2 is also a stable process as the norms of the inverses of the matrices which appear are bounded by a constant depending only on the smallest angle in T. $\square$

The points listed in 1) of Lemma 6 are marked with $\odot$ in Fig. 4, while those in 2) are marked with $\otimes$. The point $\xi_{331}^{T_1}$ is marked with $\ominus$, and the tips of the special smoothness conditions in b) are marked with arrowheads. The reason for allowing a choice in 3) is to provide additional flexibility in the use of the element.

Remark. If s is a spline in $C^2(\Omega \setminus T)$, where T is as in Lemma 6, then s can be uniquely extended to Δ_T to produce a spline in $C^2(\Omega)$. In particular, we use the C^3 smoothness at the v_i to compute the coefficients corresponding to domain points marked with $\odot$ in Fig. 4. We use the C^2 smoothness across edges to compute the coefficients corresponding to points marked with $\otimes$. If $\xi_{331}^{T_1} \in \mathcal{M}$, we can compute the corresponding coefficient (marked with $\ominus$ in the figure) by requiring $\tau^3_{4,\langle v_1,v_2\rangle} s = 0$. Similarly, if $\xi_{313}^{T_2} \in \mathcal{M}$, we can compute the corresponding coefficient from $\tau^3_{6,\langle v_2,v_3\rangle} s = 0$.

For completeness, we consider the case where two edges of T lie on $\partial\Omega$. Let $\Delta_T := T$. The proof of the following lemma is evident.

Lemma 7. *Let $\mathcal{S}_T = \mathcal{P}_7$. Then* dim $\mathcal{S}_T = 36$ *and the set $\mathcal{M} = \mathcal{D}_T$ is a stable MDS for $\mathcal{S}_T$.*

Remark. This macro-element can be used to extend a spline into a triangle T by using C^7 continuity across the shared edge $\langle v_2, v_3 \rangle$. Alternatively, all coefficients except for c_{700} can be computed from C^6 continuity across $\langle v_2, v_3 \rangle$, and c_{700} can be chosen equal to the function value z_{v_1} at v_1.

5.2. Four-sided macro-elements

Let $Q := T \cup \tilde{T}$, where $T := \langle v_1, v_2, v_4 \rangle$ and $\tilde{T} := \langle v_2, v_3, v_4 \rangle$ are a pair of adjacent triangles. There are four cases depending on how many edges of Q lie on $\partial\Omega$. Our first result concerns the case where no edges of Q lie on $\partial\Omega$. Let w_1 and w_2 be the barycenters of T and $\tilde{T}$, and let $\triangle_Q$ be the subtriangulation consisting of the six triangles $T_1 := \langle w_1, v_1, v_2 \rangle$, $T_2 := \langle w_2, v_2, v_3 \rangle$, $T_3 := \langle w_2, v_3, v_4 \rangle$, $T_4 := \langle w_1, v_4, v_1 \rangle$, $T_5 := \langle w_1, v_2, v_4 \rangle$, and $T_6 := \langle w_2, v_4, v_2 \rangle$.

Lemma 8. *Let $\mathcal{S}_Q$ be the subspace of all splines $s \in \mathcal{S}_7^{2,3}(\triangle_Q)$ satisfying the following additional smoothness conditions:*

a) $s \in C^6(w_i)$ *for* $i = 1, 2$,

b) $\tau^j_{j,\langle v_1, w_1 \rangle} s = 0$ *for* $j = 5, 6, 7$,

c) $\tau^5_{5,\langle v_3, w_2 \rangle} s = 0$.

Then dim $\mathcal{S}_T = 53$, *and the set $\mathcal{M}$ containing the domain points*

1) $\{D_3^{T_i}(v_i)\}_{i=1}^4$,

2) $\{\xi^{T_i}_{133}, \xi^{T_i}_{232}, \xi^{T_i}_{223}\}_{i=1}^4$,

3) $\xi^{T_4}_{331}$ *or* $\xi^{T_1}_{313}$

is a stable MDS for $\mathcal{S}_Q$.

Proof: First we show that $\mathcal{M}$ is a determining set. Suppose that we set the coefficients c_ξ of $s \in \mathcal{S}_Q$ to zero for all $\xi \in \mathcal{M}$. Then we claim that all other coefficients must be zero. First we observe that by the C^3 supersmoothness at the vertices v_i, all coefficients corresponding to domain points in the disks $D_3(v_i)$ must be zero for all $i = 1, \ldots, 4$. It then follows as in the proof of Lemma 6 that all of the unset coefficients in T must be zero. Now by the C^2 smoothness across the edge $\langle v_2, v_4 \rangle$, we see that the coefficients corresponding to domain points $\xi^{T_6}_{133}, \xi^{T_6}_{232}, \xi^{T_6}_{223}$ are also zero. At this point we have shown that all of the coefficients of $s|_{\tilde{T}}$ corresponding to the minimal determining set of Lemma 5 are zero, and thus all coefficients of $s|_{\tilde{T}}$ are zero.

To show that $\mathcal{M}$ is a minimal determining set, we now demonstrate that its cardinality is equal to the dimension of $\mathcal{S}_Q$. It is easy to see that

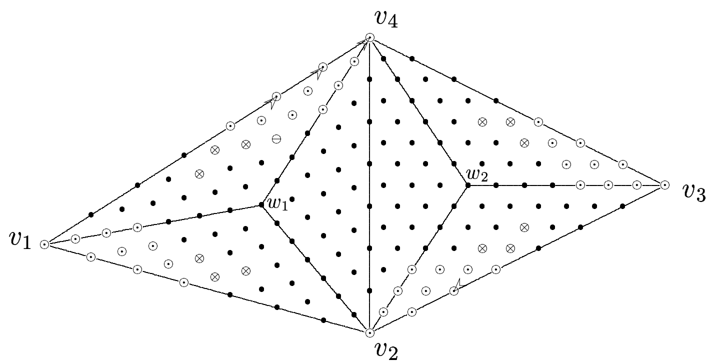

Fig. 5. The split and MDS of Lemma 8.

$\#\mathcal{M}$ is equal to 53. Let $\triangle_1$ be the triangulation obtained by applying the Clough-Tocher split to triangle T. Then (cf. [8]) the dimension of the superspline space $\mathcal{S}_7^{2,3}(\triangle_1) \cap C^6(w_1)$ is 40, and it follows easily that the dimension of $\mathcal{S}_7^{2,3}(\triangle_Q) \cap C^6(w_1) \cap C^6(w_2)$ is 57. Our space $\mathcal{S}_Q$ is the subspace that satisfies the additional 4 smoothness conditions b) and c). Thus, $57-4 \le \dim \mathcal{S} \le \#\mathcal{M} = 53$, and we conclude that the dimension of $\mathcal{S}$ is 53 and $\mathcal{M}$ is a MDS. The fact that $\mathcal{M}$ is stable follows as in Lemma 6. □

The points listed in 1) of Lemma 8 are marked with $\odot$ in Fig. 5, while those in 2) are marked with $\otimes$. The point $\xi_{331}^{T_4}$ is marked with $\ominus$, and the tips of the special smoothness conditions in b)–c) are marked with arrowheads. The reason for allowing a choice in 3) is to provide additional flexibility in the use of the element.

Remark. If s is a spline in $C^2(\Omega \setminus Q)$ with Q as in Lemma 8, then s can be uniquely extended to $\triangle_Q$ to produce a spline in $C^2(\Omega)$. In particular, we use C^3 smoothness at the v_i to compute the coefficients corresponding to domain points marked with $\odot$ in Fig. 5, and C^2 smoothness across edges to compute those corresponding to points marked with $\otimes$. If $\xi_{331}^{T_4} \in \mathcal{M}$, we can compute the corresponding coefficient (marked with $\ominus$ in the figure) by setting the additional smoothness condition $\tau^3_{4,\langle v_4,v_1\rangle} s = 0$. If $\xi_{313}^{T_1} \in \mathcal{M}$, we can compute the corresponding coefficient by setting the additional smoothness condition $\tau^3_{6,\langle v_1,v_2\rangle} s = 0$.

Our next lemma concerns the case where one edge of Q lies on $\partial\Omega$. Let $\triangle_Q$ be the split of Q used in Lemma 8, and assume that the edge $\langle v_3, v_4\rangle$ lies on $\partial\Omega$.

Lemma 9. *Let $\mathcal{S}_Q$ be the subspace of all splines $s \in \mathcal{S}_7^{2,3}(\triangle_Q)$ satisfying the following additional smoothness conditions:*

a) $s \in C^6(w_i)$ *for* $i = 1, 2$,

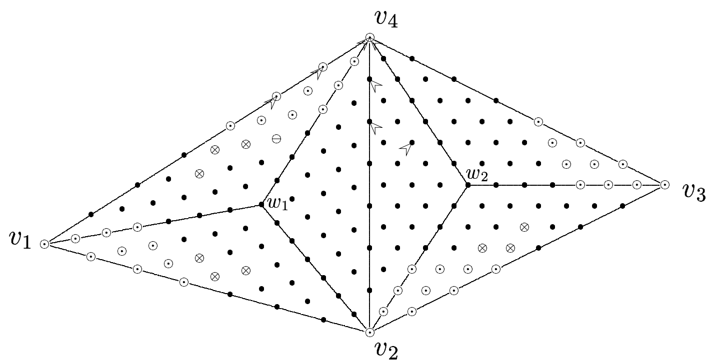

Fig. 6. The split and MDS of Lemma 9.

b) $\tau^j_{j,\langle v_1,w_1\rangle} s = 0$ *for* $j = 5, 6, 7$,

c) $\tau^3_{4,\langle v_4,v_2\rangle} s = 0$,

d) $\tau^j_{j,\langle v_2,w_2\rangle} s = 0$ *for* $j = 5, 6, 7$.

Then $\dim \mathcal{S} = 50$, *and the set* $\mathcal{M}$ *containing the domain points*

1) $D_3^{T_i}(v_i)$, $i = 1, \ldots, 4$,

2) $\{\xi^{T_i}_{133}, \xi^{T_i}_{232}, \xi^{T_i}_{223}\}_{i=1,2,4}$

2) $\xi^{T_4}_{331}$ *or* $\xi^{T_1}_{313}$

is a stable MDS for $\mathcal{S}_Q$.

Proof: The proof is similar to the proof of Lemma 8, and is based on first computing coefficients corresponding to domain points in triangle T, then for those in triangle $\tilde{T}$ using Lemma 6. For a more detailed analysis, see [20]. □

The points listed in 1) of Lemma 9 are marked with $\odot$ in Fig. 6, while those in 2) are marked with $\otimes$. The point $\xi^{T_4}_{331}$ is marked with $\ominus$, and the tips of the special smoothness conditions in b)–d) are marked with arrowheads. The reason for allowing a choice in 3) is to provide additional flexibility in the use of the element.

Remark. If s is a spline in $C^2(\Omega\backslash Q)$ with Q as in Lemma 9, then s can be uniquely extended to $\triangle_Q$ to produce a spline in $C^2(\Omega)$. In particular, we use C^3 smoothness at the v_i to compute the coefficients corresponding to domain points marked with $\odot$ in Fig. 6, and C^2 smoothness across edges $\langle v_1, v_2\rangle$, $\langle v_2, v_3\rangle$, and $\langle v_4, v_1\rangle$ to compute those corresponding to domain points marked with $\otimes$. If $\xi^{T_4}_{331} \in \mathcal{M}$, we can compute the corresponding coefficient (marked with $\ominus$ in the figure) by setting the additional smoothness condition $\tau^3_{4,\langle v_4,v_1\rangle} s = 0$. If $\xi^{T_1}_{313} \in \mathcal{M}$, we can compute the corresponding coefficient by setting the additional smoothness condition $\tau^3_{6,\langle v_1,v_2\rangle} s = 0$.

Our next lemma concerns the case where two adjacent edges of Q lie on $\partial\Omega$. Suppose $\langle v_1, v_2\rangle$ and $\langle v_2, v_3\rangle$ lie on the boundary of Ω. Let $w = (v_2 + v_3 + v_4)/3$ and let $\triangle_Q$ be the triangulation of Q consisting of the four triangles $T_1 := \langle v_1, v_2, v_4\rangle$, $T_2 := \langle w, v_2, v_3\rangle$, $T_3 := \langle w, v_3, v_4\rangle$, and $T_4 := \langle w, v_4, v_2\rangle$.

Lemma 10. *Let $\mathcal{S}_Q$ be the subspace of all splines $s \in \mathcal{S}^{2,3}_7(\triangle_Q)$ satisfying the following additional smoothness conditions:*

a) $s \in C^6(w)$,

b) $\tau^j_{j,\langle v_4,w\rangle} s := 0$ *for* $j = 5, 6, 7$,

c) $\tau^3_{6,\langle v_4,v_2\rangle} s := 0$,

Then $\dim \mathcal{S}_Q = 49$, *and the set $\mathcal{M}$ containing the domain points*

1) $\mathcal{D}^{T_1}_7$,

2) $D^{T_3}_3(v_3)$,

3) $\{\xi^{T_3}_{133}, \xi^{T_3}_{232}, \xi^{T_3}_{223}\}$

is a stable MDS for $\mathcal{S}_Q$.

Proof: First we show that $\mathcal{M}$ is a determining set. Suppose that we set the coefficients c_ξ of $s \in \mathcal{S}_Q$ to zero for all $\xi \in \mathcal{M}$. Then all coefficients corresponding to domain points in T_1 are zero. We now claim that all other coefficients must be zero. First we observe that by the C^3 supersmoothness at the vertices, all coefficients corresponding to domain points in the disks $D_3(v_i)$ must be zero for $i = 2, 3, 4$. By the C^2 smoothness across the edge $\langle v_4, v_2\rangle$ it follows that the coefficients corresponding to $\xi^{T_4}_{133}, \xi^{T_4}_{232}, \xi^{T_4}_{223}$ are zero. Condition c) implies that the coefficient corresponding to $\xi^{T_4}_{313}$ is also zero. Then all coefficients of $s|_{\tilde{T}}$ with $\tilde{T} = \langle v_2, v_3, v_4\rangle$ corresponding to the points in the minimal determining set described in Lemma 6 are zero. It follows that all coefficients of s corresponding to domain points in $\triangle_Q$ must be zero, and thus $\mathcal{M}$ is a determining set.

To show that $\mathcal{M}$ is a minimal determining set, we now demonstrate that its cardinality is equal to the dimension of $\mathcal{S}_Q$. It is easy to see that $\#\mathcal{M}$ is equal to 49. Let $\triangle_2$ be the triangulation obtained by applying the Clough-Tocher split to triangle $\tilde{T}$. Then (cf. [8]) the dimension of the superspline space $\mathcal{S}^{2,3}_7(\triangle_2) \cap C^6(w)$ is 40, and it follows easily that the dimension of $\mathcal{S}^{2,3}_7(\triangle_Q) \cap C^6(w)$ is 53. Taking account of the 4 special conditions b)–c), we have $53 - 4 \le \dim \mathcal{S} \le \#\mathcal{M} = 49$, and we conclude that the dimension of $\mathcal{S}$ is 49 and $\mathcal{M}$ is a MDS. Its stability follows as before. □

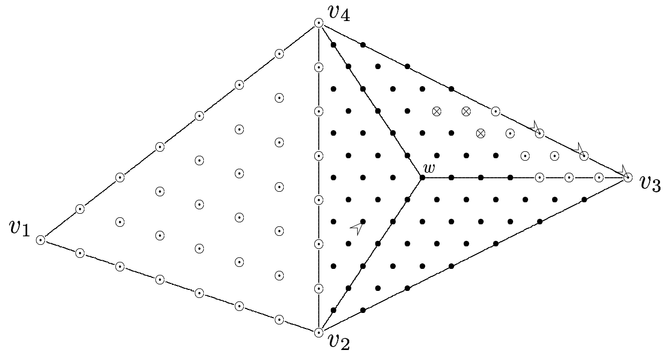

Fig. 7. The split and MDS of Lemma 10.

The points listed in 1)–2) of Lemma 10 are marked with $\odot$ in Fig. 7, while those in 3) are marked with $\otimes$. The tips of the special smoothness conditions in b)–c) are marked with arrowheads.

Remark. If s is a spline belonging to $C^2(\Omega \setminus Q)$ with Q as in Lemma 10, then s can be uniquely extended to $\triangle_Q$ to produce a spline in $C^2(\Omega)$. In particular, we use C^3 smoothness at the vertex v_3 and C^7 smoothness across the edge $\langle v_4, v_1 \rangle$ to compute the coefficients marked with $\odot$ in Fig. 7. We use C^2 smoothness across the edge $\langle v_3, v_4 \rangle$ to compute the coefficients marked with $\otimes$.

Our final lemma in this section deals with the case when two nonadjacent edges of Q lie on $\partial\Omega$. Let $\langle v_2, v_3 \rangle$ and $\langle v_4, v_1 \rangle$ be the edges of Q on $\partial\Omega$. Let $\triangle_Q$ be the split of Q used in Lemma 10.

Lemma 11. *Let $\mathcal{S}_Q$ be the subspace of all splines s in $\mathcal{S}_7^{2,3}(\triangle_Q)$ satisfying the following additional smoothness conditions:*

a) $s \in C^6(w)$,

b) $\tau^j_{j,\langle v_4, w \rangle} s := 0$ *for* $j = 5, 6, 7$,

c) $\tau^3_{6,\langle v_4, v_2 \rangle} s := 0$,

Then $\dim \mathcal{S}_Q = 49$, *and the set $\mathcal{M}$ containing the domain points*

1) $\{D_3^{T_3}(v_i)\}_{i=3}^4$,

2) $E_3^T(\langle v_1, v_2 \rangle)$,

3) $\{\xi^{T_3}_{313}, \xi^{T_3}_{322}, \xi^{T_3}_{223}\}$

is a stable MDS.

Proof: The proof is similar to the proof of Lemma 10, and is based on first computing coefficients corresponding to domain points in triangle T,

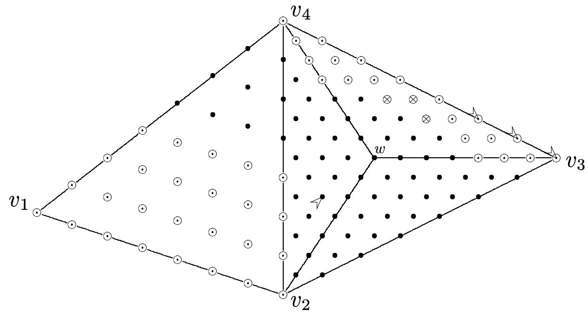

Fig. 8. The split and MDS of Lemma 11.

then in triangle $\tilde{T}$ using Lemma 6. For a more detailed analysis, see [20]. □

The points listed in 1)–2) of Lemma 11 are marked with $\odot$ in Fig. 8, while those in 3) are marked with $\otimes$. The tips of the special smoothness conditions in b)–c) are marked with arrowheads.

Remark. If s is a spline belonging to $C^2(\Omega \setminus Q)$ with Q as in Lemma 11, then s can be uniquely extended to $\triangle_Q$ to produce a spline in $C^2(\Omega)$. In particular, we use C^3 smoothness at the vertices v_3 and v_4, and C^3 smoothness across the edge $\langle v_1, v_2 \rangle$ to compute the coefficients marked with $\odot$ in Fig. 8. We use C^2 smoothness across the edge $\langle v_3, v_4 \rangle$ to compute the coefficients marked with $\otimes$.

§6. Construction of a Lagrange Interpolation Pair

In this section we describe an algorithm for solving Problem 1. Given an arbitrary triangulation $\triangle$ and a black & white coloring of its triangles obtained from Algorithm 4, there are two basic steps. In the first step, we construct the set P, while in the second step we construct the spline space $\mathcal{S}$.

Algorithm 12. *Let $\triangle$ be a triangulation which has been colored by Algorithm 4. Let P be the set of points obtained from the following steps:*

1) *Define all black triangles to be unmarked.*
2) *Repeat until no longer possible: choose an unmarked black triangle T that does not touch any marked triangle. Put $\mathcal{A}_0^T := \mathcal{D}_T$ in P and mark T.*
3) *Repeat until no longer possible: choose an unmarked black triangle T that touches some marked triangle at only one vertex u in T. Put $\mathcal{A}_1^T := \mathcal{D}_T \setminus D_3^T(u)$ in P and mark T.*

4) *Repeat until no longer possible: choose an unmarked black triangle T that touches marked triangles at two vertices u, v of T. If T is not a neighbor of a marked triangle, then put $\mathcal{A}^T_{2a} = \mathcal{D}_T \setminus [D^T_3(u) \cup D^T_3(v)]$ in P. Otherwise, put $\mathcal{A}^T_{2b} := \mathcal{D}_T \setminus [D^T_3(u) \cup D^T_3(v) \cup E^T_2(\langle u, v\rangle)]$ in P. In both cases, mark T.*

5) *Repeat until no longer possible: choose an unmarked black triangle T that touches marked triangles at all three vertices u, v, w of T. If T is not a neighbor of a marked triangle, then put $\mathcal{A}^T_{3a} := \mathcal{D}_T \setminus [D^T_3(u) \cup D^T_3(v) \cup D^T_3(w)]$ in P. If T shares one edge, say $\langle v, w\rangle$, with a marked neighbor, put $\mathcal{A}^T_{3b} := \mathcal{D}_T \setminus [D^T_3(u) \cup D^T_3(v) \cup D^T_3(w) \cup E^T_2(\langle v, w\rangle)]$ in P. In both cases, mark T.*

6) *Include all vertices of $\triangle$ which are not already in P.*

It is easy to see that the set P produced by Algorithm 12 contains all of the vertices of $\triangle$. Clearly, all vertices of black triangles are added to P in the course of carrying out steps 1) – 5). We call these black vertices. A vertex v is added in step 6) if and only if it is a vertex of a white triangle T, but is not a vertex of any black triangle. We call these white vertices. There are no white vertices in the triangulation of Fig. 2, but it is easy to create examples where they occur.

Algorithm 12 provides an ordering $T_1, \ldots, T_{n_b}$ for the black triangles of $\triangle$. For each black triangle we choose one of the point sets listed in (10). For later use, we denote the set of black triangles chosen in step 2) of Algorithm 12 by $\mathcal{T}_0$. Similarly, we write $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ for those chosen in steps 3)–5).

The order in which the point constellations are chosen in Algorithm 12 is important to ensure that our Lagrange interpolation method is local. In this connection we have the following lemma.

Lemma 13. *Suppose $\mathcal{T}_i$ are the classes of triangles created by Algorithm 12. Then*

1) *No two triangles in the class $\mathcal{T}_0$ can touch each other.*

2) *If two triangles in the same class $\mathcal{T}_i$, $1 \le i \le 3$, touch each other at a vertex v, then they must also touch a triangle in one of the classes $\mathcal{T}_0, \ldots, \mathcal{T}_{i-1}$ at the same vertex.*

Proof: The first assertion is obvious. Suppose T_1 and T_2 are two triangles in class $\mathcal{T}_i$ with $1 \le i \le 3$ that touch at a vertex v, and suppose they do not touch any triangle in $\mathcal{T}_j$ with $j < i$ at the same vertex v. Suppose T_1 is marked before T_2 in Algorithm 12. Then before T_1 was marked, T_2 must have had $4 - i$ vertices that were not shared with marked triangles. But then T_2 would have been chosen to be in $\mathcal{T}_{i-1}$. This completes the proof of 2). □

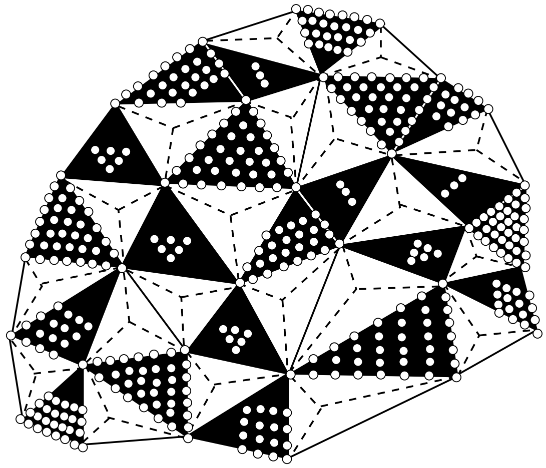

Fig. 9. An interpolation set P for the spline space $\mathcal{S}$.

A simple count shows that

$$\#P = 10\ V_B + V_W + 6\ N_1 + 9\ N_2, \tag{12}$$

where V_B and V_W are the numbers of black and white vertices in $\mathcal{V}$, N_1 is the number of black triangles which appear singly, and N_2 is the number of pairs of black triangles.

Example 14. Let $\triangle$ be the triangulation of 41 triangles with the coloring shown in Fig. 9.

Discussion: Here $V_B = 30$, $V_W = 0$, $N_1 = 13$ and $N_2 = 4$, and so by (12) the cardinality of P is 414. Algorithm 12 chooses the point constellation $\mathcal{D}_T$ six times, $\mathcal{A}_1^T$ four times, $\mathcal{A}_{2a}^T$ three times, $\mathcal{A}_{2b}^T$ once, $\mathcal{A}_{3a}^T$ four times and $\mathcal{A}_{3b}^T$ three times. This gives a total of 414 points in P. These points are shown as dots in the figure. □

As a first step towards defining our spline space $\mathcal{S}$, we need to define an appropriate triangulation $\triangle^*$. We do this by splitting some of the white triangles of $\triangle$.

Algorithm 15. *Given a triangulation $\triangle$ of Ω which has been colored by Algorithm 4, let $\triangle^*$ be the triangulation obtained by performing the following steps:*

1) *For each single white triangle T, apply one of the splits described in Lemmas 5–7, depending on the number of edges of T on $\partial\Omega$.*
2) *For each pair Q of adjacent white triangles, apply one of the splits described in Lemmas 8–11, depending on the number and configuration of edges of Q on $\partial\Omega$.*

For the triangulation in Example 14, Algorithm 15 applies the Clough-Tocher split to all 20 white triangles. These splits are shown in Fig. 9 with dotted lines.

Definition 16. *Let $\mathcal{S}$ be the set of all splines $s \in \mathcal{S}_7^{2,3}(\triangle^*)$ with the following additional properties:*

1) *For each single white triangle T in $\triangle$, $s|_T$ satisfies the additional smoothness conditions described in Lemmas 5–7.*
2) *For each pair Q of adjacent white triangles in $\triangle$, $s|_Q$ satisfies the additional smoothness conditions described in Lemmas 8–11.*
3) *For each single white triangle in $\triangle$ which has two black neighbors, $\tau_{4,e}^3 s = 0$, where $e = \langle v_1, v_2 \rangle$ in the notation of Lemma 6.*
4) *For each single white triangle in $\triangle$ which has exactly one black neighbor, $\tau_{m,e}^n s = 0$ for $n \le m \le 7$ and $0 \le n \le 6$, where e is the edge shared with the black neighbor.*
5) *For each pair of white triangles T and $\tilde{T}$ in $\triangle$ which have four black neighbors, $\tau_{4,e}^3 s = 0$, where $e = \langle v_4, v_1 \rangle$ in the notation of Lemma 8.*
6) *For each pair of white triangles T and $\tilde{T}$ in $\triangle$ which have exactly three black neighbors, $\tau_{4,e}^3 s = 0$, where $e = \langle v_4, v_1 \rangle$ in the notation of Lemma 9.*
7) *For each pair of white triangles T and $\tilde{T}$ in $\triangle$ with exactly two black neighbors that touch, $\tau_{m,e}^n s = 0$ for $n \le m \le 7$ and $0 \le n \le 6$, where $e = \langle v_4, v_1 \rangle$ in the notation of Lemma 10.*
8) *For each pair of white triangles T and $\tilde{T}$ in $\triangle$ with exactly two black neighbors that do not touch, $\tau_{m,e}^n s = 0$ for $n \le m \le 7$ and $0 \le n \le 3$, where $e = \langle v_1, v_2 \rangle$ in the notation of Lemma 11.*

We are now ready to prove the main result of this paper, namely that the set P of points constructed in Algorithm 12 and the space $\mathcal{S}$ of Definition 16 form a Lagrange interpolating pair with $\dim \mathcal{S} = \#P$ as given by (12).

Theorem 17. *Let P and $\mathcal{S}$ be as above. Then for any given $\{z_\eta\}_{\eta \in P}$, there is a unique spline $s \in \mathcal{S}$ such that $s(\eta) = z_\eta$, for all $\eta \in P$. The interpolation process is* local *in the sense that for every triangle $T \in \triangle^*$ and every domain point $\xi \in \mathcal{D}_T$, there exists a set $\Gamma_\xi \subset P \cap \operatorname{star}^5(T)$ such that the B-coefficient c_ξ of s depends only on the values of $\{z_\eta\}_{\eta \in \Gamma_\xi}$. Moreover, the interpolation process is* stable *in the sense that there exists a constant C depending only on the smallest angle in $\triangle$ such that*

$$|c_\xi| \le C \max_{\eta \in \Gamma_\xi} |z_\eta|, \qquad \text{all } \xi \in \mathcal{D}_T. \tag{13}$$

Proof: We show how to compute the B-coefficients of s, one triangle at a time. First we deal with the black triangles in the order $T_1, \ldots, T_{n_b}$ assigned by Algorithm 12. Let $\mathcal{T}_0, \ldots, \mathcal{T}_3$ be the classes of black triangles created by the algorithm. We say that a vertex of $\triangle$ is a type-k vertex if it is a vertex of a triangle in $\mathcal{T}_k$, but not a vertex of any triangle in $\mathcal{T}_j$ with $0 \le j < k$.

We begin with the triangle $T := T_1$ which lies in $\mathcal{T}_0$. Clearly, the coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_T}$ are uniquely defined as the solution of the system of 36 equations corresponding to interpolation at the points $\mathcal{A}_0^T := \mathcal{D}_T$. This system corresponds to the matrix M_0 introduced in Sect. 3. In this case $\Gamma_\xi = \mathcal{D}_T = \text{star}^0(T)$. Moreover, (13) holds with $C_0 := \|M_0^{-1}\| = 412.02$. Since by Lemma 13, triangles in $\mathcal{T}_0$ do not touch each other, we can uniquely compute the coefficients of $s|_T$ for all remaining triangles in class $\mathcal{T}_0$ in the same way. Once this is done, s is uniquely determined on all of the triangles of class $\mathcal{T}_0$. Now for each type-0 vertex u of $\triangle$, we can use the smoothness condition $s \in C^3(u)$ to determine the B-coefficients of s corresponding to domain points ξ in each of the sets $D_3^T(u)$, where T is a triangle sharing the vertex u. For these ξ, we have $\Gamma_\xi \subset \text{star}(T)$ and (13) holds with a constant $\tilde{C}_0$ depending on C_0 and the smallest angle in $\triangle$.

Now let $T := T_i \in \mathcal{T}_1$. By the definition of $\mathcal{T}_1$ coupled with Lemma 13, T must share exactly one vertex, say u, with at least one triangle T_u in $\{T_1, \ldots, T_{i-1}\} \cap \mathcal{T}_0$. This means that u is a type-0 vertex, and hence the B-coefficients of $s|_T$ corresponding to $D_3^T(u)$ are already known. By Theorem 3 the remaining coefficients of $s|_T$ are uniquely determined by the interpolation conditions corresponding to the 26 points in $\mathcal{A}_1^T$. These coefficients can be computed by solving the system $M_1 x = y$, where M_1 is the matrix introduced in Sect. 3, x is the vector with components $\{c_\eta\}_{\eta \in \mathcal{A}_1^T}$ in lexicographical order, and y is the vector with components $\{z_\eta - \sum_{\xi \in D_3^T(u)} c_\xi B_\xi^7(\eta)\}_{\eta \in \mathcal{A}_1^T}$ in the same order. It follows that $\Gamma_\xi \subset T \cup T_u \subset \text{star}^1(T)$, and (13) holds with $C_1 := \|M_1^{-1}\|(1 + \tilde{C}_0 \|M_0^{-1}\|)$. Now for each type-1 vertex u of $\triangle$, we can use the smoothness condition $s \in C^3(u)$ to determine the B-coefficients of s corresponding to domain points ξ in each of the sets $D_3^T(u)$, where T is a triangle sharing the vertex u. For these ξ, we have $\Gamma_\xi \subset \text{star}^2(T)$ and (13) holds with a constant $\tilde{C}_1$ depending on C_1 and the smallest angle in $\triangle$.

Next, let $T := T_i \in \mathcal{T}_2$. By definition of $\mathcal{T}_2$, T must share exactly two vertices, say u, v, with one or more triangles in $\{T_1, \ldots, T_{i-1}\} \cap (\mathcal{T}_0 \cup \mathcal{T}_1)$. Lemma 13 implies that u and v are both vertices of type 0 or 1, and hence the B-coefficients of $s|_T$ corresponding to $D_3^T(u)$ and $D_3^T(v)$ are already known. If T is not a neighbor of any of the triangles $\{T_1, \ldots, T_{i-1}\}$, then the remaining coefficients of $s|_T$ are uniquely determined by the interpolation conditions corresponding to the 16 points in the set $\mathcal{A}_{2a}^T$ defined in (10). These coefficients can be computed by solving a linear system with the matrix M_{2a} introduced in Sect. 3. It follows that for all $\xi \in \mathcal{D}_T$, $\Gamma_\xi \subset \text{star}^2(T)$, and (13) holds with a constant C_2 depending on $\tilde{C}_1$ and $\|M_{2a}^{-1}\|$. If T shares the edge $e := \langle u, v \rangle$ with some triangle $\tilde{T}$ in $\{T_1, \ldots, T_{i-1}\}$, then the B-coefficients of $s|_T$ in the sets $E_m^T(e)$, $m = 0, 1, 2$, are uniquely determined by the C^2 smoothness conditions across e. Then the remaining coefficients can again be uniquely computed by interpolation at the points of $\mathcal{A}_{2b}^T$, which in this case contains the 13 points of the set $\mathcal{A}_{2b}$ defined in (10), see also Fig. 1. Again we get $\Gamma_\xi \subset \text{star}^2(T)$, and (13) holds with a constant C_2 depending on $\tilde{C}_1$ and $\|M_{2b}^{-1}\|$. Now for each type-2 vertex u of $\triangle$, we can use the smoothness condition $s \in C^3(u)$ to determine the B-coefficients of s corresponding to domain points ξ in each of the sets $D_3^T(u)$, where T is a triangle sharing the vertex u. For these ξ, we have $\Gamma_\xi \subset \text{star}^3(T)$ and (13) holds with a constant $\tilde{C}_2$ depending on C_2 and the smallest angle in $\triangle$.

The next step is to consider $T := T_i = \langle u, v, w \rangle \in \mathcal{T}_3$. Now T touches triangles in $\{T_1, \ldots, T_{i-1}\}$ at all three vertices, which by Lemma 13 must be of type 0,1 or 2, i.e. in $\mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2$. Thus, the coefficients of $s|_T$ corresponding to $D_3^T(u)$, $D_3^T(v)$, and $D_3^T(w)$ are already known. Now depending on whether T shares an edge with a triangle in $\{T_1, \ldots, T_{i-1}\}$, we can uniquely compute the remaining coefficients of $s|_T$ by interpolation at the 6 points in $\mathcal{A}_{3a}$ or the 3 points in $\mathcal{A}_{3b}$. In either case we get $\Gamma_\xi \subset \text{star}^3(T)$, and (13) holds with a constant C_3 depending on $\tilde{C}_2$, either $\|M_{3a}^{-1}\|$ or $\|M_{3b}^{-1}\|$, and the smallest angle in $\triangle$.

It remains to discuss the white triangles in $\triangle^*$. Coefficients in these triangles are computed either by smoothness conditions across edges of neighboring triangles or by Lemma 2. Thus, if ξ is a point in such a triangle T, then $\Gamma_\xi \subset \text{star}^5(T)$, and (13) again holds with a constant depending only on the smallest angle in $\triangle$. $\square$

Fig. 10 shows an example of a triangulation $\triangle$ where the worst case of Theorem 17 occurs, *i.e.*, there is a triangle T and a point $\xi \in T$ (marked with $\odot$ in the figure) with $\Gamma_\xi = \text{star}^5(T)$. Points in P are shown as black dots in this figure, while domain points where coefficients are determined from smoothness conditions are marked with a $*$. In particular, if we set $z_\eta = 0$ for all $\eta \in P$ except for the z corresponding to the point marked with a $\otimes$, it is easy to see that the coefficient c_ξ can be nonzero for the point ξ marked with $\odot$. Note that as shown in the proof of Theorem 17, for most ξ the set Γ_ξ is smaller than $\text{star}^5(T)$.

Theorem 18. *Let P be the set of domain points defined in Algorithm 12, and let $\mathcal{S}$ be the spline space in Definition 16. Then P is a stable local minimal determining set for $\mathcal{S}$.*

Proof: It is easy to see that P is a determining set. Since $\#P = \dim \mathcal{S}$, it follows that it is a minimal determining set. The proof that P is local

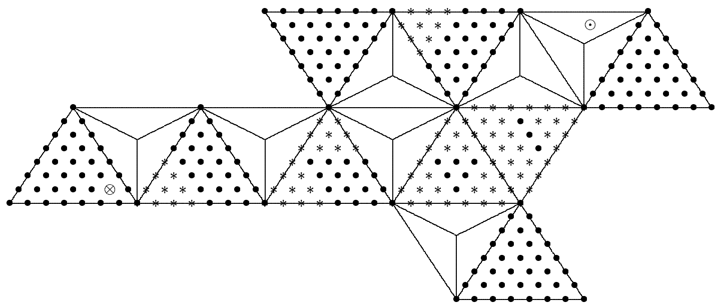

Fig. 10. An example where $\Gamma_\xi = \mathrm{star}^5(T)$ in Theorem 17.

and stable follows along the lines of the proof of Theorem 17. □

§7. Bounds on the Error of Interpolation

Suppose $P, \mathcal{S}$ is the Lagrange interpolation pair of the previous section, where $\mathcal{S}$ is defined over a triangulation $\triangle^*$ of a planar domain Ω. Then for every $f \in C(\Omega)$, there is a unique spline $s = \mathcal{I}f \in \mathcal{S}$ which satisfies $s(\eta) = \mathcal{I}f(\eta) = f(\eta)$, for all $\eta \in P$. This defines a linear projector $\mathcal{I}$ mapping $C(\Omega)$ onto $\mathcal{S}$. We now give an error bound for $f - \mathcal{I}f$ and its derivatives in the infinity norm. Similar results hold for general p-norms.

Theorem 19. *Suppose f lies in the Sobolev space $W_\infty^{m+1}(\Omega)$ for some $0 \le m \le 7$. Then*

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}f)\|_\Omega \le K |\triangle|^{m+1-\alpha-\beta} |f|_{m+1,\Omega} \tag{14}$$

for $0 \le \alpha + \beta \le m$. Here $|\cdot|_{m+1,\Omega}$ is the usual Sobolev semi-norm, and $|\triangle|$ is the mesh size of $\triangle$. The constant K depends only on the smallest angle in $\triangle$.

Proof: Fix $0 \le m \le 7$, and let $f \in W_\infty^{m+1}(\Omega)$. Let T be some triangle in $\triangle^*$, and let $\Omega_T := \mathrm{star}^5(T)$. Then it is well known (cf. Lemma 4.6 in [7] or Lemma 4.3.8 in [3]) that there exists a polynomial $q := q_f \in \mathcal{P}_m$ such that

$$\|D_x^\alpha D_y^\beta (f - q)\|_{\Omega_T} \le K_0 \; |\Omega_T|^{m+1-\alpha-\beta} |f|_{m+1,\Omega_T}, \tag{15}$$

for $0 \le \alpha + \beta \le m$, where K_0 is an absolute constant. Since $|\Omega_T| \le 11 \; |\triangle|$, it follows that

$$\|D_x^\alpha D_y^\beta (f - q)\|_{\Omega_T} \le K_1 \; |\triangle|^{m+1-\alpha-\beta} |f|_{m+1,\Omega_T}, \tag{16}$$

with $K_1 := (11)^{m+1} K_0$. Since $\mathcal{I}q = q$,

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}f)\|_T \le \|D_x^\alpha D_y^\beta (f - q)\|_T + \|D_x^\alpha D_y^\beta \mathcal{I}(f - q)\|_T.$$

By the Markov inequality [23],

$$\|D_x^\alpha D_y^\beta \mathcal{I}(f-q)\|_T \le K_2\ |\triangle|^{-(\alpha+\beta)} \|\mathcal{I}(f-q)\|_T, \tag{17}$$

for some K_2 depending on the smallest angle in T. Let $\mathcal{I}(f-q)|_T = \sum_{\xi \in \mathcal{D}_T} c_\xi B_\xi^7$. Then, since the Bernstein basis polynomials form a partition of unity, we have

$$\|\mathcal{I}(f-q)\|_T \le \max_{\xi \in \mathcal{D}_T} |c_\xi|.$$

But by Theorem 17,

$$|c_\xi| \le C \max_{\eta \in P \cap \Omega_T} |(f-q)(\eta)| \le C\|f-q\|_{\Omega_T}, \qquad \xi \in \mathcal{D}_T.$$

Combining the above inequalities leads immediately to (14). □

§8. Remarks

Remark 1. The problem of constructing Lagrange interpolation pairs for C^1 splines has been investigated in [11–13,15–18] using splines of various degrees on either triangulations or triangulated quadrangulations. A coloring of the triangulations or quadrangulations played an important role in all of the constructions. While our method here also uses coloring, it is different from the C^1 methods in that we first choose interpolation points for the black triangles, and then use certain macro-elements to extend the spline to the white triangles.

Remark 2. Here we have established error bounds for our interpolation method directly. These results could also be established using the weak interpolation techniques developed in [5,18].

Remark 3. Following the approach of this paper, it is also possible to create Lagrange interpolation pairs using C^2 splines of degree five. In particular, we can replace the macro-elements in Sect. 5 which are based on Clough-Tocher splits by alternative macro-elements based on double Clough-Tocher splits. For example, the macro-element in Lemma 5 can be replaced by the macro-element in Theorem 9.1 of [2]. The macro-element in Lemma 6 can be replaced with a similar element where three additional smoothness conditions are required, see [19]. Using these two triangular elements, we can then build four-sided elements in the same way as done here. Alternatively, we also use C^2 splines of degree five based on the split in [22], or the classical C^2 macro-elements which are based on polynomials of degree nine and do not require any splits.

Remark 4. In treating the black triangles, it is possible to use polynomials of degree less than seven, and then degree-raise them before extending into the white triangles. For example, on triangles of class $\mathcal{T}_0$, we could even use linear polynomials (which would require no additional interpolation points at all in the triangles of class $\mathcal{T}_0$). The resulting algorithm would be simpler, but of course there would be a corresponding loss in approximation order.

Remark 5. Local Lagrange interpolation methods are useful for the construction and reconstruction of surfaces and scattered data fitting problems, especially since they do not require any derivatives. For example, the method here could be used in a two-stage process, where in the first stage one constructs a C^0 linear spline based on a very fine triangulation which in turn is interpolated by our C^2 method on a coarser triangulation. See [12,16,18] for examples of such two-stage methods.

Remark 6. In this paper, we have made heavy use of the Bernstein-Bézier representation of splines as a theoretical tool. But the Bernstein-Bézier representation is also of practical importance since all of the computations needed to construct a spline can be done directly with the Bernstein-Bézier-coefficients. In particular, there is no need to construct basis functions for any of the spaces used here.

Remark 7. As noted above, the matrices which arise in the various linear systems arising in the computation of our interpolating spline do not depend on the size or shape of triangles in the triangulation. This means that there are only a small number of fixed matrices which can be precomputed and inverted once and for all.

Remark 8. Given a Lagrange interpolation pair $P, \mathcal{S}$, it is clear that for each $\xi \in P$, there exists a unique spline L_ξ such that

$$L_\xi(\eta) = \delta_{\xi,\eta}, \qquad \eta \in P.$$

These are the fundamental splines or cardinal splines associated with P. Following the arguments in the proof of Theorem 17, it can be seen that for all $\xi \in P \cap T$, the support of L_ξ is contained in $\mathrm{star}^5(T)$, see Fig. 10.

Remark 9. In Theorem 18, we showed that the set of domain points P of Algorithm 12 is a local stable MDS for the spline space $\mathcal{S}$ of Definition 16. It follows that for each point $\xi \in P$, there exists a unique spline B_ξ such that $c_\xi = 1$ and $c_\eta = 0$ for all $\eta \in P \setminus \xi$. These basis functions are different from the basis functions in Remark 8, and in general have smaller supports.

Acknowledgments. L. L. Schumaker was supported in part by the Army Research Office under grant DAAD-190210059.

References

1. Alfeld, P. and L. L. Schumaker, Smooth Macro-elements based on Clough-Tocher triangle splits, Numer. Math. **90** (2002), 597–616.
2. Alfeld, P. and L. L. Schumaker, Upper and lower bounds on the dimension of superspline spaces, Constr. Approx. **19** (2003), 145–161.
3. Brenner, S. C. and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, New York, 1994.
4. Cowen, L.J., W. Goddard, and C. E. Jesurum, Coloring with defect, in: *Proceedings of the 8th Annual ACM Symposium on discrete algorithms*, SIAM, 1997, 548-557.
5. Davydov, O., G. Nürnberger, and F. Zeilfelder, Bivariate spline interpolation with optimal approximation order, Constr. Approx. **17** (2001), 181–208.
6. Jensen, T. R. and B. Toft, *Graph Coloring Problems*, Wiley, New York, 1995.
7. Lai, M.-J. and L. L. Schumaker, On the approximation power of bivariate splines, Advances in Comp. Math. **9** (1998), 251–279.
8. Lai, M.-J. and L. L. Schumaker, Macro-elements and stable bases for splines on Clough-Tocher triangulations, Numer. Math. **88** (2001), 105–119.
9. Lai, M.-J. and L. L. Schumaker, Quadrilateral macro-elements, SIAM J. Numer. Anal. **39** (2002), 1732–1748.
10. Lovász, L., On decomposition of graphs, Studia Sci. Math. Hungar. **1** (1966), 237–238.
11. Nürnberger, G., L. L. Schumaker, and F. Zeilfelder, Local Lagrange interpolation by bivariate C^1 cubic splines, in *Mathematical Methods for Curves and Surfaces III, Oslo, 2000*, T. Lyche and L. L. Schumaker (eds), Vanderbilt University Press, Nashville, 2001, 393–404.
12. Nürnberger, G., L. L. Schumaker, and F. Zeilfelder, Lagrange interpolation by C^1 cubic splines on triangulations of separable quadrangulations, in *Approximation Theory X: Splines, Wavelets, and Applications*, C. K. Chui, L. L. Schumaker, and J. Stöckler (eds), Vanderbilt University Press, Nashville, 2002, 405–424.
13. Nürnberger, G., L. L. Schumaker, and F. Zeilfelder, Lagrange interpolation by C^1 cubic splines on triangulated quadrangulations, Advances in Comp. Math, to appear.
14. Nürnberger, G. and F. Zeilfelder, Developments in bivariate spline interpolation, J. Comput. Appl. Math. **121** (2000), 125–152.
15. Nürnberger, G. and F. Zeilfelder, Local Lagrange interpolation by cubic splines on a class of triangulations, in *Trends in Approximation*

Theory, K. Kopotun, T. Lyche, and M. Neamtu (eds), Vanderbilt University Press, Nashville, 2001, 341-350.

16. Nürnberger, G. and F. Zeilfelder, Local Lagrange interpolation on Powell-Sabin triangulations and terrain modelling, in *Recent Progress in Multivariate Approximation*, W. Haußmann, K. Jetter, and M. Reimer (eds.), Birkhäuser ISNM 137, Basel, 2001, 227–244.
17. Nürnberger, G. and F. Zeilfelder, Fundamental splines on triangulations, in *Proc. Conference on Multivariate Approximation, 2002,* W. Haußmann, K. Jetter and J. Stöckler, (eds.), Wiley-VCH, Berlin, 215–233.
18. Nürnberger, G. and F. Zeilfelder, Lagrange interpolation by bivariate C^1 splines with optimal approximation order, Advances in Comp. Math, to appear.
19. Rayevskaya, V., Lagrange Interpolation and n-sided macro-element spaces for splines on triangulations, dissertation, Vanderbilt, 2003.
20. Rayevskaya, V. and L. L. Schumaker, Multi-sided macro-element spaces based on Clough-Tocher triangle splits, submitted.
21. Schumaker, L. L., A conjecture on determinants of Bernstein-Basis polynomials, private communication, 2003.
22. Wang, T., A C^2-quintic spline interpolation scheme on triangulation, Comput. Aided Geom. Design **9** (1992), 379–386.
23. Wilhelmsen, D. R., A Markov inequality in several dimensions, J. Approx. Theory **11** (1974), 216–220.

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