Local Lagrange Interpolation by Bivariate C^1 Cubic Splines

Günther Nürnberger, Larry L. Schumaker,
and Frank Zeilfelder

Abstract. Lagrange interpolation schemes are constructed based on C^1 cubic splines on certain triangulations obtained from checkerboard quadrangulations.

§1. Introduction

Given a triangulation $\triangle$ of a simply connected polygonal domain Ω, the space of C^1 cubic splines is defined by

$$\mathcal{S}_3^1(\triangle) := \{s \in C^1(\Omega) : \; s|_T \in \mathcal{P}_3, \text{ all } T \in \triangle\},$$

where $\mathcal{P}_3$ is the space of cubic bivariate polynomials.

In this paper we are interested in constructing spline interpolation methods that are based on a given set of Lagrange data and which deliver full approximation power. It is well known that to work with $\mathcal{S}_3^1(\triangle)$ successfully, we have to restrict our attention to special classes of triangulations. Indeed, for general triangulations, at this point it is not known whether interpolation at all of the vertices of $\triangle$ is even possible, and the dimension of $\mathcal{S}_3^1(\triangle)$ is also unknown. Moreover, it is known [3] that the space is defective in the sense that it does not give full approximation power on some triangulations (including the very regular type-I triangulations). This implies that in general it does not have a stable local basis.

There are several classes of triangulations where the situation is simplified. First, one can work on the refined triangulation $\triangle_{CT}$ which is obtained from $\triangle$ by splitting each triangle into three subtriangles. The classical Clough-Tocher C^1 cubic element can then be constructed locally from values and gradients at each of the vertices of $\triangle$. If certain cross-derivative information is also available, the method gives full approximation power, see

Mathematical Methods in CAGD: Oslo 2000
Tom Lyche and Larry L. Schumaker (eds.), pp. 0—1.

ISBN 0-8265-xxxx-x.

e.g. [2,12]. Another class of triangulations where Hermite interpolation can be performed easily with $\mathcal{S}_3^1(\triangle)$ are the triangulations $\triangle$ which are obtained from a quadrangulation by drawing in both diagonals, see [9,10,17]. To use these interpolation methods where only Lagrange data is available, we need enough data to estimate the required derivatives accurately.

In this paper we give a direct construction of a C^1 cubic spline interpolant which uses only Lagrange data. For other work on Lagrange interpolation methods based on spline spaces, see [4–5, 13–16].

The paper is organized as follows. In Sect. 2 we present a basic definition and some notation. The new concept of Lagrange minimal determining sets is introduced in Sect. 3 and several useful lemmas are established. Sect. 4 presents the main results for checkerboard triangulations, and in Sect. 5 we establish analogous results for certain reduced checkerboard triangulations. We conclude the paper with a numerical example and remarks.

§2. Notation and preliminaries

One of the keys to our discussion is the idea of a minimal determining set for a spline space. The concept was introduced in [1], and has since been heavily used in the multivariate spline literature. Here we need a more general form.

Definition 1. *Suppose Λ is a set of linear functionals defined on $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$. Then $\mathcal{M} \subseteq \Lambda$ is called a* determining set for $\mathcal{S}$ *provided that for any $s \in \mathcal{S}$, $\lambda s = 0$ all $\lambda \in \mathcal{M}$ implies $s \equiv 0$. The set $\mathcal{M}$ is called a* minimal determining set (MDS) for $\mathcal{S}$ *provided Λ does not contain any smaller determining set.*

Another way to describe a minimal determining set is to note that it is a set such that setting λs for all $\lambda \in \mathcal{M}$ uniquely determines s. It is easy to see, cf. [1], that if $\mathcal{M}$ is a MDS for $\mathcal{S}$, then $\dim \mathcal{S} = \#\mathcal{M}$. An MDS can also be used to construct a basis for $\mathcal{S}$. Indeed, suppose $\mathcal{M}$ is an MDS. Then for each $\lambda \in \mathcal{M}$, there is a unique spline $B_\lambda \in \mathcal{S}$ such that

$$\gamma B_\lambda = \delta_{\lambda,\gamma}, \qquad \text{all } \gamma \in \mathcal{M}. \tag{1}$$

The splines $\{B_\lambda\}_{\lambda \in \mathcal{M}}$ are called the dual basis splines corresponding to $\mathcal{M}$. We are especially interested in choosing $\mathcal{M}$ so that the dual basis splines have local support.

In [1] and the rest of the subsequent spline literature, Λ was always taken to consist of linear functionals which pick off Bernstein-Bézier coefficients. The essential difference in this paper is that we will use point evaluation functionals instead.

While we intend to work with Lagrange data, it is still useful to write polynomials of degree d in their Bernstein-Bézier form

$$p = \sum_{i+j+k=d} c_{ijk}^T B_{ijk}^d, \tag{2}$$

where B_{ijk}^T are the *Bernstein polynomials* of degree d associated with T. This is called the B-representation of p, and the c_{ijk}^T are called its B-coefficients.

Assuming $T := \langle u_1, u_2, u_3 \rangle$, it is common to associate these coefficients with the domain points $\xi^T_{ijk} := (iu_1 + ju_2 + ku_3)/d$. The point ξ^T_{d00} is at the vertex u_1 while the points $\xi^T_{d-1,1,0}$ and $\xi^T_{d-1,0,1}$ are said to be on the ring $R^T_1(u_1)$. We set $D^T_1(u_1) = \{\xi^T_{d00}, \xi^T_{d-1,1,0}, \xi^T_{d-1,0,1}\}$, with similar definitions for u_2, u_3.

§3. Lagrange minimal determining sets

If $\mathcal{M}$ is a set points such that the corresponding point evaluation functionals form a MDS for $\mathcal{S}$, we call $\mathcal{M}$ a Lagrange MDS for $\mathcal{S}$. We prove our first result for general d, although we intend to apply it for $d = 3$.

Lemma 2. *The set of all domain points in T is a Lagrange minimal determining set for the space $\mathcal{P}_d$.*

Proof: Let $n := (d+2)(d+1)/2$, and let $B_1, \ldots, B_n$ be the Bernstein polynomials B^d_{ijk} written in lexicographical order. Let c be the vector of coefficients in the same order, and let $b := (p(t_1), \ldots, p(t_n))$, where $t_1, \ldots, t_n$ are the values ξ^T_{ijk} in the same order. Then the coefficients must solve the system $Ac = b$, where $A_{ij} := B_i(t_j)$ for $i, j = 1, \ldots, n$. It is easy to show that the determinant of the matrix A is nonzero and depends only on d. This means that c is stably and uniquely determined by b. $\square$

It follows from the proof of Lemma 2 and the fact that $\sum_{i+j+k=d} B^d_{ijk} \equiv 1$ that $\{B^d_{ijk}\}$ is a stable basis for $\mathcal{P}_d$ in the sense that there exists a nonzero constant K_1 such that $\|p\|_\infty \le \|c\|_\infty$ and $\|c\|_\infty \le K_1 \|p\|_\infty$ for all $p \in \mathcal{P}_d$. Indeed, we can take $K_1 := \|A^{-1}\|$.

We give two examples of Lagrange minimal determining sets for C^1 cubic splines.

Lemma 3. *Suppose $\triangle$ consists of two triangles $T_1 := \langle u_1, u_2, u_3 \rangle$ and $T_2 := \langle u_1, u_3, u_4 \rangle$ sharing the edge $\langle u_1, u_3 \rangle$. Then the set*

$$\mathcal{M} := \Big(\bigcup_{i=1}^{3} D^{T_1}_1(u_i) \Big) \cup D^{T_2}_1(u_4) \cup \{\xi^{T_1}_{111}\}$$

is a Lagrange MDS for $\mathcal{S}^1_3(\triangle)$.

Proof: By Lemma 2, the points of $\mathcal{M}$ uniquely determine all ten B-coefficients of $s|_{T_1}$. Writing $s|_{T_2}$ in B-form, we see that by the C^1 smoothness conditions, all of its B-coefficients are determined except for $c := (c_{300}, c_{210}, c_{201})$. But then $Gc = b$, where $b := (p(\xi_{300}), p(\xi_{210}), p(\xi_{201}))$ and

$$G := \begin{pmatrix} B^3_{300}(\xi_{300}) & B^3_{210}(\xi_{300}) & B^3_{201}(\xi_{300}) \\ B^3_{300}(\xi_{210}) & B^3_{210}(\xi_{210}) & B^3_{201}(\xi_{210}) \\ B^3_{300}(\xi_{201}) & B^3_{210}(\xi_{201}) & B^3_{201}(\xi_{201}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{8}{27} & \frac{4}{9} & 0 \\ \frac{8}{27} & 0 & \frac{4}{9} \end{pmatrix}.$$

Thus, all B-coefficients of s are uniquely determined. $\square$

The construction in Lemma 3 is stable in the sense that the maximum coefficient of s is bounded by $K \max_{\xi \in \mathcal{M}} |p(\xi)|$, where K is a constant depending only on the smallest angle in $\triangle$. In this case we say that $\mathcal{M}$ is a stable Lagrange MDS for $\mathcal{S}^1_3(\triangle)$.

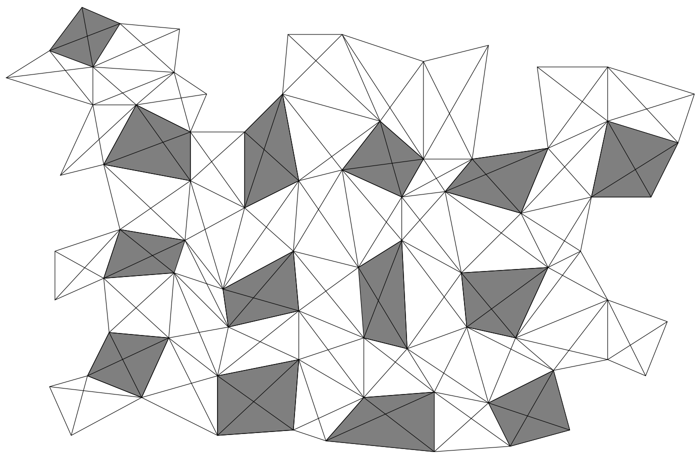

Fig. 1. The set $\mathcal{G}$ for a checkerboard triangulation.

Lemma 4. *Suppose $\triangle$ consists of four triangles $T_i := \langle u, u_i, u_{i+1}\rangle$ obtained by inserting both diagonals into a quadrilateral $Q := \langle u_1, u_2, u_3, u_4\rangle$. Then the set*

$$\mathcal{M} := \Big(\bigcup_{i=1}^{2} D_1^{T_1}(u_i)\Big) \cup D_1^{T_1}(u) \cup D_1^{T_2}(u_3) \cup D_1^{T_4}(u_4) \cup \{\xi_{111}^{T_1}\}$$

is a stable Lagrange MDS for $\mathcal{S}_3^1(\triangle)$.

Proof: Applying Lemma 3 to $T_1 \cup T_2$, it follows that the B-coefficients of s associated with domain points in $T_1 \cup T_2$ are uniquely and stable determined by the data. A similar argument shows that same holds for T_4. Then the coefficients in T_3 can be stably computed from the C^1 smoothness conditions. □

§4. Checkerboard triangulations

Definition 5. *Suppose $\diamondsuit$ is a quadrangulation consisting of quadrilaterals with largest interior angle less than π. Suppose that the quadrilaterals can be colored black and white in such a way that any two quadrilaterals sharing an edge have the opposite color. Then we call $\diamondsuit$ a* checkerboard *quadrangulation. The triangulation $\triangle$ which is obtained by drawing in both diagonals of all quadrilaterals will be called a* checkerboard *triangulation.*

Suppose $\mathcal{B}$ and $\mathcal{W}$ denote the sets of black and white quadrilaterals of $\diamondsuit$, respectively. Throughout this paper, we assume that all interior vertices of $\diamondsuit$ are of degree four. This assumption implies that there exists $\mathcal{G} \subset \mathcal{B}$ such that for every interior vertex v of $\diamondsuit$, there is a unique quadrilateral $Q \in \mathcal{G}$ sharing the vertex v. For $i = 1, 2, 3$, let $\mathcal{W}_i$ be the set of white quadrilaterals which share i edges with black quadrilaterals. Let $n_B = \#\mathcal{B}$ and $n_i := \#\mathcal{W}_i$ for $i = 1, 2, 3$, and let n_V be the total number of vertices of $\diamondsuit$. Fig. 1 shows a typical checkerboard triangulation in which the quadrilaterals in the set $\mathcal{G}$ are shaded grey. Note that we have not colored the other black quadrilaterals.

Theorem 6. *Suppose $\triangle$ is a checkerboard triangulation. Then*

$$\dim \mathcal{S}_3^1(\triangle) = 3n_V + 4n_B + 3n_1 + 2n_2 + n_3. \tag{3}$$

Moreover, the following set $\mathcal{M}$ of domain points is a stable Lagrange MDS:

1) *if $Q \in \mathcal{G}$, choose points as in Lemma 4,*
2) *if $Q \in \mathcal{B} \setminus \mathcal{G}$, choose points as in Lemma 4, leaving out the points in the sets $D_1^T(v)$ whenever v is a vertex of Q which is interior to $\diamondsuit$,*
3) *Suppose $Q := \langle u_1, u_2, u_3, u_4 \rangle \in \mathcal{W}$ and let $e_i := \langle v_i, v_{i+1} \rangle$ for $i = 1, 2, 3, 4$.*
    a) *if $Q \in \mathcal{W}_3$, choose the point $\xi_{300}^{T_1}$,*
    b) *if Q shares two edges with black quadrilaterals, say e_1, e_2, choose the points $\xi_{300}^{T_1}, \xi_{210}^{T_1}$ and the points in $D_1^{T_3}(v_4)$,*
    c) *if Q shares one edge with black quadrilaterals, say e_1, choose the points $\xi_{300}^{T_1}, \xi_{210}^{T_1}, \xi_{201}^{T_1}$ and $D_1^{T_2}(v_3)$, $D_1^{T_4}(v_4)$.*

Proof: To establish that $\mathcal{M}$ is a Lagrange MDS, suppose $s \in \mathcal{S}_3^1(\triangle)$ and that we are given values for $s(\xi)$ for all $\xi \in \mathcal{M}$. We need to show that all of the B-coefficients of s are uniquely determined. By Lemma 4, all B-coefficients of s associated with domain points lying in quadrilaterals $Q \in \mathcal{G}$ are uniquely determined. Now consider $Q \in \mathcal{B} \setminus \mathcal{G}$. For each vertex $v \in Q$ which is an interior vertex of $\diamondsuit$, the B-coefficients corresponding to domain points in the disk $D_1^T(v)$ are already uniquely determined by C^1 continuity from the neighboring pieces. Leaving the corresponding basis functions out, we can then argue exactly as in Lemma 4 to see that all B-coefficients of s corresponding to the remaining domain points in Q are uniquely determined.

Now suppose $Q \in \mathcal{W}$. If Q shares four edges with black quadrilaterals, then using the C^1 continuity, it is easy to see that all B-coefficients of s corresponding to domain points in Q are uniquely determined. If Q shares the three edges e_1, e_2, e_4 with black quadrilaterals, then all B-coefficients of $s|_{T_1}$ are uniquely determined by C^1 continuity except for $c_{300}^{T_1}$ which is uniquely determined by 3a). If Q shares the two edges e_1 and e_2 with black quadrilaterals, then the C^1 conditions imply that all of the B-coefficients of $s|_{T_1}$ are uniquely determined except for $c_{300}^{T_1}$ and $c_{210}^{T_1}$. These can be determined from the data of 3b) by solving a 2×2 system. In case 3c), all of the B-coefficients of $s|_{T_1}$ are uniquely determined by the C^1 conditions except for those associated with domain points in $D^{T_1}(v_Q)$, where v_Q is the crossing point of the two diagonals. Using the data of 3c) and solving the same 3×3 system as in the proof of Lemma 3 shows that these coefficients are also uniquely determined. The C^1 continuity and the additional data of 3c) can be used to uniquely determine the B-coefficients corresponding to the remaining domain points in Q.

Since we have shown that $\mathcal{M}$ is a MDS, it follows that $\dim \mathcal{S}_3^1(\triangle) = \#\mathcal{M}$. We have chosen three points for each vertex. This contributes $3n_V$ to the count. All black quadrilaterals Q contain $\xi_{111}^{T_1}$ and the three points in $D_1^{T_1}(v_Q)$, where v_Q is the crossing point of the two diagonals of Q. For each $Q \in \mathcal{W}_i$ with $1 \le i \le 3$, we have included $4 - i$ additional points.

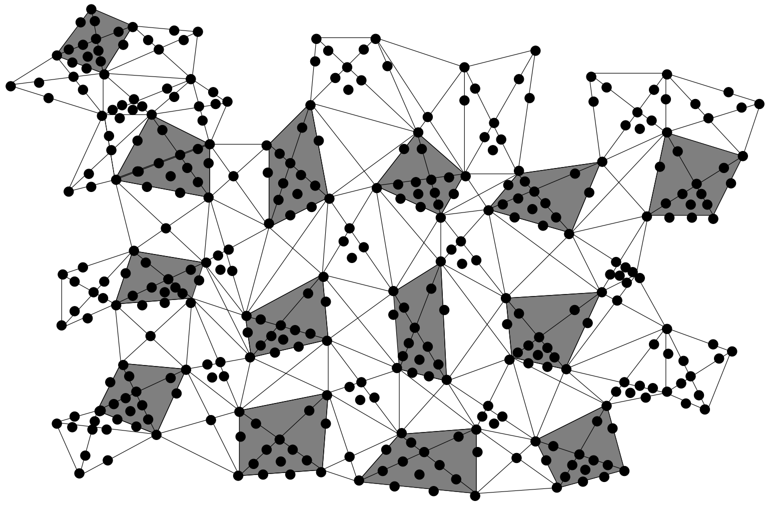

Fig. 2. The Lagrange MDS for the triangulation of Fig. 1.

Finally, we note that all of the above computations are stable in the sense that the size of the computed B-coefficients is bounded by a constant depending only on the smallest angle in the triangulation $\triangle$. This follows from the fact that the computations of Lemmas 2–4 are stable, and the fact that computing coefficients from C^1 smoothness conditions is automatically stable, cf. eg. [7]. $\square$

Fig. 2 shows the Lagrange MDS of Theorem 6 for the checkerboard triangulation of Fig. 1. We now examine the dual basis splines corresponding to $\mathcal{M}$. Given $\xi \in \mathcal{M}$, B_ξ is defined to be the spline in $\mathcal{S}_3^1(\triangle)$ such that $B_\xi(\xi) = 1$ and $B_\xi(\eta) = 0$ for all other points $\eta \in \mathcal{M}$.

Corollary 7. *Let $\triangle$ be a checkerboard triangulation, and let $\mathcal{M}$ be the set defined in Theorem 6. Then the dual basis splines corresponding to $\mathcal{M}$ form a stable local basis for $\mathcal{S}_3^1(\triangle)$.*

Proof: The proof of Theorem 6 shows that all B-coefficients of B_ξ are uniquely and stable determined. It remains to discuss the support of B_ξ. Suppose ξ lies in a quadrilateral Q_ξ. Then we claim

1) $\operatorname{supp}(B_\xi) = Q$ if $\xi \in \mathcal{W}$,
2) $\operatorname{supp}(B_\xi) \subset \operatorname{star}(Q_\xi)$ if $\xi \in \mathcal{B} \setminus \mathcal{G}$,
3) $\operatorname{supp}(B_\xi) \subset \operatorname{star}^2(Q_\xi)$ otherwise.

Here $\operatorname{star}(Q)$ is the union of all quadrilaterals which intersect with Q in at least one point, and $\operatorname{star}^2(Q) := \operatorname{star}(\operatorname{star}(Q))$. These assertions follow immediately from the checkerboard nature of the quadrangulation and the observation that B_ξ vanishes identically on Q whenever

1) $Q \neq Q_\xi$ and $Q \in \mathcal{G}$,
2) $Q \neq Q_\xi$ and $Q \in \mathcal{B}$ does not intersect Q_ξ. $\square$

We are now ready to discuss interpolation. Suppose $\triangle$ is a checkerboard triangulation, and that B_ξ are the dual basis functions of Corollary 7 corresponding to the Lagrange MDS $\mathcal{M}$ for $\mathcal{S}_3^1(\triangle)$ defined in Theorem 6. Given any function defined on Ω, let

$$\mathcal{I}f := \sum_{\xi\in\mathcal{M}} f(\xi)\, B_\xi. \tag{4}$$

By the duality (1) of the basis functions B_ξ, it is clear that the cubic spline $\mathcal{I}f$ interpolates f at all the points of $\mathcal{M}$, *i.e.*,

$$\mathcal{I}f(\xi) = f(\xi), \qquad \xi \in \mathcal{M}. \tag{5}$$

This includes in particular all vertices of $\diamondsuit$. We now give an error bound for this interpolation method.

Theorem 8. *There exists a constant C depending only on the smallest angle in $\triangle$ such that if f is in the Sobolev space $W_\infty^{m+1}(\Omega)$ with $0 \le m \le 3$,*

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}f)\|_{\infty,\Omega} \le C\ |\triangle|^{m+1-\alpha-\beta}\ |f|_{m+1,\infty,\Omega}, \tag{6}$$

for all $0 \le \alpha + \beta \le m$. Here $|\triangle|$ is the maximum of the diameters of the triangles in $\triangle$.

Proof: We apply Theorem 5.1 of [11]. Clearly, $\mathcal{I}p = p$ for all cubic polynomials. The hypothesis (5.3) of that theorem is trivial since $|f(\xi)| \le \|f\|_{T_\xi}$, where T_ξ is the triangle which contains ξ. $\square$

The analog of this theorem also holds for the p-norm, $1 \le p < \infty$, see Remark 13. The result of Theorem 8 can also be established with the weak-interpolation methods described in [6].

§5. Reduced checkerboard triangulations

In this section we triangulate a checkerboard quadrangulation in a different way which involves fewer triangles but still leads to Lagrange interpolating C^1 cubic splines with full approximation power.

Definition 9. *Suppose $\diamondsuit$ is a checkerboard quadrangulation, and let $\mathcal{B}$ and $\mathcal{W}$ be the sets of black and white quadrilaterals, respectively. Let $\triangle$ be the triangulation obtained by drawing in one diagonal of each black quadrilateral, and both diagonals of each white quadrilateral. We call $\triangle$ a* reduced checkerboard *triangulation.*

The proof of the following theorem is almost the same as that of Theorem 6 and Corollary 7.

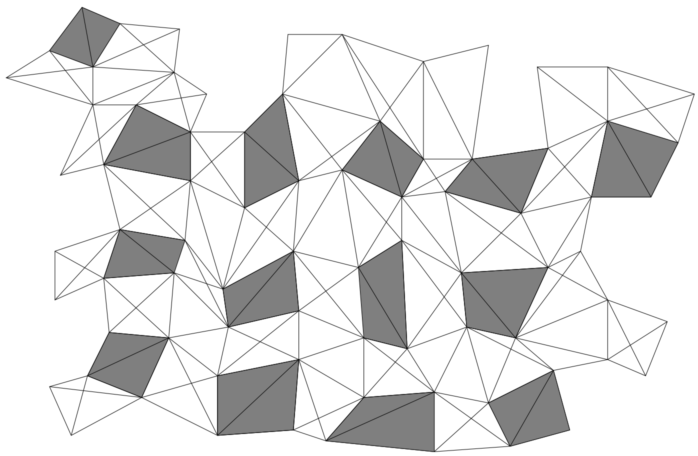

Fig. 3. A reduced checkerboard triangulation.

Theorem 10. *Suppose $\triangle$ is a reduced checkerboard triangulation. Then*

$$\dim \mathcal{S}_3^1(\triangle) = 3n_V + n_B + 3n_1 + 2n_2 + n_3, \tag{7}$$

where n_V, n_B, n_1, n_2, n_3 are as in Theorem 6. Moreover, the following set $\mathcal{M}$ of domain points is a stable Lagrange MDS:

1) *if $Q \in \mathcal{G}$, choose points as in Lemma 3,*
2) *if $Q \in \mathcal{B} \setminus \mathcal{G}$, choose points as in Lemma 3, leaving out the points in the sets $D_1^T(v)$ whenever v is a vertex of Q which is interior to $\diamondsuit$,*
3) *if Q is a white quadrilateral, choose points as in 3) of Theorem 6.*

The corresponding dual basis $\{B_\xi\}_{\xi \in \mathcal{M}}$ is a stable local basis for $\mathcal{S}_3^1(\triangle)$.

Using the basis functions B_ξ of this theorem, the C^1 cubic spline $\mathcal{I}f$ defined in (4) clearly satisfies the interpolation conditions (5). As before, $\mathcal{I}p = p$ for all cubic polynomials, and the error bound given in Theorem 8 also holds.

§6. Numerical example

We illustrate our method by interpolating Franke's test function

$$\begin{aligned} f(x,y) = &\tfrac{3}{4} e^{-\frac{1}{4}((9x-2)^2+(9y-2)^2)} + \tfrac{3}{4} e^{-\frac{1}{49}(9x+1)^2 - \frac{1}{10}(9y+1)} + \\ &\tfrac{1}{2} e^{-\frac{1}{4}((9x-7)^2+(9y-3)^2)} - \tfrac{1}{5} e^{-(9x-4)^2-(9y-7)^2} \end{aligned}$$

on the unit square $[0,1] \times [0,1]$. The first test was done on a sequence of uniform quadrangulations associated with the vertices $(i/N, j/N)$ for $i,j = 0, N$, where $N = 2^n + 1$ for $n = 1, \ldots, 9$. For each n, we computed the maximal error (using 25 points per quadrilateral). In the following table we list the number of data points (which is also the dimension of the corresponding spline space), the size of $|\triangle_n|$, the maximal error E_n, and the rate of convergence $\ln(E_{n-1}/E_n)/\ln(|\triangle_{n-1}|/|\triangle_n|)$. (Note that for this sequence of checkerboard

n	# Data	$\lvert\triangle_n\rvert$	Error	Rate
1	69	3.33E-01	2.12E-01	
2	177	2.00E-01	5.58E-02	2.61
3	549	1.11E-01	1.09E-02	2.78
4	1917	5.88E-02	2.78E-03	2.15
5	7149	3.03E-02	2.38E-04	3.70
6	27597	1.54E-02	2.00E-05	3.65
7	108429	7.75E-03	1.37E-06	3.92
8	429837	3.89E-03	8.83E-08	3.98
9	1711629	1.95E-03	5.57E-09	4.00

Tab. 1. Results for a sequence of uniform quadrangulations.

n	# Data	$\lvert\triangle_n\rvert$	Error	Rate
1	69	4.43E-01	3.53E-01	
2	177	2.96E-01	1.16E-01	2.76
3	549	1.70E-01	2.39E-02	2.85
4	1917	8.99E-02	2.31E-03	3.67
5	7149	4.58E-02	2.72E-04	3.17
6	27597	2.36E-02	2.51E-05	3.59
7	108429	1.19E-02	3.56E-06	2.85
8	429837	6.11E-03	2.69E-07	3.87
9	1711629	3.97E-03	1.92E-08	3.84

Tab. 2. Results for a sequence of randomized quadrangulations.

triangulations, $|\triangle_n|$ is not exactly one-half of $|\triangle_{n-1}|$). The table shows that the method achieves the convergence rate of four.

As a second test, we deformed the quadrangulations using a random number generator. Each vertex was deformed by a sufficiently small amount to maintain the topology of the quadrangulation and to insure convext quadrilaterals. This changed the values of $|\triangle_n|$, of course, and also affected the smallest angle in the triangulation which no doubt has some effect on the constant in the error bound. Table 2 shows the corresponding results which also show a convergence rate of four.

§7. Remarks

Remark 11. It is well known from graph coloring theory [8], Theorem 14, that a quadrangulation admits a black/white coloring if and only if all interior vertices are even. However, the set $\mathcal{G}$ does not exist for every such quadrangulation. For a simple example, consider a quadrangulation with one interior vertex of degree six surrounded by interior vertices of degree four. This is the reason why we require that all interior vertices of a checkerboard quadrangulation be of degree four.

Remark 12. In this paper we have focused on Lagrange interpolation. Clearly, our methods can also be used to create C^1 cubic splines which satisfy Hermite interpolation conditions where a function value and gradient values are specified at the vertices of $\triangle$.

Remark 13. Using Theorem 5.1 of [11], it is straightforward to establish the analog of Theorem 8 for arbitrary p-norms. However, in this case the constant C may also depend on the Lipschitz constant of the boundary if Ω is nonconvex.

Remark 14. The problem of extending the current results to more general classes of quadrangulations is currently under study. It seems to involve some difficult coloring problems which have not been addressed in the graph-theory literature.

Acknowledgments. The second author was supported by the National Science Foundation under grant DMS-9803340 and by the Army Research Office under grant DAAD-19-99-1-0160

References

1. Alfeld, P. and L. L. Schumaker, The dimension of bivariate spline spaces of smoothness r for degree $d \geq 4r+1$, Constr. Approx. **3** (1987), 189–197.
2. Alfeld, P. and L. L. Schumaker, Smooth finite elements based on Clough-Tocher triangle splits, manuscript, 2000.
3. Boor, C. de and K. Höllig, Approximation order from bivariate C^1-cubics: a counterexample, Proc. Amer. Math. Soc. **87** (1983), 649–655.
4. Davydov, O. and G. Nürnberger, Interpolation by C^1 splines of degree $q \geq 4$ on triangulations, J. Comput. Appl. Math., to appear.
5. Davydov, O., G. Nürnberger, and F. Zeilfelder, Cubic spline interpolation on nested polygon triangulations, in *Curve and Surface Fitting: Saint-Malo 1999,* A. Cohen, C. Rabut, and L. L. Schumaker (eds), Vanderbilt University Press, Nashville TN, 2000, 161–170.
6. Davydov, O., G. Nürnberger, and F. Zeilfelder, Bivariate spline interpolation with optimal approximation order, Constr. Approx., to appear.
7. Davydov, O. and L. L. Schumaker, On stable local bases for bivariate polynomial spline spaces, submitted, 1999.
8. Jensen, T. R. and B. Toft, *Graph Coloring Problems,* Wiley, New York, 1995.
9. Fraeijs de Veubeke, B., A conforming finite element for plate bending, J. Solids Structures **4** (1968), 95–108.
10. Lai, M.-J., Scattered data interpolation and approximation using bivariate C^1 piecewise cubic polynomials, Comput. Aided Geom. Design **13** (1996), 81–88.

11. Lai, M.-J. and L. L. Schumaker, On the approximation power of bivariate splines, Advances in Comp. Math. **9** (1998), 251–279.
12. Lai, M.-J. and L. L. Schumaker, Macro-elements and stable local bases for splines on Clough-Tocher triangulations, manuscript, 1999.
13. Nürnberger, G. and F. Zeilfelder, Spline interpolation on convex quadrangulations, in *Approximation Theory IX, Vol. 2: Computational Aspects,* Charles K. Chui and Larry L. Schumaker (eds), Vanderbilt University Press, Nashville TN, 1998, 259–266.
14. Nürnberger, G. and F. Zeilfelder, Interpolation by spline spaces on classes of triangulations, J. Comput. Appl. Math. **119** (2000), 347–376.
15. Nürnberger, G. and F. Zeilfelder, Developments in bivariate spline interpolation, J. Comput. Appl. Math. **121** (2000), 125–152.
16. Nürnberger, G. and F. Zeilfelder, Local Lagrange interpolation by cubic splines on a class of triangulations, Proc. Conf. Trends in Approximation Theory, submitted.
17. Sander, G., Bornes supérieures et inférieures dans l'analyse matricielle des plaques en flexion-torsion, Bull. Soc. Royale Sciences Liège **33** (1964), 456–494.

Günther Nürnberger
Institut für Mathematik
Universität Mannheim, D-618131 Mannheim, GERMANY
`nuernberger@euklid.math.uni-mannheim.de`

Larry L. Schumaker
Department of Mathematics, Vanderbilt Univerisity
Nashville, TN 37205, USA
`s@mars.cas.vanderbilt.edu`

Frank Zeilfelder
Max Planck Institut für Informatik
Stuhlsatzenhausweg 85
D-66123 Saarbrücken, GERMANY
`zeilfeld@mpi-sb.mpg.de`