Bounds on Projections onto Bivariate Polynomial Spline Spaces with Stable Bases

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Abstract. We derive L_∞ bounds for norms of projections onto bivariate polynomial spline spaces on regular triangulations with stable local bases. We then apply this result to derive error bounds for best L_2- and ℓ_2-approximation by splines on quasi-uniform triangulations.

§1. Introduction

Let $X \subseteq L_\infty(\Omega)$ be a linear space defined a set Ω with polygonal boundary. Suppose $\langle \cdot, \cdot \rangle$ is a semi-definite inner-product on X with associated semi-norm $\| \cdot \|$. We assume that

$$\langle f, g \rangle = 0, \qquad \text{whenever } fg = 0 \text{ on } \Omega, \tag{1.1}$$

$$\|f\| \le \|g\|, \qquad \text{whenever } |f(x)| \le |g(x)| \text{ for all } x \in \Omega. \tag{1.2}$$

Suppose $\mathcal{S} \subseteq X$ is a linear space of polynomial splines (bivariate piecewise polynomials) defined on a regular triangulation $\triangle$ of Ω (two triangles intersect only at a common vertex or along a common edge). We assume that $\mathcal{S}$ is a Hilbert space with respect to $\langle \cdot, \cdot \rangle$.

Let $P : X \to \mathcal{S}$ be the projection of X onto $\mathcal{S}$ defined by the minimization problem

$$\|f - Pf\| = \min_{s \in \mathcal{S}} \|f - s\|. \tag{1.3}$$

It is well known that for every $f \in X$, Pf satisfies

$$\langle f - Pf, s \rangle = 0, \qquad \text{for all } s \in \mathcal{S}. \tag{1.4}$$

The main purpose of this paper is to bound the norm

$$\|P\|_\infty := \sup_{f \neq 0} \frac{\|Pf\|_{L_\infty(\Omega)}}{\|f\|_{L_\infty(\Omega)}}$$

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of P under appropriate assumptions on X and the spline space $\mathcal{S}$. In those cases where we know the approximation power of $\mathcal{S}$, this leads to error bounds for $\|f - Pf\|_\infty$ for smooth functions f.

The problem of estimating the L_∞ norm of projections onto spline spaces has been considered in a number of earlier papers – see [2,3,5,9] for the univariate case and [6,10] for the bivariate case, and references therein. This paper may be regarded as a continuation of [10], the main difference being that we obtain results under much weaker assumptions on the spline spaces than previously required (see Remark 8.1).

§2. Local Estimates in the X-norm

Throughout this paper we assume the triangles T in $\triangle$ are closed and bounded. In addition, we suppose that X is such that for every $f \in X$,

1) $f \cdot \chi_T \in X$ for every triangle T in $\triangle$, where χ_T is the characteristic function of T,
2) $f = \sum_{T\in\triangle} f_T$ for some $f_T \in X$ with $\mathrm{supp}(f_T) \subseteq T$.

Given a vertex v of $\triangle$, we define $\mathrm{star}^1(v)$ to be the union of all triangles in $\triangle$ which share the vertex v, and set

$$\mathrm{star}^\ell(v) := \bigcup\{\mathrm{star}^1(w) : \ w \text{ is a vertex of } \mathrm{star}^{\ell-1}(v)\}, \qquad \ell > 1.$$

Theorem 2.1. *Suppose that $\mathcal{S}$ is a spline space with a basis $\mathcal{B} := \{\phi_\xi\}_{\xi\in\mathcal{M}}$ such that for some constants $0 < K_1 \le K_2 < \infty$,*

$$K_1 \sum_{\xi\in\mathcal{M}} |c_\xi|^2 \le \|\sum_{\xi\in\mathcal{M}} c_\xi\phi_\xi\|^2 \le K_2 \sum_{\xi\in\mathcal{M}} |c_\xi|^2, \tag{2.1}$$

for all $(c_\xi)_{\xi\in\mathcal{M}}$. Let ℓ be the smallest integer such that for each $\xi \in \mathcal{M}$, there is a vertex v_ξ of $\triangle$ with $\mathrm{supp}(\phi_\xi) \subseteq \mathrm{star}^\ell(v_\xi)$. Let f be a function in X with support on a triangle T in $\triangle$. Suppose τ is another triangle which lies outside of $\mathrm{star}^q(T)$ for some $q \ge 1$. Then

$$\|Pf \cdot \chi_\tau\| \le C_1\sigma^q\|f\|, \tag{2.2}$$

for some constants $0 < \sigma < 1$ and C_1 depending only on ℓ and the ratio K_2/K_1.

Proof: Given a triangle T, we define $\mathrm{star}^0(T) = T$, and set

$$\mathrm{star}^\ell(T) := \bigcup\{\mathrm{star}^\ell(w) : \ w \text{ is a vertex of } T\}, \qquad \ell \ge 1.$$

Let

$$\mathcal{M}_0^T := \{\xi \in \mathcal{M} : \ T \cap \mathrm{supp}(\phi_\xi) \neq \emptyset\},$$

$$\mathcal{M}_k^T := \{\xi \in \mathcal{M} : \ \mathrm{supp}(\phi_\xi) \cap \mathrm{star}^{k(2\ell+1)}(T) \neq \emptyset\},$$

$$\mathcal{N}_0^T := \mathcal{M}_0^T,$$

$$\mathcal{N}_k^T := \mathcal{M}_k^T \setminus \mathcal{M}_{k-1}^T.$$

Suppose

$$Pf := \sum_{\xi\in\mathcal{M}} c_\xi \phi_\xi,$$

and let

$$u_k := \sum_{\xi\in\mathcal{M}_k^T} c_\xi\phi_\xi, \qquad w_k := Pf - u_k, \qquad a_k := \sum_{\xi\in\mathcal{N}_k^T} c_\xi^2,$$

for $k \geq 0$. By (2.1),

$$\sum_{j\geq k+1} a_j = \sum_{\xi\notin\mathcal{M}_k^T} c_\xi^2 \leq \frac{\|w_k\|^2}{K_1}.$$

Now, $\langle f - Pf, w_k\rangle = 0$ by (1.4), and $\langle f, w_k\rangle = 0$ since $w_k \equiv 0$ in T and $f \equiv 0$ outside of T. Using the fact that $\mathrm{supp}(w_k) \cap \bigcup_{\xi\in\mathcal{M}_{k-1}^T} \mathrm{supp}(\phi_\xi) = \emptyset$ for $k \geq 1$, it follows that

$$\|w_k\|^2 = \langle Pf - u_k, w_k\rangle = \langle f - u_k, w_k\rangle = -\langle u_k, w_k\rangle$$
$$= -\langle \sum_{\xi\in\mathcal{N}_k^T} c_\xi\phi_\xi, w_k\rangle \leq \|\sum_{\xi\in\mathcal{N}_k^T} c_\xi\phi_\xi\| \, \|w_k\|,$$

and thus by (2.1),

$$\|w_k\|^2 \leq \|\sum_{\xi\in\mathcal{N}_k^T} c_\xi\phi_\xi\|^2 \leq K_2 a_k.$$

Combining the above, we have

$$\sum_{j\geq k+1} a_j \leq \frac{K_2}{K_1}\, a_k, \qquad k \geq 0. \tag{2.3}$$

Then applying Lemma 2 in [2] (see also [9]) with $\gamma := K_2/K_1$, we see that

$$a_k \leq (\gamma+1)\rho^k a_0,$$

where $\rho := \gamma/(\gamma+1)$. Since $\|Pf\|^2 \leq \|f\|^2$, we have

$$a_0 \leq \sum_{j\geq 0} a_j = \sum_{\xi\in\mathcal{M}} c_\xi^2 \leq \frac{1}{K_1}\|Pf\|^2 \leq \frac{1}{K_1}\|f\|^2.$$

Let τ be a triangle of $\triangle$ which lies outside of $\mathrm{star}^q(T)$. If $1 \leq q \leq 2\ell$, then

$$\|Pf\cdot\chi_\tau\| \leq \|Pf\| \leq \|f\|,$$

and (2.2) holds for any $0 < \sigma < 1$ with $C_1 = 1$. Suppose now $q \geq 2\ell+1$, and let $k := \lceil \frac{q}{2\ell+1} \rceil$. Then $\mathrm{supp}(\phi_\xi) \cap \tau \neq \emptyset$ implies $\xi \notin \mathcal{M}_k^T$, and thus by (2.1),

$$\|Pf\cdot\chi_\tau\|^2 \leq \|\sum_{\xi\notin\mathcal{M}_k^T} c_\xi\phi_\xi\|^2 \leq K_2 \sum_{\xi\notin\mathcal{M}_k^T} c_\xi^2 = K_2 \sum_{j\geq k+1} a_j \leq \frac{K_2^2}{K_1}\, a_k.$$

Combining the above estimates yields (2.2) with $\sigma := \rho^{\frac{1}{4\ell+2}}$. $\square$

§3. Main results

Let P be the projector defined in (1.3). In this section we first estimate $\|P\|_\infty$ in a fairly general setting, and then specialize to the situation where $\triangle$ is a quasi-uniform triangulation. We recall that throughout this paper we are assuming that for every $f \in X$, there exist functions $f_T \in X$ with $\operatorname{supp}(f_T) \subseteq T$ and $f = \sum_{T\in\triangle} f_T$.

Theorem 3.1. *Suppose $\mathcal{S}$ is a polynomial spline space defined on a triangulation $\triangle$, and that $\mathcal{B} := \{\phi_\xi\}_{\xi\in\mathcal{M}}$ is a basis for $\mathcal{S}$ such that (2.1) holds. In addition, suppose that there exist constants $0 < K_3 \le K_4 < \infty$ (depending only on $\mathcal{S}$ and the X-norm) such that*

$$K_3 \|s\|_{L_\infty(T)} \le \|s \cdot \chi_T\|, \quad s \in \mathcal{S}, \quad T \in \triangle, \tag{3.1}$$

$$\|f_T\| \le K_4 \|f\|_{L_\infty(T)}, \quad f \in X, \quad T \in \triangle. \tag{3.2}$$

Then

$$\|P\|_\infty \le C_2 := \frac{K_4}{K_3}\Big(n_0 + C_1 \sum_{q\ge1} \sigma^q n_q\Big), \tag{3.3}$$

where

$$n_0 := \max_{\tau\in\triangle} \#\, \operatorname{star}^1(\tau), \qquad n_q := \max_{\tau\in\triangle} \#\, \big[\operatorname{star}^{q+1}(\tau) \setminus \operatorname{star}^q(\tau)\big],$$

and C_1, σ are the constants in (2.2).

Proof: Let τ be a fixed triangle in $\triangle$, and let

$$\Omega_0^\tau := \operatorname{star}^1(\tau), \qquad \Omega_q^\tau := \operatorname{star}^{q+1}(\tau) \setminus \operatorname{star}^q(\tau).$$

Suppose $f = \sum_{T\in\triangle} f_T$ with $\operatorname{supp}(f_T) \subseteq T$ for all T. Since P is a linear operator, by (3.1)

$$\|Pf\|_{L_\infty(\tau)} \le \sum_{T\in\triangle} \|Pf_T\|_{L_\infty(\tau)} \le \frac{1}{K_3} \sum_{T\in\triangle} \|Pf_T \cdot \chi_\tau\|.$$

Then by (3.2),

$$\begin{aligned}\|Pf\|_{L_\infty(\tau)} &\le \frac{1}{K_3} \sum_{q\ge0} \sum_{T\in\Omega_q^\tau} \|Pf_T \cdot \chi_\tau\| \le \frac{1}{K_3}\Big[\sum_{T\in\Omega_0^\tau} \|f_T\| + \sum_{q\ge1}\sum_{T\in\Omega_q^\tau} C_1 \sigma^q \|f_T\|\Big] \\ &\le \frac{K_4}{K_3}\Big[n_0 + C_1 \sum_{q\ge1} \sigma^q n_q\Big] \|f\|_{L_\infty(\Omega)}.\end{aligned}$$

Taking the supremum over all $\tau \in \triangle$ and all $f \in X$, we get (3.3). $\square$

This result leads immediately to the following error bound.

Corollary 3.2. *Let P be a projection onto a spline space $\mathcal{S}$ as in Theorem 3.1. Then for any function f in $L_\infty(\Omega)$,*

$$\|f - Pf\|_{L_\infty(\Omega)} \le (1 + C_2)\, d(f, \mathcal{S})_{L_\infty(\Omega)}, \tag{3.4}$$

where

$$d(f, \mathcal{S})_{L_\infty(\Omega)} := \inf_{s\in\mathcal{S}} \|f - s\|_{L_\infty(\Omega)}.$$

§4. Quasi-uniform triangulations

In order to get more information on the constant C_2 appearing in (3.3), we now restrict ourselves to quasi-uniform triangulations.

Definition 4.1. *Let $0 < \beta < \infty$. A triangulation $\triangle$ is said to be β-*quasi-uniform *provided that*

$$\frac{|\triangle|}{\rho_\triangle} \leq \beta, \tag{4.1}$$

where $|\triangle|$ is the maximum of the diameters of the triangles in $\triangle$, and $\rho_\triangle$ is the minimum of the radii of the incircles of triangles of $\triangle$.

We note that if a triangulation is β-quasi-uniform, then the smallest angle in $\triangle$ is bounded below by a positive constant depending on β. The converse is not true, *i.e.*, a triangulation whose triangles do not have small angles can still have triangles of vastly different sizes. It is well-known that univariate quasi-uniform partitions play an essential role in univariate spline approximation theory. Quasi-uniform triangulations are also commonly used in deriving error bounds for finite-element methods.

The following result gives a bound on the size of the constants n_q appearing in (3.3) in terms of β.

Lemma 4.2. *Suppose $\triangle$ is a β-quasi-uniform triangulation. Then for all $q \geq 0$,*

$$n_q \leq \frac{(2q+3)^2\beta^2}{\pi}. \tag{4.2}$$

Proof: For each triangle τ in $\triangle$, $\mathrm{star}^{q+1}(\tau)$ is contained in a square of side-length $(2q+3)|\triangle|$. Moreover, the area of any triangle in $\triangle$ is greater than $\pi\rho_\triangle^2$. Thus,

$$\sum_{k=0}^{q} n_k \leq \frac{|\triangle|^2(2q+3)^2}{\pi\rho_\triangle^2},$$

and (4.2) follows from (4.1). □

It follows immediately from this lemma that if $\mathcal{S}$ is a space of splines defined on a β-quasi-uniform triangulation, then the constant C_2 in (3.3) depends only on $K_2/K_1, K_4/K_3$ and β, ℓ.

§5. Spline spaces with stable local bases

The following definition is taken from [7,15,16].

Definition 5.1. *We say that a basis $\mathcal{B} := \{\phi_\xi\}_{\xi\in\mathcal{M}}$ for a space $\mathcal{S}$ of splines on a triangulation $\triangle$ is a* stable local basis *provided*

1) *there exists an integer ℓ such that for each $\xi \in \mathcal{M}$,*

$$\mathrm{supp}(\phi_\xi) \subseteq \mathrm{star}^\ell(v_\xi) \textit{ for some vertex } v_\xi \textit{ of } \triangle,$$

2) *there exist constants* $0 < K_5 \le K_6 < \infty$, *depending only on the degree* d *of* $\mathcal{P}_d$ *and the smallest angle* $\theta_\triangle$ *in the triangulation* $\triangle$, *such that for all* $(c_\xi)_{\xi \in \mathcal{M}}$,

$$K_5 \max_{\xi \in \mathcal{M}} |c_\xi| \le \| \sum_{\xi \in \mathcal{M}} c_\xi \phi_\xi \|_{L_\infty(\Omega)} \le K_6 \max_{\xi \in \mathcal{M}} |c_\xi|. \tag{5.1}$$

The following classes of splines have stable local bases in this sense:

1) The spline spaces $\mathcal{S}_d^0(\triangle)$ for all $d \ge 1$. In this case we can take $\ell = 1$.

2) The spline spaces

$$\mathcal{S}_d^r(\triangle) := \{s \in C^r(\Omega) : \; s|_T \in \mathcal{P}_d \;\text{ for all triangles } T \in \triangle\}$$

with $d \ge 3r + 2$, see [7]. In this case we can take $\ell = 3$.

3) Superspline spaces of the form

$$\mathcal{S}_d^{r,\rho}(\triangle) := \{s \in \mathcal{S}_d^r(\triangle) : \; s \in C^{\rho_v}(v) \;\text{ for all } v \in \mathcal{V}\},$$

with $d \ge 3r + 2$ and $\rho := \{\rho_v\}_{v \in \mathcal{V}}$, where ρ_v are given integers such that $r \le \rho_v \le d$, and $\mathcal{V}$ is the set of all vertices of $\triangle$, see [7]. We make the standard assumption that

$$k_v + k_u < d \quad \text{for each pair of neighboring vertices } v, u \in \mathcal{V},$$

where

$$k_v := \max\{\rho_v, \mu\}, \qquad v \in \mathcal{V},$$

with $\mu := r + \lfloor \frac{r+1}{2} \rfloor$. In this case we can also take $\ell = 3$.

4) Spline spaces of the form $\mathcal{S}_{d(r)}^r(\triangle_{PS})$ for some special values of $d(r)$, where $\triangle_{PS}$ is the Powell-Sabin refinement of an arbitrary triangulation $\triangle$, see [15]. In this case we can take $\ell = 1$.

5) Spline spaces of the form $\mathcal{S}_{d(r)}^r(\triangle_{CT})$, for some special values of $d(r)$, where $\triangle_{CT}$ is the Clough-Tocher refinement of an arbitrary triangulation $\triangle$, see [16]. Again we can take $\ell = 1$.

6) Certain other superspline spaces with $d \ge 3r + 2$ described in [4], where $\ell = \lfloor \frac{r}{2} \rfloor$ and in [14], where $\ell = 3$.

7) The spaces $\mathcal{S}_{3r}^r(\diamondsuit)$, where $\diamondsuit$ is the triangulation obtained by inserting the diagonals into each quadrangle of an arbitrary quadrangulation, see [13]. Here $\ell = 1$.

All of these spaces have full approximation power in the following sense:

Theorem 5.2. *Let $\mathcal{S}$ be a spline space of degree d with a stable local basis, and let $0 \le m \le d$. Then for all f in the Sobolev space $W_\infty^{m+1}(\Omega)$,*

$$d(f, \mathcal{S})_{L_\infty(\Omega)} \le C_3 |\triangle|^{m+1} |f|_{m+1,\infty,\Omega}.$$

Here the constant C_3 depends only on d and the smallest angle $\theta_\triangle$ in $\triangle$ if Ω is convex, and also on the Lipschitz constant $L_{\partial\Omega}$ of the boundary of Ω if Ω is not convex.

Proof: The proof is a minor modification of the proof of Theorem 1.1 of [14]. For an alternate proof based on interpolation, see Theorem 2.2 of [12]. □

§6. Best L_2-approximation

In this section we derive error bounds for best L_2-approximation by spaces of splines with stable local bases. In particular, given such a spline space $\mathcal{S}$ and a function $f \in L_2(\Omega)$, we are interested in error bounds for $f - Pf$, where Pf is defined by

$$\|f - Pf\|_{L_2(\Omega)} = \inf_{s \in \mathcal{S}} \|f - s\|_{L_2(\Omega)}. \tag{6.1}$$

Our aim is to apply the above results with $X := L_2(\Omega) \cap L_\infty(\Omega)$ equipped with the usual L_2-inner-product $\langle f, g \rangle := \int_\Omega fg$. Clearly, this inner-product satisfies the conditions (1.1) and (1.2).

Lemma 6.1. *Suppose $\mathcal{S}$ is a spline space defined on a triangulation $\triangle$, and that $\mathcal{B} := \{\phi_\xi\}_{\xi \in \mathcal{M}}$ is a stable local basis for $\mathcal{S}$. Then there exist constants $0 < K_7 \le K_8 < \infty$ depending only on d, ℓ and the smallest angle $\theta_\triangle$ in $\triangle$ such that*

$$K_7 \min_{T \in \triangle} A_T \sum_{\xi \in \mathcal{M}} |c_\xi|^2 \le \int_\Omega \Big| \sum_{\xi \in \mathcal{M}} c_\xi \phi_\xi \Big|^2 \le K_8 \max_{T \in \triangle} A_T \sum_{\xi \in \mathcal{M}} |c_\xi|^2, \tag{6.2}$$

for all $(c_\xi)_{\xi \in \mathcal{M}}$, where A_T denotes the area of T.

Proof: Following the proof of Theorem 2.3 of [7] (see also [14]), we have

$$k_7 A_T \sum_{\xi \in \Sigma_T} |c_\xi|^2 \le \int_T \Big| \sum_{\xi \in \Sigma_T} c_\xi \phi_\xi \Big|^2 = \int_T \Big| \sum_{\xi \in \mathcal{M}} c_\xi \phi_\xi \Big|^2 \le k_8 A_T \sum_{\xi \in \Sigma_T} |c_\xi|^2, \tag{6.3}$$

where $\Sigma_T := \{\xi : \ T \subseteq \operatorname{supp}(\phi_\xi)\}$, and where the constants k_7, k_8 depend only on d and $\theta_\triangle$. As shown in [14], the cardinality of Σ_T is bounded by a constant which depends only on $\theta_\triangle$ and ℓ. Then replacing A_T by $\min_{T \in \triangle} A_T$ on the left and by $\max_{T \in \triangle} A_T$ on the right and summing over all triangles T, we get (6.2). □

Theorem 6.2. *Suppose $\mathcal{S}$ is a polynomial spline space which satisfies the hypotheses of Lemma 6.1, and let P be the best L_2-approximation operator defined in (6.1). Then $\|P\|_\infty \le C_4$, where C_4 depends only on d, K_8/K_7, β, and ℓ.*

Proof: We apply Theorem 3.1. Lemma 6.1 implies (2.1) holds with $K_1 := K_7 \min_{T\in\triangle} A_T$ and $K_2 := K_8 \max_{T\in\triangle} A_T$. Then the fact that $\triangle$ is a β-quasi-uniform triangulation implies that the ratio K_2/K_1 depends only on d, ℓ and β. We now verify hypotheses (3.1) and (3.2), and examine the ratio K_4/K_3 of the constants appearing there. To establish (3.1), we note that for every $s \in \mathcal{S}$ and $T \in \triangle$, $s \cdot \chi_T$ is a polynomial of degree at most d on the triangle T. Then by mapping T to a standard triangle and using the equivalence of all norms on a finite-dimensional linear space, we have

$$\int_T s^2 \ge K_9 A_T \, \|s\|^2_{L_\infty(T)}, \tag{6.4}$$

where A_T is the area of T and K_9 depends only on d and the smallest angle in T, see e.g. [14, p. 256]. This gives (3.1) with $K_3 := (K_9 A_T)^{1/2}$. On the other hand,

$$\int_T f^2 \le A_T \, \|f\|^2_{L_\infty(T)} \tag{6.5}$$

for all $f \in X$. This gives (3.2) with $K_4 := A_T^{1/2}$. It follows immediately that K_4/K_3 is bounded by a constant depending only on d and β. $\square$

Combining this result with Theorem 5.2 (cf. Corollary 3.2), we immediately get the following error bound for the best L_2-approximation operator P.

Theorem 6.3. *Let $\mathcal{S}$ be one of the spaces listed in Sect. 5, and suppose $\triangle$ is β-quasi-uniform. Then for all $f \in W^{m+1}_\infty(\Omega)$ with $0 \le m \le d$,*

$$\|f - Pf\|_{L_\infty(\Omega)} \le C_5 |\triangle|^{m+1} |f|_{m+1,\infty,\Omega}.$$

Here the constant C_5 depends only on d, ℓ and β if Ω is convex, and also on the Lipschitz constant $L_{\partial\Omega}$ of the boundary of Ω if Ω is not convex.

§7. Best ℓ_2-approximation

In this section we derive error bounds for best ℓ_2-approximation by spaces of splines $\mathcal{S}$ with stable local bases. Suppose $D := \{t_i\}_{i=1}^m$ is a set of scattered data points in Ω. Given a function $f \in X := L_\infty(\Omega)$, we are interested in error bounds for $f - Pf$, where Pf is defined by

$$\|f - Pf\|_{\ell_2(\Omega)} = \inf_{s\in\mathcal{S}} \|f - s\|_{\ell_2(\Omega)}.$$

Here $\|\cdot\|_{\ell_2(\Omega)}$ is the ℓ_2-norm corresponding to

$$\langle f, g\rangle := \sum_{i=1}^m f(t_i) g(t_i),$$

which is a semi-definite inner-product on X. It is easy to see that (1.1) and (1.2) are satisfied.

Throughout this section we suppose that $\triangle$ is a β-quasi-uniform triangulation of a set Ω, and that for some fixed integer N,

$$\dim \mathcal{P}_d := \binom{d+2}{2} \le \#(D \cap T) \le N, \qquad \text{for all } T \in \triangle. \tag{7.1}$$

Suppose that there exists a constant $0 < K_3^*$ such that

$$K_3^* \|s\|_{L_\infty(T)} \le \Big[\sum_{t_i \in D\cap T} s(t_i)^2 \Big]^{1/2}, \qquad \text{for all } s \in \mathcal{P}_d \text{ and } T \in \triangle. \tag{7.2}$$

Let

$$\kappa := \max_{t_i \in D} \#\{T : \ t_i \in T\}.$$

Lemma 7.1. *Suppose $\{\phi_\xi\}_{\xi\in\mathcal{M}}$ is a stable local basis for $\mathcal{S}$. Then for all $\{c_\xi\}_{\xi\in\mathcal{M}}$,*

$$K_1^* \sum_{\xi\in\mathcal{M}} |c_\xi|^2 \le \sum_{t_i\in D} \Big[\sum_{\xi\in\mathcal{M}} c_\xi \phi_\xi(t_i) \Big]^2 \le K_2^* \sum_{\xi\in\mathcal{M}} |c_\xi|^2, \tag{7.3}$$

where

$$K_1^* := \frac{(K_3^*)^2 K_7 \min_{T\in\triangle} A_T}{\kappa \max_{T\in\triangle} A_T}$$

$$K_2^* := \frac{K_8 N \max_{T\in\triangle} A_T}{K_9 \min_{T\in\triangle} A_T},$$

where K_7, K_8 are as in (6.2) and K_9 is the constant in (6.4).

Proof: Let $s := \sum_{\xi\in\mathcal{M}} c_\xi \phi_\xi$. Then by (6.4) and (7.1),

$$\sum_{t_i\in D} s(t_i)^2 \le \sum_{T\in\triangle} \sum_{t_i\in D\cap T} s(t_i)^2 \le \sum_{T\in\triangle} N \|s\|^2_{L_\infty(T)} \le \frac{N}{K_9} \sum_{T\in\triangle} \frac{1}{A_T} \int_T s^2,$$

and the right-hand side of (7.3) follows from the right-hand side of (6.2). Now by (7.2) and (6.5),

$$\begin{aligned} \sum_{t_i\in D} s(t_i)^2 &\ge \sum_{T\in\triangle} \frac{1}{\kappa} \sum_{t_i\in D\cap T} s(t_i)^2 \ge \frac{(K_3^*)^2}{\kappa} \sum_{T\in\triangle} \|s\|^2_{L_\infty(T)} \\ &\ge \frac{(K_3^*)^2}{\kappa} \sum_{T\in\triangle} \frac{1}{A_T} \int_T |s|^2, \end{aligned}$$

and the left-hand side of (7.3) follows from the left-hand side of (6.2). $\square$

Theorem 7.2. *Suppose $\mathcal{S}$ is a polynomial spline space which satisfies the hypotheses of Lemma 7.1, and let P be the best ℓ_2-approximation operator associated with a set of points D satisfying (7.1) and (7.2). Then $\|P\|_\infty \le C_6$, where C_6 depends only on d, $K_2^*/K_1^*, K_4^*/K_3^*, \beta$, and ℓ, where $K_4^* := N^{1/2}$.*

Proof: The result follows immediately from Theorem 3.1 with $K_i := K_i^*$ for $i = 1, \dots, 4$, since

$$\sum_{t_i \in D \cap T} f(t_i)^2 \le N \|f\|^2_{L_\infty(T)}, \qquad \text{for all } T \in \triangle \text{ and } f \in L_\infty(\Omega). \quad \square$$

Combining this with Theorem 5.2, we get the following error bound for best ℓ_2 approximation of functions in the Sobolev spaces $W_\infty^{m+1}(\Omega)$.

Theorem 7.3. *Let $\mathcal{S}$ be as in Theorem 6.3, and suppose that $\triangle$ is β-quasi-uniform. Let P be the best ℓ_2 approximation operator associated with the scattered data D satisfying (7.1), (7.2). Then for all $f \in W_\infty^{m+1}(\Omega)$,*

$$\|f - Pf\|_{L_\infty(\Omega)} \le C_7 |\triangle|^{m+1} |f|_{m+1,\infty,\Omega}.$$

Here the constant C_7 depends only on d, ℓ, β, N, C_6, and K_3^* if Ω is convex, and also on the Lipschitz constant $L_{\partial\Omega}$ of the boundary of Ω if Ω is not convex.

We conclude this section by describing a set of scattered data points for which Theorem 7.3 applies. In particular, we show how to choose D so that (7.1) and (7.2) are satisfied.

Theorem 7.4. *Suppose $\triangle$ is a β-quasi-uniform triangulation, and fix $0 < N_0$. For each $T := \langle v_1, v_2, v_3 \rangle \in \triangle$, let*

$$D_T := \left\{ \xi_{ijk}^T := \frac{i v_1 + j v_2 + k v_3}{d} \right\}_{i+j+k=d}$$

be the associated set of Bernstein-Bézier domain points. Let

$$D := \bigcup_{T \in \triangle} (D_T \cup \tilde{D}_T),$$

where $\tilde{D}_T$ is any set of n_T additional points in T with $n_T \le N_0$. Then (7.1) is satisfied with $N := \binom{d+2}{2} + N_0$. Moreover, (7.2) is satisfied with a constant $0 < K_3^*$ depending only on d.

Proof: Suppose $T := \langle v_1, v_2, v_3 \rangle \in \triangle$. Given $s \in \mathcal{S}$, let $s := \sum_{i+j+k=d} c_{ijk} B_{ijk}^d$ be its expansion in Bernstein-Bézier form on T. Then

$$\|s\|_{L_\infty(T)} \le \max_{i+j+k=d} |c_{ijk}| \le \|M^{-1}\|_\infty \max_{i+j+k=d} |s(\xi_{ijk}^T)|,$$

where

$$M := (B^d_{ijk}(\xi^T_{\nu,\mu,\kappa}))$$

depends only on d. Using the equivalence of the ℓ_∞ and ℓ_2 norms for vectors of length $\binom{d+2}{2}$, we have

$$\|s\|_{L_\infty(T)} \leq \|M^{-1}\|_\infty \Big[\sum_{t\in D_T} s(t)^2 \Big]^{1/2},$$

which immediately implies (7.2). □

§8. Remarks

Remark 8.1. The results in [10] were established only for spline spaces with a stable local basis which has the additional property

$$|c_\xi| \leq K A_T^{-1} \|s\|_{L_2(T)}$$

for all $\xi \in \Sigma_T$ and all T in $\triangle$. This condition implies that the basis $\{\phi_\xi\}_{\xi\in\mathcal{M}}$ is locally linearly independent (LLI), *i.e.*, for every $T \in \triangle$, the basis splines $\{\phi_\xi\}_{\xi\in\Sigma_T}$ are linearly independent on T, where $\Sigma_T := \{\xi : \; T \subseteq \mathrm{supp}(\phi_\xi)\}$, see [8]. But it has recently been shown [7] that for spline spaces with smoothness $r > 0$, stable local bases for spline spaces are not LLI in general. This limits the applicability of the results [10] substantially.

Remark 8.2. Theorem 3.1 can be generalized by replacing the triangles T appearing there by clusters of triangles – see [11] for the case of univariate splines. This would lead to error bounds for ℓ_2 approximation using spline spaces $\mathcal{S} \subseteq \mathcal{S}^0_d(\triangle)$ with sets of scattered data which contain fewer points per triangle than the set D appearing in Theorem 7.3.

Remark 8.3. Most of the results here do not require that the triangulation $\triangle$ consist of a finite number of triangles, and so Ω can even be unbounded.

Remark 8.4. In this paper we have worked with piecewise polynomials on triangulations, but an analogous treatment is possible for more general partitions, including for example quadrangulations, and also for more general piecewise functions, see [11].

Remark 8.5. For simplicity, we have presented error bounds for best L_2-approximation and best ℓ_2-approximation only in the supremum norm. Analogous results can easily be obtained in the p-norms, $1 \leq p < \infty$. Moreover, we have given error bounds only for $f - Pf$, but using the Bramble-Hilbert lemma in the standard way (cf. [14]) or the Markov inequality for bivariate polynomials, one can obtain results for derivatives of $f - Pf$.

Remark 8.6. Starting with a spherical triangulation of the surface of a sphere, it is possible to construct spaces of splines which are direct analogs of bivariate polynomial splines (instead of being piecewise polynomials, they are piecewise spherical harmonics). These classes of spherical splines have proven to be useful for scattered data fitting — see [1] for a treatment of a number of explicit methods, including ℓ_2-approximation. Recently [17], stable local bases have also been constructed for these spaces, leading to a result on their approximation power. It is easy to see that the methods and results presented here extend immediately to these classes of spherical splines.

Remark 8.7. The conditions (1.1) and (1.2) are not satisfied for all semi-definite inner-products. For example, consider $X = C^1[0,1]$ with

$$\langle f, g \rangle := \int_0^1 f'(x)\, g'(x)\, dx + \frac{f(0)g(0) - (f(1) - f(0))(g(1) - g(0))}{2}.$$

It is not hard to check that this is an inner-product on X. But then clearly (1.1) fails for the functions

$$f(x) := \begin{cases} (1-3x)^2, & \text{if } 0 \le x \le 1/3, \\ 0, & \text{otherwise,} \end{cases} \qquad g(x) := \begin{cases} (x-2/3)^2, & \text{if } 2/3 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, (1.2) fails for the functions $f(x) := x$ and $g(x) := (1+x)/2$.

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