C^1 Quintic Splines on Type-4 Tetrahedral Partitions

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Abstract. Starting with a partition of a rectangular box into subboxes, it is shown how to construct a natural tetrahedral (type-4) partition and associated trivariate C^1 quintic polynomial spline spaces with a variety of useful properties, including stable local bases and full approximation power. It is also shown how the spaces can be used to solve certain Hermite and Lagrange interpolation problems.

§1. Introduction

Let $\triangle$ be a tetrahedral partition of a set Ω in $\mathbb{R}^3$. Then for any integers $0 \le r \le d$, the associated space of polynomial splines of degree d and smoothness r is defined by

$$\mathcal{S}_d^r(\triangle) := \{s \in C^r(\Omega) : s|_T \in \mathcal{P}_d, \text{ all tetrahedra } T \in \triangle\},$$

where $\mathcal{P}_d$ is the space of trivariate polynomials of degree d. While $\mathcal{S}_d^r(\triangle)$ is clearly the natural analog of the heavily-studied bivariate polynomial splines on triangulations (see [13]), much less is known about trivariate spline spaces. Indeed, even for the case $r = 1$ (which we focus on here), we cannot calculate the dimension of $\mathcal{S}_d^1(\triangle)$ for general partitions $\triangle$, let alone construct the stable local bases which would be needed for applications.

Although there is no general theory, there are a few C^1 trivariate (super) spline spaces which have been shown to be useful for applications. These include

1) classical finite-element spaces with $d = 9$ on general tetrahedral partitions, see [18],
2) finite-element spaces with $d = 5$ on (Clough-Tocher) subpartitions of $\triangle$ where every tetrahedron in $\triangle$ is split into four subtetrahedra, see [1,11],
3) finite-element spaces with $d = 3$ on (alternative Clough-Tocher) subpartitions of $\triangle$ where every tetrahedron in $\triangle$ is split into twelve subtetrahedra, see [16],

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4) finite-element spaces with $d = 2$ on (Powell-Sabin) subpartitions of $\triangle$ where every tetrahedron in $\triangle$ is split into twenty four subtetrahedra, see [17].

The purpose of this paper is to show that if we choose $\triangle$ to be a special type of partition (which we call a type-4 partition, see Definition 2.2 below), then it is possible to construct useful C^1 superspline spaces with $d = 5$ without the additional complication of having to split all of the tetrahedra in $\triangle$. In particular, we will construct spaces with stable local bases and with full approximation power which can be used to solve certain trivariate interpolation problems.

The paper is organized as follows. In Section 2 we introduce type-4 partitions, while in Section 3 we collect several facts about trivariate polynomials. Section 4 contains dimension results on a key superspline subspace $\mathcal{S}_5^{1,2}(\triangle)$. A local basis for $\mathcal{S}_5^{1,2}(\triangle)$ is constructed in Section 5, and is used in the following section to define a certain quasi-interpolation operator which is shown to provide optimal order approximation of smooth functions. In Section 7 we examine a particularly useful superspline subspace of $\mathcal{S}_5^{1,2}(\triangle)$ which has fewer parameters but the same approximation order. This space is applied in Sections 8 and 9 to solve natural Hermite and Lagrange interpolation problems. We conclude the paper with several remarks.

§2. Type-4 Tetrahedral Partitions

Let $B := [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ be a rectangular box in $\mathbb{R}^3$, and let

$$\begin{aligned} a_1 &= x_0 < x_1 < \cdots < x_m = b_1 \\ a_2 &= y_0 < y_1 < \cdots < y_n = b_2 \\ a_3 &= z_0 < z_1 < \cdots < z_l = b_3. \end{aligned} \tag{2.1}$$

Let $\mathcal{V} := \{(x_i, y_j, z_k)\}$ and let $\mathcal{B} := \{B_{ijk}\}$ be the set of $N := m \times n \times l$ subboxes

$$B_{ijk} := [x_i, x_{i+1}] \times [y_j, y_{j+1}] \times [z_k, z_{k+1}]$$

defined by the grid (2.1).

Lemma 2.1. *The set $\mathcal{V}$ can be divided into two sets $\mathcal{V}_1$ and $\mathcal{V}_2$ such that for every vertex $v \in \mathcal{V}_\nu$, all of its vertices sharing an edge with v are in $\mathcal{V}_\mu$, where $\nu \neq \mu$.*

Proof: For $N = 1$, we can choose

$$\mathcal{V}_1 := \{(x_0, y_0, z_0), (x_0, y_1, z_1), (x_1, y_0, z_1), (x_1, y_1, z_0)\},$$

and $\mathcal{V}_2$ to be the set of four remaining vertices of B. The result for the general case follows by a triple induction on m, n, and l. $\square$

Clearly, for any partition of B into subboxes, there is more than one way to choose the sets $\mathcal{V}_1$ and $\mathcal{V}_2$. We say that vertices in $\mathcal{V}_1$ are of type-1, while those in $\mathcal{V}_2$ are of type-2.

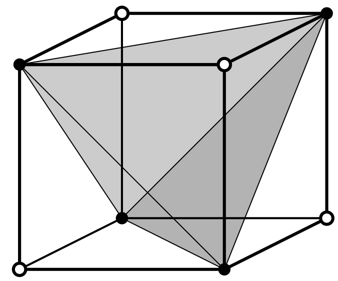

Fig. 1. The partition of a subbox into five tetrahedra.

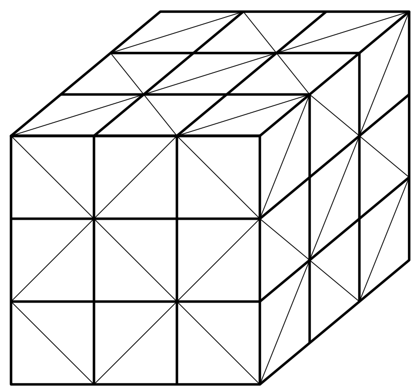

Fig. 2. A type-4 partition of a box.

Definition 2.2. *Given a partition of a box B into N subboxes as above, suppose $\triangle$ is the collection of tetrahedra which is obtained by splitting each subbox of B into five tetrahedra by connecting its four type-2 vertices with each other. We call $\triangle$ a* type-4 partition *of B.*

Figure 1 shows the split of a single subbox into five tetrahedra. Type-1 vertices are shown with white dots, while type-2 vertices are shown with black dots. The tetrahedron whose four vertices are all of type-2 is defined by the four shaded faces. We call it a type-2 tetrahedron. We refer to each of the other four tetrahedra in the subbox as a type-1 tetrahedron. Note that each type-1 tetrahedron has exactly one type-1 vertex.

Figure 2 shows the visible edges of a type-4 partition of a box B which has been partitioned into 27 subboxes. We write $\mathcal{T}_1$ and $\mathcal{T}_2$ for the sets of type-1 and type-2 tetrahedra in $\triangle$, respectively.

§3. B-form Representation of Trivariate Polynomial Splines

We make use of standard Bernstein–Bézier techniques (cf. [5,6]). We recall that given a tetrahedron $T := \langle v_1, v_2, v_3, v_4 \rangle$, any polynomial $p \in \mathcal{P}_d$ can be written in

Bernstein–Bézier (B-) form as

$$p := \sum_{i+j+k+l=d} c_{ijkl} B^d_{ijkl},$$

where $\{B^d_{ijkl}\}_{i+j+k+l=d}$ are the Bernstein polynomials of degree d associated with T. As usual, we identify the coefficients $\{c_{ijkl}\}$ with the corresponding domain points

$$\mathcal{D}_T := \{\xi^T_{ijkl} := \frac{iv_1 + jv_2 + kv_3 + lv_4}{d}\}_{i+j+k+l=d}.$$

Given an arbitrary tetrahedralization $\triangle$, let

$$\mathcal{D}_\triangle := \bigcup_{T\in\triangle} \mathcal{D}_T$$

be the corresponding set of domain points.

Clearly, every $s \in \mathcal{S}^0_d(\triangle)$ is uniquely defined by a set of coefficients $\{c_\xi\}_{\xi\in\mathcal{D}_\triangle}$. If $s \in C^r(B)$ with $r \geq 1$, then these coefficients cannot be chosen arbitrarily, but must satisfy an appropriate set of smoothness conditions. To describe these, suppose that $T := \langle v_1, v_2, v_3, v_4\rangle$ and $\tilde{T} := \langle v_5, v_2, v_3, v_4\rangle$ are two adjoining tetrahedra sharing the face $F := \langle v_2, v_3, v_4\rangle$. Suppose

$$s|_T = \sum_{i+j+k+l=d} c_{ijkl} B^d_{ijkl},$$

$$s|_{\widetilde{T}} = \sum_{i+j+k+l=d} \tilde{c}_{ijkl} \tilde{B}^d_{ijkl},$$

where $\{\widetilde{B}^d_{ijkl}\}_{i+j+k+l=d}$ are the Bernstein polynomials of degree d associated with $\widetilde{T}$.

Given $1 \leq i \leq d$, let

$$\tau^i_{jkl} := c_{ijkl} - \sum_{\nu+\mu+\kappa+\ell=i} \tilde{c}_{\nu,j+\mu,k+\kappa,l+\ell} \tilde{B}^i_{\nu\mu\kappa\ell}(v_1). \tag{3.1}$$

for all $j + k + l = d - i$. Following [4], we call τ^i_{jkl} a smoothness functional of order i. Note that for a given pair of adjoining tetrahedra, this functional is uniquely associated with the domain point $\xi^T_{ijkl} \in \mathcal{D}_T$. It is well known that the spline s is C^r continuous across the face F if and only if

$$\tau^i_{jkl} s = 0, \qquad \text{for all } j + k + l = d - i \text{ and } i = 0, \ldots, r.$$

For later use, we recall (see [10]) that there exists a constant K_1 depending only on d such that for every unit vector u and every $p \in \mathcal{P}_d$, the directional derivative $D_u p$ satisfies

$$\|D_u p\|_T \leq \frac{K_1}{\rho_T} \|p\|_T \tag{3.2}$$

where ρ_T is the diameter of the largest disk contained in T. Here $\|\cdot\|_T$ denotes the supremum norm on T.

§4. The Superspline Subspace $\mathcal{S}_5^{1,2}(\triangle)$

Despite its simple form, the space $\mathcal{S}_5^1(\triangle)$ is not suitable for our purposes — indeed, we do not even have a formula for its dimension. To overcome this difficulty, we focus instead on the superspline subspace

$$\mathcal{S}_5^{1,2}(\triangle) := \{s \in \mathcal{S}_5^1(\triangle) : s \in C^2(v) \text{ for all vertices of } \triangle\},$$

where $C^2(v)$ means that all polynomial pieces of s defined on tetrahedra sharing the vertex v have common derivatives up to order 2. Our aim in this section is to prove the following theorem.

Theorem 4.1. *Suppose B is a box which has been subdivided into N subboxes, and that $\triangle$ is an associated type-4 partition. Then*

$$\dim \mathcal{S}_5^{1,2}(\triangle) = 10N_2 + 2E_2 + 4N + 17(N_{1,1} + N_{1,2}) + 14N_{1,4} + 5N_{1,8},$$

where N_2 is the number of type-2 vertices in $\triangle$, E_2 is the number of edges of type-2 tetrahedra in $\triangle$, and $N_{1,i} := \#\mathcal{V}_{1,i}$ with

$$\mathcal{V}_{1,i} := \{v \in \mathcal{V}_1 : v \text{ is a vertex of exactly } i \text{ tetrahedra in } \triangle\} \tag{4.1}$$

for $i \in \{1, 2, 4, 8\}$.

In preparation for the proof of this result, we first recall some standard notation. Given a tetrahedron $T := \langle v, u, w, t\rangle$ and an integer $0 \le m \le 5$, we define the ring $R_m^T(v)$ of radius m around v to be the set of domain points of the form $\xi^T_{5-m,j,k,l}$ with $j + k + l = m$. We call the set $D_m^T(v) := R_0^T(v) \cup \cdots \cup R_m^T(v)$ the disk of radius m around v. If v is a vertex of $\triangle$, we define the ring $R_m(v)$ of radius m around v to be the union of the rings $R_m^T(v)$ over all tetrahedra sharing the vertex v. Finally, we define the disk of radius m around v to be $D_m(v) := R_0(v) \cup \cdots \cup R_m(v)$.

Given a vertex v of $\triangle$, we recall that star(v) is just the set of all tetrahedra in $\triangle$ which share the vertex v. Given any edge $\langle v, u\rangle$ in $\triangle$, we define the set of domain points at level i with respect to the edge $\langle v, u\rangle$ to be

$$L^i_{\langle v,u\rangle} := R_{5-i}(u) \cap \text{star}(v), \qquad 0 \le i \le 5.$$

We now identify certain subsets of the set $\mathcal{D}_\triangle$ of domain points which will be used to construct a minimal determining set for the spline space $\mathcal{S}_5^{1,2}(\triangle)$, *i.e.*, a subset $\mathcal{M}$ of $\mathcal{D}_\triangle$ such that prescribing the B-coefficients corresponding to $\xi \in \mathcal{M}$ uniquely determines a spline $s \in \mathcal{S}_5^{1,2}(\triangle)$. Let

$$\mathcal{M}_{2,1} := \bigcup_{v \in \mathcal{V}_2} \{D_2^T(v) : T \text{ is some tetrahedron in } \mathcal{T}_2 \text{ attached to } v\},$$

$$\mathcal{M}_{2,2} := \bigcup_{\langle v,u\rangle \in \mathcal{E}_2} \{D_3^T(v) \cap D_3^T(u) : T \text{ is some tetrahedron in } \mathcal{T}_2 \text{ containing } \langle v,u\rangle\},$$

$$\mathcal{M}_{2,3} := \bigcup_{T \in \mathcal{T}_2} \{\xi^T_{2111}, \xi^T_{1211}, \xi^T_{1121}, \xi^T_{1112}\},$$

where $\mathcal{E}_2$ is the set of edges of type-2 tetrahedra. In addition, let

$$\mathcal{M}_{1,1} := \bigcup_{v \in \mathcal{V}_{1,1}} \{\mathcal{D}_T \setminus \left[D_2^T(v) \cup D_2^T(u) \cup D_2^T(w) \cup R_4(t)\right] : T = \langle v, u, w, t\rangle \in \mathcal{T}_1\}.$$

For each vertex $v \in \mathcal{V}_1$, let u_v be such that it is a vertex of a maximal number of tetrahedra in star (v). For each $k = 0, 1, 2$, let $\mathcal{M}_{1,2}^k(v)$, $\mathcal{M}_{1,4}^k(v)$, and $\mathcal{M}_{1,8}^k(v)$ be the subsets of $L^k_{\langle v, u_v\rangle}$ shown with grey dots in Figures 3–6, and let

$$\mathcal{M}_{1,2} := \bigcup_{v \in \mathcal{V}_{1,2}} \bigcup_{k=0}^{2} \mathcal{M}_{1,2}^k(v),$$

$$\mathcal{M}_{1,4} := \bigcup_{v \in \mathcal{V}_{1,4}} \bigcup_{k=0}^{1} \mathcal{M}_{1,4}^k(v),$$

$$\mathcal{M}_{1,8} := \bigcup_{v \in \mathcal{V}_{1,8}} \bigcup_{k=0}^{1} \mathcal{M}_{1,8}^k(v).$$

Note that $\#\mathcal{M}_{2,1} = 10N_2$, $\#\mathcal{M}_{2,2} = 2E_2$, $\#\mathcal{M}_{2,3} = 4N$, $\#\mathcal{M}_{1,1} = 17N_{1,1}$, $\#\mathcal{M}_{1,2} = 17N_{1,2}$, $\#\mathcal{M}_{1,4} = 14N_{1,4}$, and $\#\mathcal{M}_{1,8} = 5N_{1,8}$. Theorem 4.1 is an immediate consequence of the following result.

Theorem 4.2. *Let $\triangle$ be a type-4 tetrahedral partition of a box B. Then the set*

$$\mathcal{M} := \mathcal{M}_{2,1} \cup \mathcal{M}_{2,2} \cup \mathcal{M}_{2,3} \cup \mathcal{M}_{1,1} \cup \mathcal{M}_{1,2} \cup \mathcal{M}_{1,4} \cup \mathcal{M}_{1,8}$$

is a minimal determining set for $\mathcal{S}_5^{1,2}(\triangle)$.

Proof: To show that $\mathcal{M}$ is a minimal determining set, suppose that $s \in \mathcal{S}_5^{1,2}(\triangle)$ and that the coefficients corresponding to the points in $\mathcal{M}$ have been prescribed. We need to show that all remaining coefficients are *uniquely* defined by smoothness conditions. First we observe that for all $v \in \mathcal{V}_2$, the coefficients in $D_2(v)$ are uniquely defined by $\mathcal{M}_{2,1}$ and the C^2 continuity at v. The remaining coefficients in any type-2 tetrahedron are uniquely determined by $\mathcal{M}_{2,2}$, $\mathcal{M}_{2,3}$, and the C^1 continuity.

Now consider the remaining coefficients in type-1 tetrahedra. In general, each such tetrahedron T shares exactly one face $F := \langle u_1, u_2, u_3\rangle$ with a type-2 tetrahedron. Suppose v is the type-1 vertex of T which is opposite the face F. Then clearly all coefficients on $R_4^T(v)$ are uniquely determined by the C^1 continuity conditions across F. Note that all coefficients in $D_2(u_i)$ are already determined since $u_i \in \mathcal{V}_2$ for $i = 1, 2, 3$.

We divide the analysis into four cases. First we consider the case where T is a type-1 tetrahedron attached to a vertex $v \in \mathcal{V}_{1,1}$. In this case all remaining coefficients are determined by $\mathcal{M}_{1,1}$.

Suppose now that $T_1 := \langle v, u_1, u_2, u_v \rangle$ and $T_2 := \langle v, u_1, u_3, u_v \rangle$ are two type-1 tetrahedra sharing the face $F := \langle v, u_1, u_v \rangle$, where $v \in \mathcal{V}_{1,2}$. Note that the faces $\langle v, u_1, u_3 \rangle$, $\langle v, u_1, u_2 \rangle$, $\langle v, u_v, u_3 \rangle$, and $\langle v, u_v, u_2 \rangle$ all lie on the boundary of B. We now discuss the coefficients corresponding to domain points on the levels $L^k_{\langle v,u_v \rangle}$ for $k = 0, \ldots, 5$. It suffices to consider $k = 0, 1, 2$, since the coefficients corresponding to the points on the other levels are already uniquely determined. For level $L^0_{\langle v,u_v \rangle}$ we refer to Figure 3. Coefficients corresponding to black dots are determined from smoothness conditions across faces shared with neighboring type-2 tetrahedra. The 9 points in $\mathcal{M}^0_{1,2}(v)$ are shown as grey dots. The coefficients on this level corresponding to points marked with $\otimes$ are determined by C^1 continuity across the face F, while the coefficient corresponding to the point marked with a white dot is determined by a C^2 smoothness condition at v. Level $L^1_{\langle v,u_v \rangle}$ is also shown in Figure 3. As before, the coefficients corresponding to black dots are already determined. The 6 points in $\mathcal{M}^1_{1,2}(v)$ are shown as grey dots, and the C^1 smoothness conditions across F uniquely determine the remaining three coefficients at this level (corresponding to the points marked with $\otimes$). Finally, we consider $L^2_{\langle v,u_v \rangle}$ as shown in Figure 6. In this case, $\mathcal{M}^2_{1,2}(v)$ contains the two points shown in grey, and the C^1 smoothness across F uniquely determines the coefficients corresponding to the two points marked with $\otimes$.

Suppose we now have four type-1 tetrahedra sharing an edge $e := \langle v, u_v \rangle$ where $v \in \mathcal{V}_{1,4}$. Figure 4 shows the domain points in $L^0_{\langle v,u_v \rangle}$. Suppose we set the 10 coefficients corresponding to $\mathcal{M}^0_{1,4}(v)$, shown in grey in the figure. Then the C^1 smoothness conditions across the faces $F_i := \langle u_v, u_i, v \rangle$ uniquely determine the coefficients corresponding to points marked with $\otimes$. Note that one of these coefficients is determined twice, but we get the same value either way. Now the C^2 smoothness conditions at v uniquely determine the two coefficients marked with white dots. On level $L^1_{\langle v,u_v \rangle}$ we set the 4 points in $\mathcal{M}^1_{1,4}$ shown in grey in Figure 4. These uniquely determine the coefficients corresponding to the remaining points at this level. Finally, on level $L^2_{1,4}(v)$, shown in Figure 6 (right), we use C^1 continuity across the faces F_i to determine the four coefficients corresponding to the points marked with $\otimes$. The coefficient corresponding to the intersection of the diagonals is determined twice, but we get the same value either way.

To conclude the proof, suppose now that we have eight type-1 tetrahedra sharing a vertex $v \in \mathcal{V}_{1,8}$. Let u_v be one of the other 6 (type-2) vertices, and let $\tilde{u}_v$ be the corresponding vertex such that $u_v, v, \tilde{u}_v$ are collinear. Let the remaining vertices be $u_1, \ldots, u_4$. We first consider the domain points on level $L^0_{\langle v,u_v \rangle}$, see Figure 5. The four coefficients corresponding to domain points marked with $\odot$ are uniquely determined by C^1 continuity conditions across the faces $G_i := \langle v, u_i, u_{i+1} \rangle$ for $i = 1, \ldots, 4$, where $u_5 = u_1$. Now for each $i = 1, \ldots, 4$, the coefficient corresponding to the domain point marked with an $\otimes$ and lying on $\langle v, u_i \rangle$ is determined by a C^1 continuity condition across the face $F_i := \langle u_v, u_i, v \rangle$. Now suppose we set the two coefficients corresponding to the set $\mathcal{M}^0_{1,8}(v)$. Then the four coefficients corresponding to domain points marked with white dots are uniquely determined by the C^2 smoothness at v. The remaining three coefficients corresponding to points marked with $\otimes$ are determined by C^1 continuity across the faces F_i. Now consider $L^1_{\langle v,u_v\rangle}$, see Figure 5. Here the coefficient marked with a white dot is determined by the C^2 continuity at v. Then setting the three coefficients in $\mathcal{M}^1_{1,8}(v)$ (marked in grey), the remaining coefficients at this level can be uniquely computed from C^1 continuity conditions across the faces F_i. Finally, we apply the C^1 continuity conditions across the faces G_i to determine all remaining coefficients corresponding to points on level $L^1_{\langle v,\tilde{u}_v\rangle}$.

We should point out that the coefficients corresponding to the four points in $\mathcal{M}^0_{1,8}$ at level $L^0_{\langle v,u_v\rangle}$ marked with $\otimes$ and lying on the edges $\langle v,u_i\rangle$ are each determined for a second time by C^1 smoothness conditions across the faces G_i. However, no incompatibilities arise as each of these points can be regarded as the center point in a set of the form $L^2_{\langle v,u_i\rangle}$ as shown in Figure 6 (right). □

The idea behind the construction of the minimal determing set $\mathcal{M}$ is to first set enough coefficients to determine a spline $s \in \mathcal{S}^{1,2}_5(\triangle)$ on each of the type-2 tetrahedra in $\triangle$. Type-1 tetrahedra are then used to smoothly join together the resulting polynomial pieces. Only a few additional coefficients are prescribed in each of these tetrahedra.

§5. Stable Local Bases for $\mathcal{S}^{1,2}_5(\triangle)$

Let $\mathcal{M}$ be the minimal determining set for $\mathcal{S}^{1,2}_5(\triangle)$ described in Theorem 4.2. For each $\xi \in \mathcal{M}$, let B_ξ be the unique spline in $\mathcal{S}^{1,2}_5(\triangle)$ such that

$$\lambda_\eta B_\xi = \delta_{\eta,\xi}, \qquad \eta,\xi \in \mathcal{M}, \tag{5.1}$$

where λ_η is the linear functional on $\mathcal{S}^0_5(\triangle)$ which picks off the B-coefficient of s corresponding to the domain point η.

Associated with the partition of B defined in (2.1), let

$$\beta_x := \max_{\substack{1\le i,j\le m-1\\ |i-j|\le 1}} \frac{(x_{i+1}-x_i)}{(x_{j+1}-x_j)},$$

and let β_y and β_z be defined similarly. We write $\|\cdot\|$ for the ∞-norm of either a vector or a function.

Theorem 5.1. *The set of basis splines $\{B_\xi\}_{\xi\in\mathcal{M}}$ is locally supported and stable in the sense that*

1) *for each $\xi \in \mathcal{M}$, there exists a vertex v_ξ with*

$$\text{supp } B_\xi \subset \text{star}^2(v_\xi), \tag{5.2}$$

where $\text{star}^2(v) := \text{star}(\text{star}(v))$,

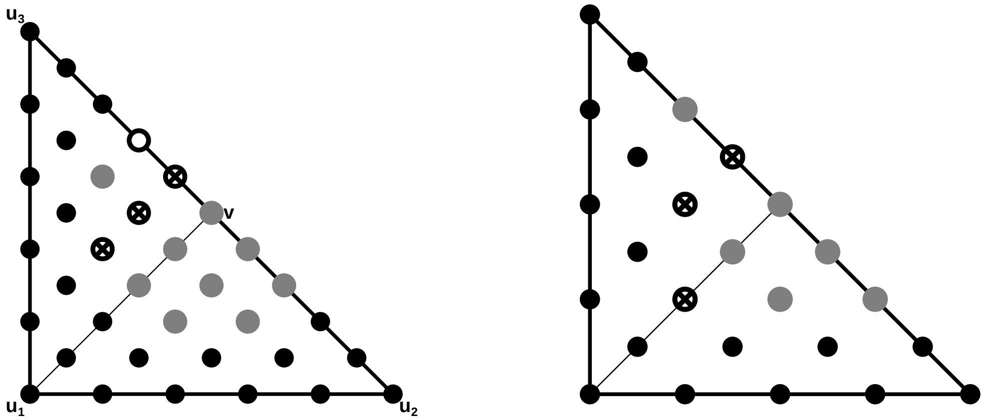

Fig. 3. The sets $\mathcal{M}^0_{1,2} \subset L^0_{\langle v,u_v\rangle}$ and $\mathcal{M}^1_{1,2} \subset L^1_{\langle v,u_v\rangle}$, $v \in \mathcal{V}_{1,2}$.

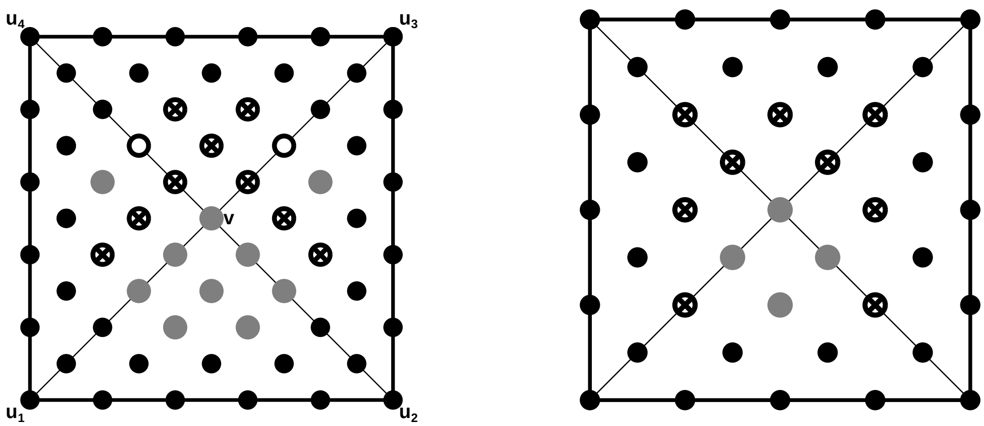

Fig. 4. The sets $\mathcal{M}^0_{1,4} \subset L^0_{\langle v,u_v\rangle}$ and $\mathcal{M}^1_{1,4} \subset L^1_{\langle v,u_v\rangle}$, $v \in \mathcal{V}_{1,4}$.

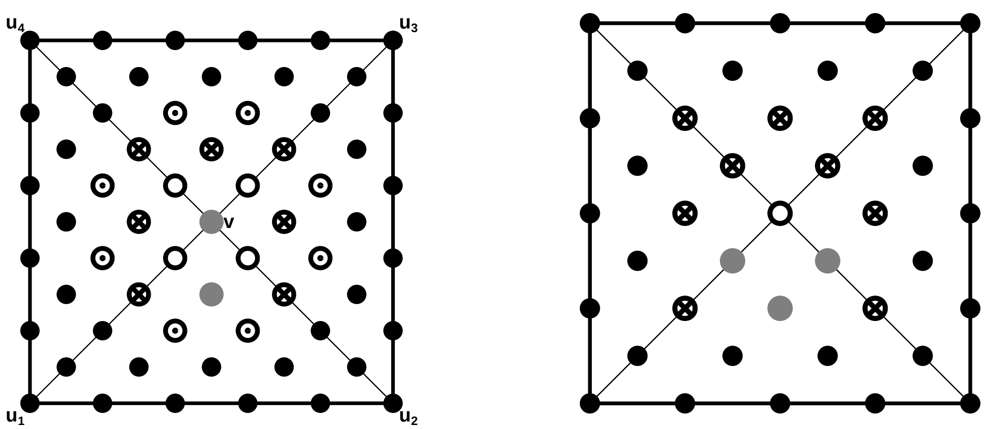

Fig. 5. The sets $\mathcal{M}^0_{1,8} \subset L^0_{\langle v,u_v\rangle}$ and $\mathcal{M}^1_{1,8} \subset L^1_{\langle v,u_v\rangle}$, $v \in \mathcal{V}_{1,8}$.

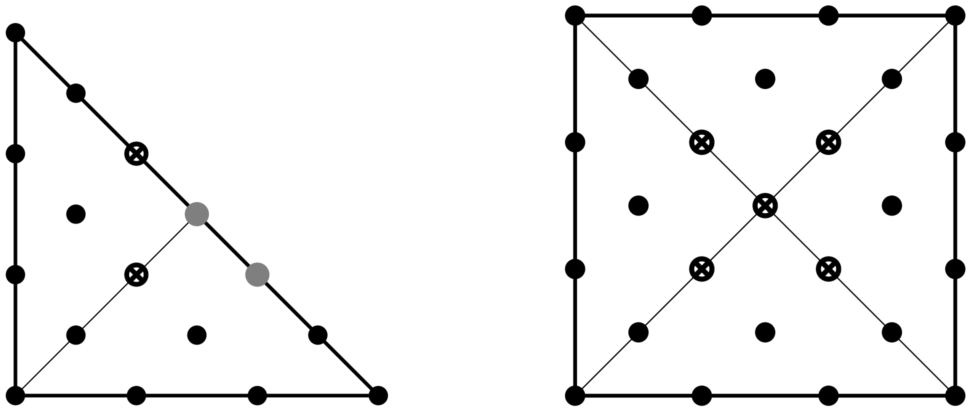

Fig. 6. The sets $\mathcal{M}^2_{1,2} \subset L^2_{\langle v,u_v\rangle}$, $v \in \mathcal{V}_{1,2}$, and $L^2_{\langle v,u_v\rangle}$, $v \in \mathcal{V}_{1,4}$.

2) *there exists constants* $0 < K_2 < K_3$, *where* K_3 *depends only on* $\beta_x, \beta_y, \beta_z$, *such that*

$$K_2\|c\| \le \Big\| \sum_{\xi\in\mathcal{M}} c_\xi B_\xi \Big\| \le K_3\|c\|, \tag{5.3}$$

for all coefficient vectors $c := \{c_\xi\}$.

Proof: First we observe that type-1 tetrahedra can only appear in clusters surrounding a vertex $v \in \mathcal{V}_{1,1}\cup\mathcal{V}_{1,2}\cup\mathcal{V}_{1,4}\cup\mathcal{V}_{1,8}$. Moreover, for any such cluster, each of its boundary faces is either on the boundary of B or is shared with a type-2 tetrahedron. By construction, all of the domain points in $\mathcal{M}_1 := \mathcal{M}_{1,1}\cup\mathcal{M}_{1,2}\cup\mathcal{M}_{1,4}\cup\mathcal{M}_{1,8}$ lie in such clusters. Thus, if $\xi \in \mathcal{M}_1$, then the corresponding basis spline B_ξ must vanish identically on all type-2 tetrahedra, and hence can only have nonzero coefficients associated with domain points in the cluster to which ξ belongs. It follows that in this case the support of B_ξ is at most star$\,(v)$.

Now consider the case where $\xi \in \mathcal{M}_{2,1}\cup\mathcal{M}_{2,2}\cup\mathcal{M}_{2,3}$. Then $\xi \in D_3(v)$ for some type-2 vertex v, and B_ξ vanishes identically on all type-2 tetrahedra which do not have a vertex at v. Setting $c_\xi = 1$ can lead to nonzero coefficients corresponding to domain points in the tetrahedra which share the vertex v. If any of these tetrahedra belongs to a cluster of type-1 tetrahedra surrounding a vertex in $\mathcal{V}_{1,8}$, the support of B_ξ could be as large as star$^2(v)$, but no larger. This completes the proof of (5.2).

We now show that there exists a constant K_4 depending on $\beta_x, \beta_y, \beta_z$ such that

$$\|B_\xi\| \le K_4, \qquad \xi \in \mathcal{M}. \tag{5.4}$$

Fix $\xi \in \mathcal{M}$, and let $\{c_\eta\}_{\eta\in\mathcal{D}_\triangle}$ be the set of B-coefficients of B_ξ. Then $c_\xi = 1$ and $c_\eta = 0$ for all other $\eta \in \mathcal{M}$. The remaining coefficients of B_ξ are computed from C^1 or C^2 smoothness conditions across faces between neighboring tetrahedra. We need only consider smoothness conditions across faces interior to the support of B_ξ. Consider first the case where F is a face between a type-1 tetrahedron T_1 and a type-2 tetrahedron T_2. Then to compute a coefficient of $s|_{T_1}$ using a C^1 smoothness condition across F, we take a linear combination of four coefficients corresponding to points in T_2 with weights equal to the barycentric coordinates of the type-1 vertex of T_1 relative to T_2. It is easy to see that each of these coordinates has absolute value equal to $1/2$. To compute a coefficient of $s|_{T_1}$ using a C^2 smoothness condition involves taking linear combinations using products of these barycentric coordinates. Now suppose F is a face between two type-1 tetrahedra. In this case, for each such computation, all of the coefficients involved correspond to points lying on a line parallel to one of the coordinate axes, where the point corresponding to the coefficient being computed lies in one subbox, and the remaining points lie in a neighboring subbox. If this coordinate axis is the x-axis, then the required barycentric coordinates are bounded by $1 + \beta_x$.

We now establish stability. The lower bound in (5.3) follows immediately from the stability of the Bernstein basis polynomials B^T_{ijkl}, see Remark 10.1. For the upper bound, suppose T is a tetrahedron in $\triangle$, and let

$$\Sigma_T := \{\xi : \text{supp } B_\xi \cap T \neq \emptyset\}. \tag{5.5}$$

Now by the local support properties of the basis functions, there exists a constant K_5 such that

$$n_T := \#\Sigma_T \leq K_5. \tag{5.6}$$

But then

$$\Big\| \sum_{\xi \in \mathcal{M}} c_\xi B_\xi \Big\|_T = \Big\| \sum_{\xi \in \Sigma_T} c_\xi B_\xi \Big\|_T \leq K_4 K_5 \max_{\xi \in \Sigma_T} |c_\xi|. \tag{5.7}$$

This gives (5.3) with $K_3 := K_4 K_5$. □

§6. Approximation Order of $\mathcal{S}_5^{1,2}(\triangle)$

Let $\mathcal{M}$ be the minimal determining set described in Theorem 4.2 for the superspline space $\mathcal{S}_5^{1,2}(\triangle)$, and let $\{B_\xi\}_{\xi \in \mathcal{M}}$ be the corresponding stable local basis defined in Section 5. Given $\xi \in \mathcal{M}$, let T_ξ be a tetrahedron which contains ξ, and for any function $f \in C(B)$, suppose $I_\xi f$ is the quintic polynomial that interpolates f at the domain points $\mathcal{D}_{T_\xi}$. Let $\mu_\xi := \lambda_\xi I_\xi$, where as in (5.1), λ_ξ is the linear functional that for any $s \in \mathcal{S}_5^0(\triangle)$, picks off the B-coefficient of s corresponding to the domain point ξ. Clearly, μ_ξ is then a linear functional defined on $C(B)$.

Theorem 6.1. *For any $f \in C(B)$, let*

$$Qf := \sum_{\xi \in \mathcal{M}} (\mu_\xi f)\, B_\xi. \tag{6.1}$$

Then Q is a linear projection of $C(B)$ onto $\mathcal{S}_5^{1,2}(\triangle)$ such that $Qf(v) = f(v)$ for all $v \in \mathcal{V}$. Moreover, there exists a constant C_1 depending only on $\beta_x, \beta_y, \beta_z$ such that for any $f \in C(B)$ and any tetrahedron $T \in \triangle$,

$$\|Qf\|_T \leq C_1 \|f\|_{U_T}, \tag{6.2}$$

where

$$U_T := \bigcup_{\xi \in \Sigma_T} T_\xi,$$

and Σ_T *is defined in (5.5).*

Proof: It is clear that Q is a linear operator. Since $I_\xi p = p$ for any quintic polynomial p, it follows that

$$\mu_\xi s = \lambda_\xi I_\xi s|_{T_\xi} = \lambda_\xi s, \tag{6.3}$$

for any spline $s \in \mathcal{S}_5^{1,2}(\triangle)$. But then the duality (5.1) of the basis $\{B_\xi\}_{\xi \in \mathcal{M}}$ implies $Qs = s$, and thus Q is a projection. Now $Qf(v) = f(v)$ for all $v \in \mathcal{V}$ follows from the fact that $\mathcal{M}$ contains $\mathcal{V}$.

Now suppose $f \in C(B)$ and that T is a tetrahedron in $\triangle$. Then by (5.4),

$$\|Qf\|_T \le n_T K_4 \max_{\xi \in \Sigma_T} |\mu_\xi f|,$$

where n_T is the cardinality of Σ_T. Using the stability of polynomial interpolation at domain points (see Remark 10.1), we have

$$|\mu_\xi f| = |\lambda_\xi I_\xi f| \le K_6 \|f\|_{T_\xi} \le K_6 \|f\|_{U_T}.$$

This leads immediately to (6.2) with $C_1 := K_4 K_5 K_6$, where K_5 is the constant in (5.6). □

The operator Q defined by (6.1) is commonly called a quasi-interpolation operator. Suppose now that $W_\infty^m(B)$ is the usual Sobolev space with seminorm

$$|f|_{m,\infty,B} := \sum_{|\alpha|=m} \|D^\alpha f\|_B, \tag{6.4}$$

where D^α is the derivative operator in standard multi-index notation. Let $|\triangle|$ be the mesh size of $\triangle$, i.e., the maximum diameter of the tetrahedra in $\triangle$. Let

$$\beta := \max_{ijk} \frac{H_{ijk}}{h_{ijk}},$$

where H_{ijk} and h_{ijk} are the longest and shortest edges of the box B_{ijk}, respectively.

Theorem 6.2. *There exists a constant* C_2 *depending only on* $\beta_x, \beta_y, \beta_z$ *and* β *such that for every* $f \in W_\infty^{m+1}(B)$ *with* $0 \le m \le 5$,

$$\|D^\alpha(f - Qf)\|_B \le C_2 |\triangle|^{m+1-|\alpha|} |f|_{m+1,\infty,B}, \tag{6.5}$$

for all $0 \le |\alpha| \le m$.

Proof: The proof is similar to the proof of analogous results for bivariate splines on triangulations, see e.g. [9] or [12, Theorem 5.1]. Suppose $f \in W_\infty^{m+1}(B)$. Given

$T \in \triangle$, let U_T be as in Theorem 6.1, and let B_T be the smallest box containing U_T. Let ρ_T be the diameter of the largest disk which is contained in T, and let $|B_T|$ be the diameter of B_T. By the local support properties of the B_ξ, it is easy to see that $|B_T| \le 4|\triangle|$, and

$$\frac{|B_T|}{\rho_T} \le K_7, \qquad \text{all } T \in \triangle, \tag{6.6}$$

for some constant K_7 depending only on $\beta_x, \beta_y, \beta_z$ and β. By Lemma 4.3.8 of [7] with $\Omega = B_T$, there exists a quintic polynomial $q := q_{f,T}$ such that

$$\|D^\alpha(f-q)\|_{U_T} \le |(f-q)|_{|\alpha|,\infty,B_T} \le K_8 |B_T|^{m+1-|\alpha|} |f|_{m+1,\infty,B_T}, \tag{6.7}$$

were K_8 is a constant depending only on m, $\beta_x, \beta_y, \beta_z$ and β. Now since Q is a projection, combining (6.2) with the Markov inequality (3.2) gives

$$\begin{aligned} \|D^\alpha(f-Qf)\|_T &\le \|D^\alpha(f-q)\|_T + \|D^\alpha Q(f-q)\|_T \\ &\le \|D^\alpha(f-q)\|_T + \frac{K_1^{|\alpha|}}{\rho_T^{|\alpha|}} \|Q(f-q)\|_T \\ &\le 4^6 K_8 (1 + C_1 K_1^{|\alpha|} K_7^{|\alpha|}) |\triangle|^{m+1-|\alpha|} |f|_{m+1,\infty,B}. \end{aligned}$$

Taking the maximum over all tetrahedra $T \in \triangle$ gives (6.5). □

§7. The Superpline Space $\hat{\mathcal{S}}_5^{1,2}(\triangle)$

In this section we introduce a subspace of $\mathcal{S}_5^{1,2}(\triangle)$ which has the same approximation power as $\mathcal{S}_5^{1,2}(\triangle)$, but has fewer parameters, and is thus easier to use in applications. The space will be defined by enforcing certain individual smoothness conditions of the type described in (3.1). Let $\Lambda_{1,1}$ be the set of smoothness functionals which are required to make a spline $s \in \mathcal{S}_5^{1,2}(\triangle)$ be C^3 across interior faces of all type-1 tetrahedra with one vertex in $\mathcal{V}_{1,1}$. Let

$$\Lambda_{1,2} := \bigcup_{v \in \mathcal{V}_{1,2}} \bigcup_{k=0}^{2} \Lambda_{1,2}^k(v),$$

$$\Lambda_{1,4} := \bigcup_{v \in \mathcal{V}_{1,4}} \bigcup_{k=0}^{1} \Lambda_{1,4}^k(v),$$

$$\Lambda_{1,8} := \bigcup_{v \in \mathcal{V}_{1,8}} \bigcup_{k=0}^{1} \Lambda_{1,8}^k(v),$$

where $\Lambda_{i,m}^k(v)$ is the set of smoothness functionals corresponding to domain points indicated by the arrows in Figures 7–10. For each such arrow, the smoothness condition is taken across the face perpendicular to the arrow. Let

$$\Lambda := \Lambda_{1,1} \cup \Lambda_{1,2} \cup \Lambda_{1,4} \cup \Lambda_{1,8}, \tag{7.1}$$

and define

$$\hat{\mathcal{S}}_5^{1,2}(\triangle) := \{s \in \mathcal{S}_5^{1,2}(\triangle) : \tau s = 0, \text{ for all } \tau \in \Lambda\}.$$

Theorem 7.1. *Suppose $\triangle$ is a type-4 partition of a box B and that Λ is the set of special smoothness described in (7.1). Then*

$$\dim \hat{\mathcal{S}}_5^{1,2}(\triangle) = 10N_2 + 2E_2 + 4N + N_{1,8} + 4(N_{1,1} + N_{1,2} + N_{1,4}), \tag{7.2}$$

where N_2, E_2, N and the $N_{1,i}$ are as in Theorem 4.1.

Proof: We claim that

$$\hat{\mathcal{M}} := \mathcal{M}_{2,1} \cup \mathcal{M}_{2,2} \cup \mathcal{M}_{2,3} \cup \mathcal{V}_{1,8} \cup \hat{\mathcal{M}}_1, \tag{7.3}$$

where $\mathcal{M}_{2,1}$, $\mathcal{M}_{2,2}$, and $\mathcal{M}_{2,3}$ are as in Theorem 4.2, and

$$\hat{\mathcal{M}}_1 := \bigcup_{v \in \mathcal{V}_1 \setminus \mathcal{V}_{1,8}} \{D_1^T(v) : T \text{ is some tetrahedron attached to } v\} \tag{7.4}$$

is a minimal determining set for $\hat{\mathcal{S}}_5^{1,2}(\triangle)$. The proof is similar to the proof of Theorem 4.2. Suppose that $s \in \hat{\mathcal{S}}_5^{1,2}(\triangle)$, and that the coefficients corresponding to the points in $\hat{\mathcal{M}}$ have been prescribed. We need to show that all remaining coefficients are *uniquely* defined by smoothness conditions. This follows as before for all coefficients in type-2 tetrahedra. If T is a type-1 tetrahedron with a vertex $v \in \mathcal{V}_{1,1}$, then the smoothness conditions in $\Lambda_{1,1}$ will uniquely determine all unset coefficients corresponding to $\mathcal{D}_T$. If T is a type-1 tetrahedron with a vertex $v \in \mathcal{V}_{1,i}$ with $i = 2, 4, 8$, we follow the argument in the proof of Theorem 4.2 except that now the special smoothness conditions in $\Lambda_{1,i}$ should also be used in computing the remaining coefficients. In some cases, these computations involve solving a small (at most 4×4) nonsingular system of linear equations, see Lemma 2.1 in [3]. To understand how this works, compare Figures 3–6 and 7–10, where the various types of dots have the same meaning as in the proof of Theorem 4.2. In particular, each point in $\mathcal{M} \setminus \hat{\mathcal{M}}$ has been replaced by a smoothness functional in Λ. For example, seven of the grey dots in the set $\mathcal{M}_{1,4}^0$ in Figure 4 (left) have been replaced by the seven smoothness conditions indicated by arrows in Figure 8 (left). The numbers inside of circles give the order in which these smoothness conditions can be utilized to compute the remaining coefficients. □

Corresponding to $\hat{\mathcal{M}}$, we can now introduce a basis for $\hat{\mathcal{S}}_5^{1,2}(\triangle)$. For each $\xi \in \hat{\mathcal{M}}$, let $\hat{B}_\xi$ be the unique spline in $\hat{\mathcal{S}}_5^{1,2}(\triangle)$ such that

$$\lambda_\eta \hat{B}_\xi = \delta_{\eta,\xi}, \qquad \eta, \xi \in \hat{\mathcal{M}}, \tag{7.5}$$

Arguing as in the proof of Theorem 5.1, it is easy to see that $\{\hat{B}_\xi\}_{\xi \in \hat{\mathcal{M}}}$ form a stable local basis for $\hat{\mathcal{S}}_5^{1,2}(\triangle)$, *i.e.*, the analogs of (5.2) and (5.3) hold. In particular, each

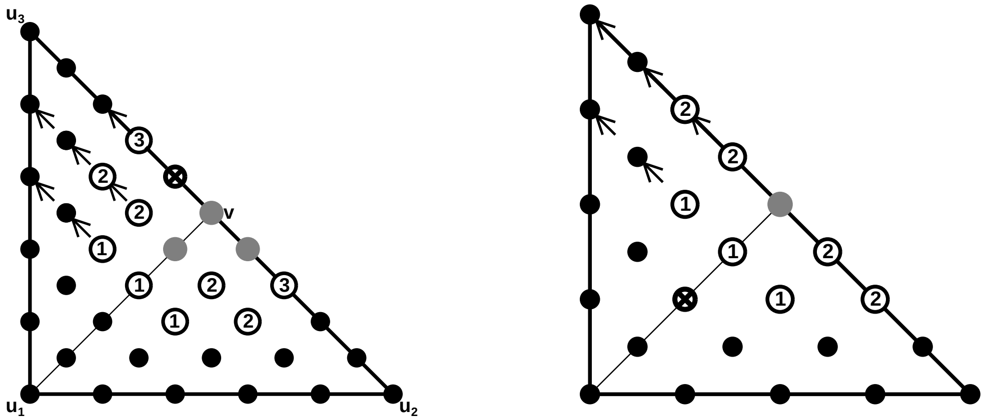

Fig. 7. The sets $\Lambda^0_{1,2}$ on $L^0_{\langle v,u_v\rangle}$ and $\Lambda^1_{1,2}$ on $L^1_{\langle v,u_v\rangle}$, $v \in \mathcal{V}_{1,2}$.

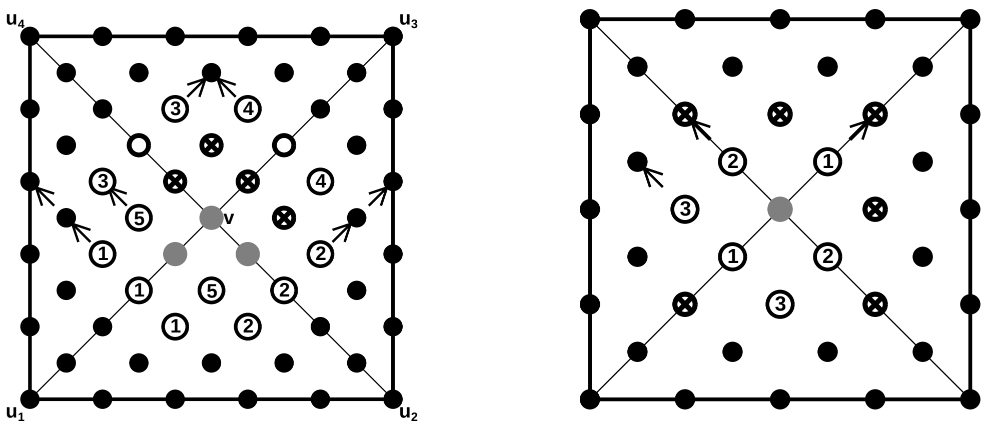

Fig. 8. The sets $\Lambda^0_{1,4}$ on $L^0_{\langle v,u_v\rangle}$ and $\Lambda^1_{1,4}$ on $L^1_{\langle v,u_v\rangle}$, $v \in \mathcal{V}_{1,4}$.

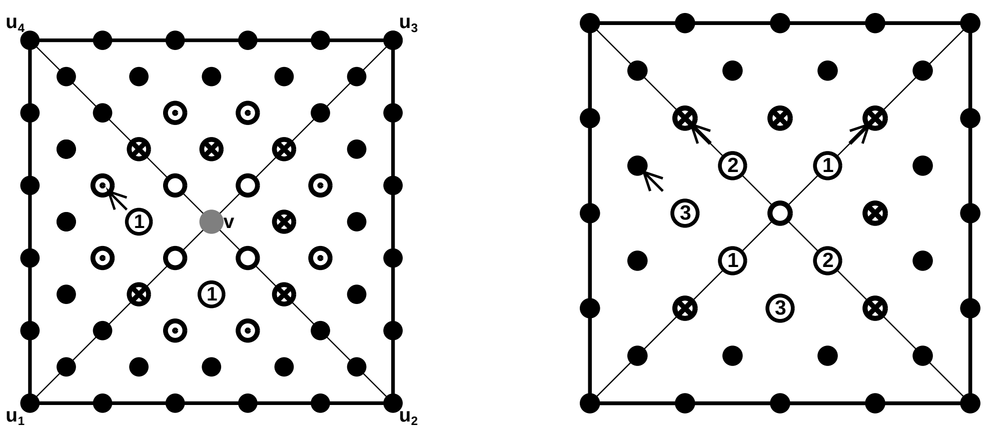

Fig. 9. The sets $\Lambda^0_{1,8}$ on $L^0_{\langle v,u_v\rangle}$ and $\Lambda^1_{1,8}$ on $L^1_{\langle v,u_v\rangle}$, $v \in \mathcal{V}_{1,8}$.

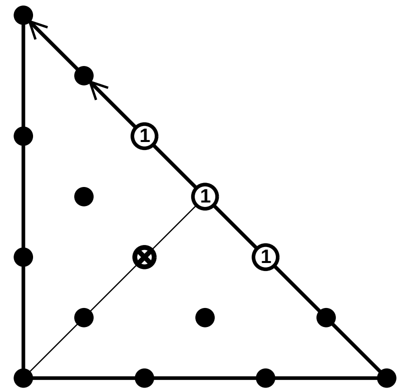

Fig. 10. The set $\Lambda^2_{1,2}$ on $L^2_{\langle v,u_v\rangle}$, $v \in \mathcal{V}_{1,2}$.

$\hat{B}_\xi$ has support which is at most star$^2(v_\xi)$ for some vertex v_ξ, and as in (5.7), there exists a constant $\hat{K}_3$ depending only on $\beta_x, \beta_y, \beta_z$ such that for all coefficient vectors $c := \{c_\xi\}$,

$$\Big\| \sum_{\xi \in \hat{\mathcal{M}}} c_\xi \hat{B}_\xi \Big\|_T \le \hat{K}_3 \max_{\xi \in \Sigma_T} |c_\xi|, \tag{7.6}$$

where

$$\Sigma_T := \{\xi : \hat{B}_\xi \cap T \neq \emptyset\}. \tag{7.7}$$

Using these basis functions and the same linear functionals appearing in (6.1), we can show that the quasi-interpolation operator

$$\hat{Q}f := \sum_{\xi \in \hat{\mathcal{M}}} (\mu_\xi f)\, \hat{B}_\xi \tag{7.8}$$

is a linear projection of $C(B)$ onto $\hat{\mathcal{S}}^{1,2}_5(\triangle)$, and that the analog of Theorem 6.2 holds, *i.e.*, $\hat{\mathcal{S}}^{1,2}_5(\triangle)$ has the same approximation power as $\mathcal{S}^{1,2}_5(\triangle)$.

§8. A Hermite Interpolation Method

We now show how to use the space $\hat{\mathcal{S}}^{1,2}_5(\triangle)$ to solve a Hermite interpolation problem. Given a type-4 partition $\triangle$, let $\mathcal{E}_2$ and $\mathcal{F}_2$ be the sets of all edges and all faces of type-2 tetrahedra, respectively. For each edge $e \in \mathcal{E}_2$, let m_e be the midpoint of e. Let T be the type-2 tetrahedron containing the edge e, and let u and v be the two vertices of T which do not lie on e. Then we define $D_{1,e}$ to be the directional derivative corresponding to the unit vector $(u - m_e)/\|u - m_e\|$, and $D_{2,e}$ to be the derivative corresponding to the unit vector $(v - m_e)/\|v - m_e\|$. For each $F \in \mathcal{F}_2$, suppose η_F is the barycentric center of F and that T is the type-2 tetrahedron containing F. Then we define D_F to be the directional derivative corresponding to the unit vector $(v - \eta_F)/\|v - \eta_F\|$, where v is the vertex of T not lying on F.

Theorem 8.1. *For any function $f \in C^2(B)$, there exists a unique spline $s \in \hat{\mathcal{S}}_5^{1,2}(\triangle)$ such that*

$$\begin{aligned} D^\alpha s(v) &= D^\alpha f(v), && \text{all } |\alpha| \le 2 \text{ and } v \in \mathcal{V}_2, && (8.1)\\ D^\alpha s(v) &= D^\alpha f(v), && \text{all } |\alpha| \le 1 \text{ and } v \in \mathcal{V}_{1,1} \cup \mathcal{V}_{1,2} \cup \mathcal{V}_{1,4}, && (8.2)\\ s(v) &= f(v), && \text{all } v \in \mathcal{V}_{1,8}, && (8.3)\\ D_{j,e}s(m_e) &= D_{j,e}f(m_e), && \text{all } e \in \mathcal{E}_2 \text{ and } j = 1,2, && (8.4)\\ D_F s(\eta_F) &= D_F f(\eta_F), && \text{all } F \in \mathcal{F}_2. && (8.5) \end{aligned}$$

Proof: Since the number of data in (8.1)–(8.5) is equal to the dimension of $\hat{\mathcal{S}}_5^{1,2}(\triangle)$, it suffices to show that these data determine the coefficients $\{c_\xi\}_{\xi \in \hat{\mathcal{M}}}$, where $\hat{\mathcal{M}}$ is the minimal determining set defined in (7.3) for $\hat{\mathcal{S}}_5^{1,2}(\triangle)$. The values $\{D^\alpha f(v)\}_{|\alpha|\le 2}$ determine the coefficients of s corresponding to $D_2(v)$. For example, if $\{c_{ijkl}\}_{i+j+k+l=5}$ are the coefficients of s restricted to a tetrahedron $T = \langle v, u, w, t\rangle$, then (see e.g. [5]),

$$\begin{aligned} c_{5000} &= f(v),\\ c_{4100} &= \tfrac{1}{5} D_{u-v} f(v) + c_{5000},\\ c_{3200} &= \tfrac{1}{20} D^2_{u-v} f(v) + 2c_{4100} - c_{5000},\\ c_{3110} &= \tfrac{1}{20} D_{w-v} D_{u-v} f(v) + c_{4010} + c_{4100} - c_{5000}. \end{aligned} \tag{8.6}$$

Thus, the data in (8.1) determine all coefficients c_ξ with $\xi \in \mathcal{M}_{2,1}$. Similarly, the data in (8.2) determine all coefficients corresponding to points in $\hat{\mathcal{M}}_1$. The coefficients corresponding to points in $\mathcal{V}_{1,8}$ are determined from (8.3), and in particular, $c_v = f(v)$ for all $v \in \mathcal{V}_{1,8}$.

It remains to consider points ξ in $\mathcal{M}_{2,2} \cup \mathcal{M}_{2,3}$. If $\xi \in \mathcal{M}_{2,2}$, then it lies on a face F of a type-2 tetrahedron T, and the corresponding coefficient is determined by (8.4). The points of $\mathcal{M}_{2,3}$ lie in the interior of type-2 tetrahedra. In particular, each such tetrahedron T contains four such points. We need to compute the corresponding coefficients. For each of the four faces of T, there is exactly one interpolation condition (8.5) which involves these coefficients, and we are led to a 4×4 linear system with nonsingular matrix

$$M := \begin{pmatrix} 1 & 1 & 1 & 0\\ 0 & 1 & 1 & 1\\ 1 & 0 & 1 & 1\\ 1 & 1 & 0 & 1 \end{pmatrix}, \tag{8.7}$$

and the proof is complete. □

The interpolation method described in Theorem 8.1 defines a linear operator S mapping $C^2(B)$ into $\hat{\mathcal{S}}_5^{1,2}(\triangle)$. Our next theorem shows that this operator provides optimal order approximation.

Theorem 8.2. *There exists a constant C_3 depending only on $\beta_x, \beta_y, \beta_z$ and β such that for every $f \in W_\infty^{m+1}(B)$ with $2 \le m \le 5$,*

$$\|D^\alpha(f - Sf)\|_B \le C_3 |\triangle|^{m+1-|\alpha|} |f|_{m+1,\infty,B}, \tag{8.8}$$

for all $0 \le |\alpha| \le m$.

Proof: Given a tetrahedron T in $\triangle$, let ρ_T, U_T, and B_T be as in the proof of Theorem 6.2. As noted there, there exists a quintic polynomial $q := q_{f,T}$ such that (6.7) holds. Let $S(f-q)|_T = \sum_{\xi \in \Sigma_T} c_\xi \hat{B}_\xi$. We claim that there exists a constant K_9 depending only on $\beta_x, \beta_y, \beta_z$ such that for all $\xi \in \Sigma_T$,

$$|c_\xi| \le K_9(|f - q|_{0,\infty,T_\xi} + |\triangle||f - q|_{1,\infty,T_\xi} + |\triangle|^2 |f - q|_{2,\infty,T_\xi}), \tag{8.9}$$

where T_ξ is the tetrahedron which contains ξ. If $\xi \in D_2(v)$, this follows immediately from (8.6) It is easy to see that the other coefficients determined by the data in (8.1)–(8.5) satisfy the same bound. Now combining (8.9) with (3.2) and (7.6), we have

$$\|D^\alpha S(f - q)\|_T \le \frac{\hat{K}_3 K_9 K_1^{|\alpha|}}{\rho_T^{|\alpha|}} \sum_{i=0}^{2} |\triangle|^i |f - q|_{i,\infty,U_T}. \tag{8.10}$$

Since S is a projection, by (6.7),

$$\begin{aligned}\|D^\alpha(f - Sf)\|_T &\le \|D^\alpha(f - q)\|_T + \|D^\alpha S(f - q)\|_T \\ &\le 3 \cdot 4^6 K_8 (1 + \hat{K}_3 K_9 K_1^{|\alpha|} K_7^{|\alpha|}) |\triangle|^{m+1-|\alpha|} |f|_{m+1,\infty,B}.\end{aligned}$$

Taking the maximum over all tetrahedra $T \in \triangle$ gives (8.8). □

As a test of the behaviour of this interpolation method, we have written a FORTRAN program to construct the spline $\hat{Q}f$ interpolating a given function $f \in C^2(B)$ as in Theorem 8.1. To test the method, we considered the unit box $B = [0,1]^3$ and a sequence of type-4 partitions $\triangle_k$ corresponding to a uniform partition of B with $m = n = l = 2^k$ with $k = 0, \ldots, 4$. This corresponds to halving the size of $|\triangle|$ at each step, and allows the calculation of the approximate rate of convergence of the method. Table 1 shows our results for the test function

$$f(x, y, z) := e^{x+y+z} \tag{8.11}$$

for $k = 0, \ldots, 4$. The second and third columns contain the number of tetrahedra $N_\triangle$ and dimension of $\hat{\mathcal{S}}_5^{1,2}(\triangle)$ for each k. The third column lists the differences between f and the interpolating splines $S_k f$ on $\triangle_k$, measured in the maximum norm on a large set of points in B. The fourth column shows the computed rates of convergence, which is clearly approaching the expected value of 6.

k	$N_\triangle$	Dim	Error	Rate
0	5	68	3.9821962736175109E-03	
1	40	314	2.6793753612786020E-04	3.893595718364550
2	320	1857	5.3219535196546985E-06	5.653796985995959
3	2560	12779	9.3991310734509170E-08	5.823284710777310
4	20480	94863	1.5623520255303447E-09	5.910735929431061

Tab. 1. Maximum errors and rates of convergence for test function (8.11).

§9. A Lagrange Interpolation Method

For bivariate splines, there has been a lot of recent work on the problem of finding spline spaces which can be used to construct Lagrange interpolation methods (using point evaluation data only) which provide full approximation power, see e.g. [14,15] and references therein. In this section we show that the spline space $\hat{\mathcal{S}}_5^{1,2}(\triangle)$ can also be used for Lagrange interpolation in the trivariate setting.

Let B be a box which has been partitioned into subboxes as in (2.1). For simplicity, suppose m, n, l are all odd. For the more general case, see Remark 10.4. Let $\triangle$ be a corresponding type-4 tetrahedralization of B, and define $\mathcal{T}_{2,\nu}$ to be the collection of type-2 tetrahedra that lie in boxes B_{ijk} that have exactly ν odd subscripts. Let $\mathcal{E}_{2,\nu}$ be the set of edges of tetrahedra in $\mathcal{T}_{2,\nu}$ for $\nu = 0, 1, 2, 3$. Then it is easy to see that no two tetrahedra in $\mathcal{T}_{2,0}$ can touch, and moreover, every type-2 vertex of $\triangle$ is a vertex of some tetrahedron in $\mathcal{T}_{2,0}$. Let

$$P_{2,0} := \bigcup_{T \in \mathcal{T}_{2,1}} (\mathcal{D}_T \setminus \mathcal{A}_T)$$

$$P_{2,1} := \bigcup_{\langle v,u \rangle \in \mathcal{E}_{2,1} \setminus \mathcal{E}_{2,0}} \{D_3^T(v) \cap D_3^T(u) : T \in \mathcal{T}_{2,1} \text{ contains } \langle v, u \rangle\},$$

$$P_{2,2} := \bigcup_{\langle v,u \rangle \in \mathcal{E}_{2,2} \setminus (\mathcal{E}_{2,0} \cup \mathcal{E}_{2,1})} \{D_3^T(v) \cap D_3^T(u) : T \in \mathcal{T}_{2,2} \text{ contains } \langle v, u \rangle\},$$

$$P_{2,3} := \bigcup_{T \in \mathcal{T}_2} \mathcal{A}_T,$$

where $\mathcal{A}_T := \{\xi_{2111}^T, \xi_{1211}^T, \xi_{1121}^T, \xi_{1112}^T\}$. Finally, let

$$P := P_{2,0} \cup P_{2,1} \cup P_{2,2} \cup P_{2,3} \cup \mathcal{V}_{1,8} \cup \hat{\mathcal{M}}_1,$$

where $\mathcal{V}_{1,8}$ and $\hat{\mathcal{M}}_1$ are defined in (4.1) and (7.4), respectively.

Theorem 9.1. *For any function $f \in C(B)$, there exists a unique spline $s \in \hat{\mathcal{S}}_5^{1,2}(\triangle)$ such that*

$$s(p) = f(p), \qquad \text{all } p \in P. \tag{9.1}$$

Proof: It is easy to see that

$$
\begin{aligned}
&\#P_{2,0} = 52(m+1)(n+1)(l+1)/8,\\
&\#P_{2,1} = (m-1)(n+1)(l+1) + (m+1)(n-1)(l+1) + (m+1)(n+1)(l-1),\\
&\#P_{2,2} = [(m-1)(n-1)(l+1) + (m-1)(n+1)(l-1) + (m+1)(n-1)(l-1)]/2,\\
&\#P_{2,3} = 4mnl,\\
&\#\mathcal{V}_{1,8} = (m-1)(n-1)(l-1)/2,\\
&\#\hat{\mathcal{M}}_1 = 3[(m+1)(n+1) + (m+1)(l+1) + (n+1)(l+1)]
\end{aligned}
$$

Then a simple computation shows that

$$\#P = [31mnl + 19(mn + ml + nl) + 23(m + n + l) + 27]/2. \tag{9.2}$$

Inserting

$$
\begin{aligned}
N_2 &= (m+1)(n+1)(l+1)/2\\
E_2 &= (m+1)nl + m(n+1)l + mn(l+1)
\end{aligned}
$$

into (7.2), it is easy to see that (9.2) is exactly the dimension of the space $\hat{\mathcal{S}}_5^{1,2}(\triangle)$.

Since the cardinality of P is equal to the dimension of $\hat{\mathcal{S}}_5^{1,2}(\triangle)$, it suffices to show that the conditions (9.1) determine all of the coefficients of $s \in \hat{\mathcal{S}}_5^{1,2}(\triangle)$. First we observe that interpolation at the points of $P_{2,0} \cup P_{2,3}$ determine s on all tetrahedra in $\mathcal{T}_{2,0}$. By the C^2 smoothness at vertices, this determines all B-coefficients of s in the disks $D_2(v)$ where $v \in \mathcal{V}_2$.

We now show that the data in $P_{2,1}$ determines s on each of the tetrahedra in $\mathcal{T}_{2,1}$. Let T be such a tetrahedron. Then T lies in a box with exactly one odd subscript, which in turn lies between two boxes with all even subscripts. The coefficients of the type-2 tetrahedra in these boxes are already determined. Then for each face of T, the C^1 continuity conditions determine one coefficient corresponding to a domain point (in the set $\mathcal{M}_{2,2}$ defined in Sect. 4) on that face and not contained in one of the 2-disks around the vertices of T. Thus, for each such face, the coefficients corresponding to the remaining two domain points on that face can be computed by solving the 2×2 linear system arising by enforcing (9.1) for the domain points in $\mathcal{P}_{2,1}$. So far all of the coefficients of s_T are determined except for the four coefficients corresponding to $\mathcal{A}_T$. These can be computed by solving the nonsingular 4×4 system that arises from enforcing the interpolation conditions at the points of $\mathcal{A}_T$.

Next we show that the data in $P_{2,2}$ determines s on each of the tetrahedra in $\mathcal{T}_{2,2}$. Let T be such a tetrahedron. Then T shares four of its edges with tetrahedra in $\mathcal{T}_{2,0} \cup \mathcal{T}_{2,1}$ whose coefficients have already been determined. The two remaining edges lie on opposite faces of the box containing T. Then for each face of T, the C^1 continuity conditions determine the coefficients corresponding to two domain points in $\mathcal{M}_{2,2}$. Thus, the coefficient corresponding to the remaining domain point on that face can be computed by enforcing (9.1) for the domain points in $\mathcal{P}_{2,2}$. At this point all of the coefficients of s_T are determined except for the four coefficients corresponding to $\mathcal{A}_T$. These can be computed as before.

We now consider tetrahedra in $\mathcal{T}_{2,3}$. Such a tetrahedron T lies in a box which is surrounded on all sides by boxes containing completely determined type-2 tetrahedra. Applying the C^1 continuity conditions determines all coefficients of s_T except for those corresponding to domain points in $\mathcal{A}_T$. These can be computed as before.

To complete the proof, we now consider the coefficients of s corresponding to points which lie in clusters of type-1 tetrahedra surrounding vertices in $\mathcal{V}_1$. Since $\mathcal{V}_1 \subset P$, for each $v \in \mathcal{V}_1$ the corresponding coefficient is equal to $f(v)$. If $v \in \mathcal{V}_{1,8}$ this determines all coefficients in the cluster surrounding v as shown in the proof of Theorem 7.1. Suppose $v \in \mathcal{V}_{1,1}$ is a vertex of $T := \langle v, u, w, t \rangle$. Since s is C^3 continuous across the face $F := \langle u, w, t \rangle$, the only coefficients of $s|_T$ remaining to be determined are $c_{4100}, c_{4010}, c_{4001}$. Now $s|_{\langle v,u \rangle}$ is a univariate quintic polynomial in B-form, all of whose coefficients are known except for c_{4100}. This coefficient is determined by interpolation at ξ_{4100}. The computations of c_{4010}, c_{4001} proceed in the same way.

If $v \in \mathcal{V}_{1,4}$, then after using the first two special smoothness conditions in Figure 8 (left), we can compute the coefficients corresponding to the three grey dots on $R_1(v)$ by considering the univariate quintic polynomials obtained by restricting s to each of the edges $\langle v, u_1 \rangle, \langle v, u_2 \rangle$ and $\langle v, u_v \rangle$ and enforcing the interpolation conditions. The remaining coefficients corresponding to Figure 8 are determined from smoothness conditions as before. The situation for $v \in \mathcal{V}_{1,2}$ is similar, see Figure 7. The coefficient corresponding to the grey dot on level 1 is computed from univariate polynomial interpolation along the line $\langle v, u_v \rangle$, and the rest of the coefficients corresponding to points at this level are determined from smoothness conditions as before. On level 0, we apply the first special smoothness condition, and then use univariate interpolation along $\langle v, u_1 \rangle$ to get the coefficient corresponding to the point at $R_1(v) \cap \langle v, u_1 \rangle$. To get the four remaining coefficients corresponding to points on the edge $\langle u_2, u_3 \rangle$, we note that s restricted to this edge is a C^1 cubic univariate spline whose unknown coefficients can be computed by enforcing the four interpolation conditions at these points. $\square$

A key ingredient in the construction of the set P is the choice of the set $\mathcal{T}_{2,0}$ of tetrahedra which do not touch each other. For a related construction in the bivariate case, see [14].

The set P constructed above can be considered as a minimal determining set for the superspline space $\hat{\mathcal{S}}_5^{1,2}(\triangle)$, and a corresponding dual basis can be constructed. In particular, for each $\xi \in P$, we let $\widetilde{B}_\xi$ be the unique spline in $\hat{\mathcal{S}}_5^{1,2}(\triangle)$ such that

$$\widetilde{B}_\xi(\eta) = \delta_{\eta,\xi}, \qquad \eta, \xi \in P.$$

These are just cardinal interpolating splines. It follows from the construction of the set P that if ξ lies in some box B, then support of the basis function B_ξ is included in the set $\text{star}^2(B)$, where for any box B, $\text{star}(B)$ is the set of all boxes touching B.

The interpolation method described in Theorem 9.1 defines a linear operator S_L mapping $C(B)$ into $\hat{\mathcal{S}}_5^{1,2}(\triangle)$. Our next theorem shows that this operator provides optimal order approximation.

Theorem 9.2. *There exists a constant C_4 depending only on $\beta_x, \beta_y, \beta_z$ and β such that for every $f \in W_\infty^{m+1}(B)$ with $0 \le m \le 5$,*

$$\|D^\alpha(f - S_L f)\|_B \le C_3 |\triangle|^{m+1-|\alpha|} |f|_{m+1,\infty,B}, \tag{9.3}$$

for all $0 \le |\alpha| \le m$.

Proof: The proof is similar to the proof of Theorem 8.2 except that now (8.9) is replaced by the simpler bound

$$|c_\xi| \le K_{10} |f|_{0,\infty,T_\xi}.$$

This follows as before except that now the univariate interpolation procedures entering into the proof of Theorem 9.1 also have to be examined. □

§10. Remarks

Remark 10.1. The space of trivariate polynomials $\mathcal{P}_d$ has dimension $n := \binom{d+3}{3}$, and so the space of quintics has dimension 56. Given a tetrahedron T, suppose we arrange the Bernstein basis polynomials B_{ijkl}^d and the corresponding domain points ξ_{ijkl}^T in lexicographical order as $\phi_1, \ldots, \phi_n$ and $\eta_1, \ldots, \eta_n$, respectively. Then it follows from standard polynomial interpolation results that the collocation matrix $M := (\phi_j(\eta_i))_{i,j=1}^n$ (which depends only on d and not on T) is nonsingular. It follows that if we write $p \in \mathcal{P}_d$ in Bernstein–Bézier form, then its coefficient vector satisfies $c = M^{-1}F$, where $F = (p(\eta_1), \ldots, p(\eta_n))^T$. This immediately implies

$$\|c\| \le K_6 \|p\|_T, \tag{10.1}$$

where $K_6 = \|M^{-1}\|$. Coupled with the trivial fact that $\|p\|_T \le \|c\|$, this shows that the Bernstein–Bézier basis is stable. If f is an arbitrary continuous function on T, then the same argument shows that the coefficients of the unique polynomial p which interpolates f at the points $\eta_1, \ldots, \eta_n$ also satisfy (10.1) with $\|p\|$ replaced by $\|f\|$.

Remark 10.2. Our type-4 partition is a natural trivariate analog of the well-known type-1 partitions of bivariate grids. For uniform grids, these are examples of cross-cut partitions which are defined by cutting across a domain with straight lines, see [8]. We have the analogous situation for type-4 partitions corresponding to a uniform trivariate grid. In this case the tetrahedralization can be regarded as being formed by cutting through the box B with planes.

Remark 10.3. Let $\tilde{B}$ be a connected set obtained as the union of some collection of subboxes in the set $\mathcal{B}$ associated with the partition (2.1). Given a type-4 partition of B, let $\tilde{\triangle}$ be the corresponding type-4 partition of $\tilde{B}$. Then most of the above results carry over directly after taking into consideration some additional kinds of type-1 vertices.

Remark 10.4. For simplicity, we have restricted our discussion of Lagrange interpolation in Sect. 9 to the simple case of a box with m, n, l all odd. The results can be extended to more general collections of boxes. In this case the set P must include additional points in type-2 tetrahedra lying in boundary boxes.

Remark 10.5. In this paper we have dealt with the superspline space $\mathcal{S}_5^{1,2}(\triangle)$ and one other smaller superspline space $\hat{\mathcal{S}}_5^{1,2}(\triangle)$. However, other superspline spaces with stable local bases and full approximation power could be defined by varying the choice of special smoothness conditions being enforced.

Remark 10.6. Lai and LeMéhauté [11] recently studied spaces of C^1 quintic splines on certain tetrahedral partitions consisting of the union of octahedra (which in our notation correspond to clusters of eight tetrahedra surrounding a vertex in $\mathcal{V}_{1,8}$). These spaces are closely related, but different, from ours.

Remark 10.7. The Hermite and Lagrange interpolation methods described in Sections 8 and 9 both involve computing coefficients from smoothness conditions and certain (small) linear systems. However, if the original grid partition (2.1) is uniform, these systems are the same for every subbox, and in fact, with a little effort, one could give explicit formulae for all B-coefficients in terms of the data appearing in Theorems 8.1 and 9.1.

Remark 10.8. All of the approximation results presented here are for the supremum norm. Analogous results can be obtained for p-norms with only minor modifications to the proofs, see e.g. [12].

Remark 10.9. In Section 7 we have given an explicit stable local bases for the space $\hat{\mathcal{S}}_5^{1,2}(\triangle)$. Although this basis was used in establishing error bounds for Hermite and Lagrange interpolation, we should point out that in practice these interpolating splines can be constructed directly in B-form from the data.

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