Non-Existence of Star-supported Spline Bases

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Abstract. We consider polynomial spline spaces $\mathcal{S}_d^r(\triangle)$ of degree d and smoothness r defined on triangulations. It is known that for $d \geq 3r+2$, $\mathcal{S}_d^r(\triangle)$ possesses a basis of *star-supported* splines, i.e., splines whose supports are at most the set of triangles surrounding a vertex. Here we extend the theory by showing that for all $d \leq 3r+1$, there exist triangulations for which no such bases exist.

Keywords: multivariate splines, piecewise polynomial functions, triangulations.

§1. Introduction

Given a regular triangulation $\triangle$, let

$$\mathcal{S}_d^r(\triangle) := \{s \in C^r(\Omega) : \ s|_T \in \mathcal{P}_d \ \text{ for all triangles } T \in \triangle\},$$

where $\mathcal{P}_d$ is the space of polynomials of degree d, and Ω is the union of the triangles in $\triangle$. Such spline spaces have been heavily studied, cf. e.g. [1–10] and references therein.

Of particular interest for applications are spline spaces that possess a basis where every spline is supported only on the star of a vertex. (The star of a vertex is the set of triangles sharing that vertex.) Using such bases in applications leads to sparse linear systems. We call such splines *star-supported*. In [1] they are referred to as *minimally supported*, while in [5] they are called *vertex splines*. It is easy to see that for all $d \geq 1$, the spaces $\mathcal{S}_d^0(\triangle)$ have star-supported bases. In addition, for $r \geq 1$, it is known [6,7] that the spaces $\mathcal{S}_d^r(\triangle)$ possess bases of star-supported splines for all $d \geq 3r+2$. The following complement to this result is the main result of this paper.

Theorem 1.1. *Suppose $r \geq 1$ and $d \leq 3r+1$. Then there are triangulations $\triangle$ for which $\mathcal{S}_d^r(\triangle)$ does not have a star-supported basis.*

Our proof of this theorem is based on showing that there exist triangulations such that the number of linearly independent star-supported splines in the space

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$\mathcal{S}_d^r(\triangle)$ is less than the dimension of the space. Clearly, it suffices to work with an upper bound on the number of linearly independent star-supported splines, and a lower bound on the dimension.

Concerning the dimension of $\mathcal{S}_d^r(\triangle)$, as shown in [9],

$$\begin{aligned} \dim \mathcal{S}_d^r(\triangle) \geq & V_B(d^2+d-2rd+r^2-r)/2+V_I(d^2-3rd+2r^2) \\ & +3rd-d^2-3r(r-1)/2+1+\sigma, \end{aligned} \tag{1.1}$$

where V_B and V_I are the number of *boundary vertices* and *interior vertices* of $\triangle$, respectively, and

$$\sigma = \sum_{v \in \mathcal{V}_I} \sigma_v, \qquad \sigma_v = \sum_{j=1}^{d-r} (r+j+1-je_v)_+ \,. \tag{1.2}$$

Here $\mathcal{V}_I$ is the set of interior vertices, and e_v is the number of edges of *different slopes* attached to the vertex v.

To help simplify the proof, we shall work with *uniform type-I triangulations.* Such a triangulation is obtained by starting with a rectangular grid, which we may assume is generated by the lines $x_i = i/L$ and $y_j = j/L$ for $i, j = 0, \ldots, L$, and then drawing in all diagonals in the northeasterly direction. For a uniform type-I triangulation, the number of edges attached to each interior vertex v is six, and the number of edges with different slopes is three. Thus,

$$\sigma_v = \sum_{j=1}^{d-r} (r+1-2j)_+ \,, \qquad \text{all } v \in \mathcal{V}_I. \tag{1.3}$$

Moreover, for this type of triangulation, the number of interior vertices is significantly larger than the number of boundary vertices when L is large, and the term involving V_I dominates in (1.1). In view of this, to prove Theorem 1.1 it suffices to establish

Theorem 1.2. *Let $\triangle_H$ be the triangulation formed by the six triangles surrounding a typical interior vertex v of a type-I triangulation. Suppose $r \geq 1$ and $d \leq 3r+1$, and let*

$$\mathcal{V}_d^r(\triangle_H) = \{s \in \mathcal{S}_d^r(\triangle_H) : s \text{ vanishes up to order } r \text{ on the boundary of } \triangle_H\}\,.$$

Then

$$\dim \mathcal{V}_d^r(\triangle_H) < N_{r,d}(v) := d^2 - 3rd + 2r^2 + \sigma_v. \tag{1.4}$$

Theorem 1.2 (and thus also Theorem 1.1) is trivial in the case $d \leq r$ since in this case $\mathcal{S}_d^r(\triangle) \equiv \mathcal{P}_d$, and clearly there are no star-supported splines in the

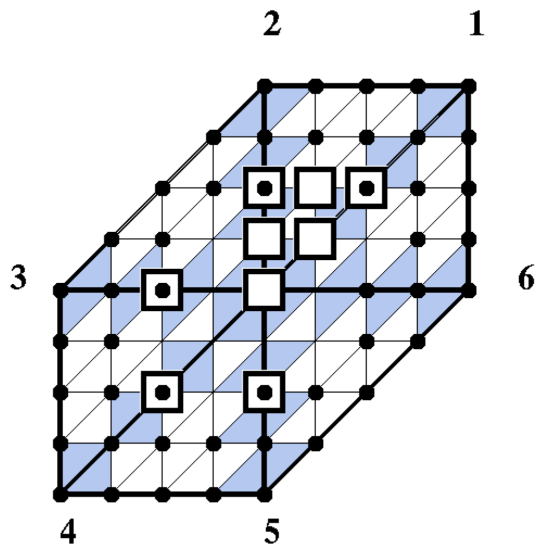

Fig. 1. A determining set for $\mathcal{V}_4^1(\triangle_H)$.

space. We give a proof of Theorem 1.2 for $r+1 \le d \le 2r$ in Sect. 2, and for $2r+1 \le d \le 3r+1$ in Sect. 4.

Throughout the paper we assume familiarity with the Bernstein-Bézier machinery as used e.g. in [1–7]. In particular, given a vertex v of $\triangle$, we recall that the *j-th ring $\mathcal{R}_j(v)$ around v* is the set of domain points at a distance j from v, while the *j-th disk $\mathcal{D}_j(v)$ around v* is the set of domain points at a distance of at most j from v. For each domain point P, we write $\lambda_P s$ for the associated coefficient of a spline s. We recall that a subset Γ of the domain points associated with a spline space $\mathcal{S}$ is called a *determining set* for $\mathcal{S}$ provided that the identically zero spline is the only spline $s \in \mathcal{S}$ whose coefficients $\lambda_P s$ are zero for all $P \in \Gamma$. We also recall that if Γ is a determining set, then $\dim \mathcal{S} \le \#\Gamma$.

We conclude this section with an example to illustrate the basic ideas. Figure 1 shows the B-net of a typical spline in the space $\mathcal{V}_4^1(\triangle_H)$. Coefficients marked with dots on the outermost two rings must be zero because we require function values and first derivatives of a spline in $\mathcal{V}_4^1(\triangle_H)$ to vanish on the boundary of $\triangle_H$. We can identify the points in rings $\mathcal{R}_0(v), \ldots, \mathcal{R}_2(v)$ with the B-net of a spline in $S_2^1(\triangle_H)$. The subset of points which are marked with a box in the figure form a minimal determining set for $\mathcal{S}_2^1(\triangle_H)$. This follows from the general theory of minimal determining sets for spline spaces on vertex stars given in [10], but can also easily be verified directly. For a spline $s \in \mathcal{V}_4^1(\triangle_H)$, not all of these coefficients can be set independently, since the smoothness conditions coupled with the boundary conditions imply that certain coefficients in the second ring must be automatically zero. In particular, the C^1 conditions indicated by the quadrilaterals in Figure 1 force the coefficients in the centers of the interior edges to vanish. We see that five coefficients associated with points in the minimal determining set of $\mathcal{S}_2^1(\triangle_H)$ must vanish. These are marked with boxes containing dots. Thus the dimension

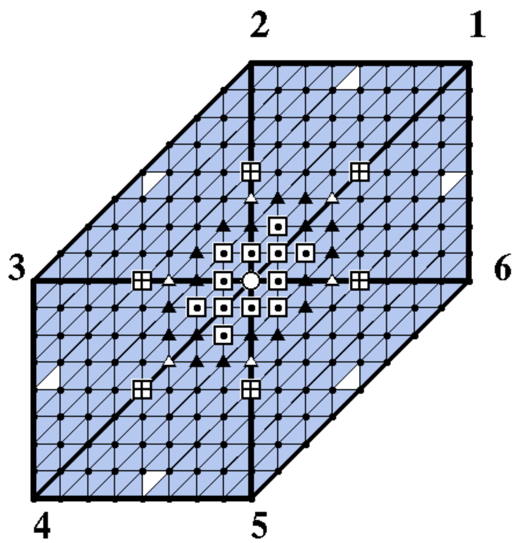

Fig. 2. Theorem 2.1 for $\mathcal{V}_8^4(\triangle_H)$.

of $\mathcal{V}_4^1(\triangle_H)$ cannot exceed the number of remaining empty boxes, which is 4. Since $N_{1,4} = 6$, this establishes Theorem 1.2 in this case.

§2. Proof of Theorem 1.1 for $r+1 \le d \le 2r$

Given an interior vertex v of a triangulation, let $\triangle_v$ be the triangulation consisting of the triangles which make up star(v). Let

$$\mathcal{V}_d^r(\triangle_v) = \{s \in \mathcal{S}_d^r(\triangle_v) : s \text{ vanishes up to order } r \text{ on the boundary of } \triangle_v\}. \quad (2.1)$$

In this section we show that for $r+1 \le d \le 2r$, $\mathcal{V}_d^r(\triangle_v)$ is trivial in the sense that it contains only the zero function. Applying this to an interior vertex v of a type-I triangulation $\triangle_I$ and noting that $N_{r,d}$ is positive, this implies Theorem 1.2 and thus also Theorem 1.1 in this case. We have the following slightly more precise result:

Theorem 2.1. *$\mathcal{V}_d^r(\triangle_v) \equiv \{0\}$ for all $d \le 2r$ if r is even, and for all $d \le 2r+1$ if r is odd.*

Proof: Consider first the case $r = 2m$. Suppose $v_1, \ldots, v_n$ are the vertices connected to v, and let $s \in \mathcal{V}_d^r(\triangle_v)$. Then by the boundary conditions, all coefficients of s associated with domain points on the rings $\mathcal{R}_{d-r}(v), \ldots, \mathcal{R}_d(v)$ are zero. Let $w_i^{(1)} = (rv + (d-r)v_i)/d$ for $i = 1, \ldots, n$. Applying Lemma 2.2 below to the rings $\mathcal{R}_j(w_1^{(1)}), \ldots, \mathcal{R}_j(w_n^{(1)})$ for $j = 1, \ldots, m$ shows that all coefficients of s are zero for domain points on these rings. Then all of the coefficients associated with points on the ring $\mathcal{R}_{d-r-1}(v)$ are zero if and only if $d \le 2r$. Now the process can be repeated based on the points $w_i^{(2)} = ((r+1)v + (d-r-1)v_i$, $i = 1, \ldots, n$. Repeating this process a total of $d - r - 1$ times, we find that all of the coefficients of s are zero.

The case $r = 2m + 1$ is similar. In the first step we apply Lemma 2.2 to the rings $\mathcal{R}_j(w_1^{(1)}), \ldots, \mathcal{R}_j(w_n^{(1)})$ for $j = 1, \ldots, m+1$. Then all coefficients associated with the ring $\mathcal{R}_{d-r+1}(v)$ are zero if and only if $d \leq 2r + 1$. We then repeat as before. $\square$

Figure 2 illustrates Theorem 2.1 for $\mathcal{V}_8^4(\triangle_H)$. The points $w_i^{(1)}$ alluded to in the proof are marked with a plus sign in a box. The boundary conditions imply that the coefficients associated with points on $\mathcal{R}_4(v)$ are zero. Then carrying out the first step of the proof, we see that the coefficients associated with the rings $\mathcal{R}_1(w_i^{(1)})$ and $\mathcal{R}_2(w_i^{(1)})$ are zero. These are marked with open triangles and with filled triangles, respectively. In the second step of the proof we get the coefficients in the rings $\mathcal{R}_2(w_i^{(2)})$ to be zero — these are marked as boxes containing a dot. Finally, in the third step, we find that the coefficient associated with the point at v (marked with an open circle) is also zero.

The following restatement of Lemma 3.3 of [7] was used in the proof of Theorem 2.1, and will also be used again later.

Lemma 2.2. *Let* $T^{[1]} = \langle v_0, v_1, v_2 \rangle$ *and* $T^{[2]} = \langle v_0, v_2, v_3 \rangle$ *be two triangles sharing the common edge* $e := \langle v_0, v_2 \rangle$. *Suppose* p_1, p_2 *are polynomials of degree* d *on* $T^{[1]}, T^{[2]}$ *which join together with* C^k *smoothness across the edge* e *for some* $0 \leq k \leq d$. *Given* $k \leq j \leq d$, *suppose that all coefficients of* p_1 *and* p_2 *in the set* $\mathcal{D}_{j-1}(v_0)$ *are zero, and define*

$$\begin{aligned} c_i &= c^{[1]}_{d-j,i,j-i} \\ c_{-i} &= c^{[2]}_{d-j,j-i,i} \end{aligned} \qquad i = 0, \ldots, j.$$

Suppose that

$$c_i = c_{-i} = 0 \quad \text{for } i = k - q + 1, \ldots, k$$

for some q *with* $m = k - 2q \geq -1$. *Suppose in addition that*

$$c_i = 0 \quad \text{for } i = 0, \ldots, m \quad \text{if } m \geq 0.$$

Then $c_i = c_{-i} = 0$ *for all* $i = 0, \ldots, k$.

Figure 3 illustrates Lemma 2.2 in the case $j = k = d = 3$ and $q = 2$. Here we are assuming that the coefficients associated with the small dots are all zero, and that the four coefficients associated with the points marked with a plus sign are also zero. Then the lemma asserts that the three points associated with the large dots must be zero.

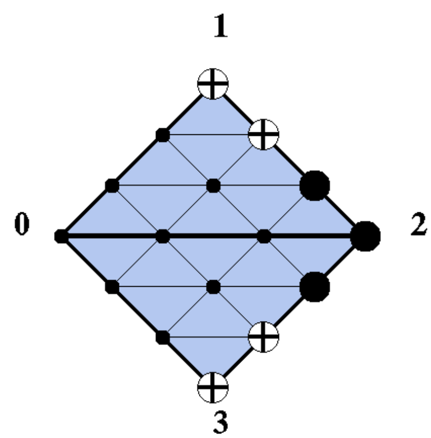

Fig. 3. Use of Lemma 2.2 for $j = k = d = 3$ and $q = 2$.

§3. Constructing Minimal Determining Sets on Cells

Let $\triangle_v$ be a triangulation which is obtained by connecting a vertex v to boundary vertices $v_1, \ldots, v_n$. Such a triangulation is called a *cell*. For $\ell = 1, \ldots, n$, let $T^{[\ell]}$ be the triangle with vertices $v, v_\ell, v_{\ell+1}$, where for convenience we identify $v_{n+1} := v_1$. We denote the Bézier points in triangle $T^{[\ell]}$ by $P^{[\ell]}_{ijk}$. We now establish the following modification of Theorem 3.3 in [10].

Theorem 3.1. *Let Γ_0 be the set of all Bézier domain points in the triangle $T^{[1]}$. Suppose $\mu_{n-e+1} < \cdots < \mu_n = n+1$ are such that the associated edges are pairwise noncollinear, and let $\mu_1 < \cdots < \mu_{n-e}$ be a complementary set so that $M = \{\mu_1, \ldots, \mu_n\} = \{2, \ldots, n+1\}$. For each $j = 1, \ldots, d-r$, let Γ_j be the first $nj - (r+j+1) + (r+j+1-je)_+$ points in the ordered set*

$$\{P^{[\mu_1]}_{d-j-r,0,j+r}, \ldots, P^{[\mu_1]}_{d-j-r,j-1,r+1}, \ldots, P^{[\mu_n]}_{d-j-r,0,j+r}, \ldots, P^{[\mu_n]}_{d-j-r,j-1,r+1}\}, \quad (3.1)$$

and let

$$\Gamma = \Gamma_0 \cup \bigcup_{j=1}^{d-r} \Gamma_j.$$

Then the set Γ is a determining set for $\mathcal{S}^r_d(\triangle_v)$.

Proof: This theorem differs from Theorem 3.3 of [10] in as much as the points in each group of (3.1) are written in reverse order. The proof of this version is nearly identical to the original one. Suppose s is a spline in $\mathcal{S}^r_d(\triangle_v)$ such that the coefficients $\lambda_P s$ corresponding to points $P \in \Gamma$ are all zero. We claim that this implies $s \equiv 0$, and thus Γ is a determining set for $\mathcal{S}^r_d(\triangle_v)$. To see this involves examining the rank of a certain block diagonal matrix A, cf. Lemma 2.1 of [10]. Here the submatrix B_j appearing in the proof of Theorem 2.2 in [10] involves different columns in the last block, and now corresponds to a Hermite-Birkhoff interpolation problem of the type described in Lemma 3.2 below. □

Lemma 3.2. *Let $\theta_1 < \cdots < \theta_{l+1}$ and let $0 < k < m$ be integers. Then the Hermite-Birkhoff interpolation problem of finding a polynomial p of degree $n := lm + k - 1$ satisfying*

$$\begin{aligned} p^{(j-1)}(\theta_i) &= r_{ij}, \qquad j = 1, \ldots, m \qquad i = 1, \ldots, l \\ p^{(m-j)}(\theta_{l+1}) &= r_{l+1,m-j}, \qquad j = 1, \ldots, k \end{aligned} \tag{3.2}$$

is poised, i.e., there exists a unique solution for every choice of the data r_{ij}.

Proof: It suffices to show that the homogeneous problem admits only the solution $p \equiv 0$. Suppose p satisfies (3.2) with 0 data. Then by Rolle's theorem, we conclude that $q := p^{(m-k)}$ has a k-tuple zero at each θ_i, $i = 1, \ldots, l$, $m - k$ zeros in each interval (θ_i, θ_{i+1}), $i = 1, \ldots, l - 1$, and an additional k-tuple zero at θ_{l+1}. Thus q has a total of $lk + (l - 1)(m - k) + k = n - (m - k) + 1$ zeros, counting multiplicities. But q is a polynomial of degree $n - (m - k)$, and hence q must be identically zero. Integrating $m - k$ times and using the fact that $p^{(j-1)}(\theta_1) = 0$ for $j = 1, \ldots, m$, we conclude that $p \equiv 0$. $\square$

We will apply Theorem 3.1 to the hexagonal triangulation $\triangle_H$ where $n = 6$ and $e = 3$. In this case we take $M = \{2, 3, 4, 5, 6, 1\}$.

§4. Proof of Theorem 1.1 for $2r + 1 \le d \le 3r + 1$

Throughout this section we assume that

$$d = 2r + k + 1 \tag{4.1}$$

with $0 \le k \le r$. Suppose v is an interior vertex of a uniform type-I triangulation, and as before, let $\triangle_H$ be the hexagonal triangulation corresponding to star(v). To prove Theorem 1.2 (and thus also Theorem 1.1), we need to show that the dimension of $\mathcal{V}_d^r(\triangle_H)$ is bounded by the number $N_{r,d}$ in (1.4). First, we observe that for this range of d,

$$\sigma_v = \begin{cases} m^2, & \text{if } r = 2m, \\ m^2 + m, & \text{if } r = 2m + 1, \end{cases} \tag{4.2}$$

and thus

$$N_{r,d} = \begin{cases} (k + m + 1)^2, & r = 2m, \\ (k + m + 1)(k + m + 2), & r = 2m + 1. \end{cases} \tag{4.3}$$

To get an upper bound on dim $\mathcal{V}_d^r(\triangle_H)$, we proceed as in the example $\mathcal{V}_4^1(\triangle_H)$ discussed in the introduction. We need to find a set Γ which determines $\mathcal{S}_d^r(\triangle_H)$ on $\mathcal{D}_{d-r-1}$, and then examine which of these points can be dropped in view of the interaction of the smoothness conditions with the boundary conditions. Note that $d - r - 1 = r + k$.

To find a set Γ which determines $\mathcal{S}_d^r(\triangle_H)$ on $\mathcal{D}_{d-r-1}$, we identify the domain points of $s \in \mathcal{S}_d^r(\triangle_H)$ lying in $\mathcal{D}_{d-r-1}$ with the domain points of a spline

Ring	$T^{[1]}$	$T^{[2]}, T^{[3]}, T^{[4]}$	$r = 2m$	$r = 2m+1$
$\mathcal{R}_0(v)$	1	0	0	0
$\mathcal{R}_1(v)$	2	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathcal{R}_r(v)$	$r+1$	0	0	0
$\mathcal{R}_{r+1}(v)$	$r+2$	1	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathcal{R}_{r+m}(v)$	$r+m+1$	m	0	0
$\mathcal{R}_{r+m+1}(v)$	$r+m+2$	$m+1$	1	0
$\mathcal{R}_{r+m+2}(v)$	$r+m+3$	$m+2$	3	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathcal{R}_{2r}(v)$	$2r+1$	r	$2m-1$	$2m$

Tab. 1. Number of points in Γ.

in $\mathcal{S}^r_{d-r-1}(\triangle_H)$, and then apply Theorem 3.1. We can choose Γ one ring at a time. The rows marked $\mathcal{R}_i(v)$ in Table 1 gives the number of points on the rings $\mathcal{R}_0(v), \ldots, \mathcal{R}_{r+k}$. For each ring $\mathcal{R}_i(v)$, Γ includes all of the points on that ring in triangle $T^{[1]}$. The number of such points is $i+1$ and is listed in the second column of the table. In addition, for each $i = r+1, \ldots, r+k$, Γ also includes the last $i-r$ points on $\mathcal{R}_i(v)$ in the triangles $T^{[2]}, T^{[3]}, T^{[4]}$. The numbers of these points are shown in the third column of the table. If $r = 2m$, Γ also includes the last $2(i-m)-1$ points on $\mathcal{R}_{r+i}(v)$ in the triangles $T^{[5]}$ for $i = m+1, \ldots, k$. These points are shown in the fourth column of the table. Finally, if $r = 2m+1$, Γ also includes the last $2(i-m-1)$ points on $\mathcal{R}_{r+i}(v)$ in the triangles $T^{[5]}$ for $i = m+2, \ldots, k$. These points are shown in the fifth column of the table.

We now show how to select a subset $\widetilde{\Gamma}$ of Γ which is a determining set for $\mathcal{V}^r_d(\triangle_H)$. Since the cardinality of $\widetilde{\Gamma}$ is an upper bound on dim $\mathcal{V}^r_d(\triangle_H)$, our proof of Theorem 1.2 will be complete if we show that $\#\widetilde{\Gamma} < N_{r,d}$. There are two cases.

Case 1 ($r = 2m$). For each $i = 1, \ldots, m$, let

$$\Gamma_{i,1} := \{\text{the } 2i \text{ domain points in } \mathcal{R}_{r+k-m+i}(v) \cap T^{[1]} \text{ closest to edge } \langle v, v_1\rangle\},$$

and set

$$\Gamma_{i,j} := \{\text{the } 2i \text{ domain points in } \mathcal{R}_{r+k-m+i}(v) \cap T^{[j-1]} \text{ closest to edge } \langle v, v_j\rangle\},$$

for $j = 2, \ldots, 6$. Define

$$\widetilde{\Gamma} := \Gamma \setminus \bigcup_{i=1}^{m} \bigcup_{j=1}^{6} \Gamma_{i,j}.$$

Note that the sets $\Gamma_{i,j}$ may contain some points which are not contained in Γ. To see that $\widetilde{\Gamma}$ is a determining set for $\mathcal{V}_d^r(\triangle_H)$, suppose s is a spline in this space with $\lambda_P s = 0$ for all $P \in \widetilde{\Gamma}$. Then since none of the points of Γ in the rings $\mathcal{R}_0(v), \ldots, \mathcal{R}_{r+k-m}(v)$ have been removed, all coefficients of s corresponding to points on these rings are zero. Now combining this with the fact that all coefficients of s on the rings $\mathcal{R}_{r+k+1}(v), \ldots, \mathcal{R}_d(v)$ are zero, Lemma 2.2 implies that all coefficients corresponding to the remaining points in $\Gamma \setminus \widetilde{\Gamma}$ are zero. But then $s \equiv 0$, and we have shown that $\widetilde{\Gamma}$ is a determining set.

We now compute the cardinality of $\widetilde{\Gamma}$. The number of points in $\widetilde{\Gamma}$ on the rings $\mathcal{R}_0(v), \ldots, \mathcal{R}_{m+k}(v)$ and lying in triangle $T^{[1]}$ is

$$\kappa_1 := \sum_{i=1}^{m+k+1} i = \binom{m+k+2}{2}.$$

The number of points in $\widetilde{\Gamma}$ on rings $\mathcal{R}_{r+1}(v), \ldots, \mathcal{R}_{m+k}(v)$ outside of $T^{[1]}$ is given by

$$\kappa_2 := 3 \sum_{i=r+1}^{m+k} (i-r) = \frac{3(k-m+1)(k-m)_+}{2}.$$

We get the factor 3 since such points occur in each of the triangles $T^{[i]}$ for $i = 2, 3, 4$. Now the number of points in $\widetilde{\Gamma}$ lying in triangle $T^{[1]}$ and on the rings $\mathcal{R}_{m+k+1}(v), \ldots, \mathcal{R}_{r+k}(v)$ is given by

$$\kappa_3 := \sum_{i=1}^{m} [m+k+i+1-4i]_+ = \sum_{i=1}^{m} [m+k+1-3i]_+ \le \frac{(m+k)(m+k-1)}{6}.$$

Finally, we count the number of points in $\widetilde{\Gamma}$ which lie outside the triangle $T^{[1]}$ and on the rings $\mathcal{R}_{m+k+1}(v), \ldots, \mathcal{R}_{r+k}(v)$. There are no such points near the edge $\langle v, v_6 \rangle$. Using the values in the fourth column of Table 1, we get

$$\kappa_4 := 3 \sum_{i=1}^{m} [m+k+i-r-2i]_+ = 3 \sum_{i=1}^{m} [k-m-i]_+ = \frac{3(k-m)(k-m-1)_+}{2}.$$

It follows that $n_{r,d} := \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4$ is an upper bound on the cardinality of $\widetilde{\Gamma}$, and

$$n_{r,d} \le \begin{cases} (2k^2 + 4km + 4k + 2m^2 + 4m + 3)/3, & \text{if } 0 \le k \le m, \\ (11k^2 - 14km + 4k + 11m^2 + 4m + 3)/3, & \text{if } m \le k \le 2m. \end{cases}$$

We claim that $n_{r,d} < N_{r,d}$ for all choices of k and m. To see this, note that $\delta(k) := N_{r,d} - n_{r,d}$ is a quadratic polynomial on each of the intervals $[0, m]$ and $[m, 2m]$. Simple calculus shows that both pieces are positive on their domains.

Case 2 $(r = 2m+1)$**.** This case is very similar to Case 1. For each $i = 0, \ldots, m$, let

$$\Gamma_{i,1} := \{\text{the } 2i+1 \text{ domain points in } \mathcal{R}_{r+k-m+i}(v) \cap T^{[1]} \text{ closest to edge } \langle v, v_1 \rangle\},$$

and set

$$\Gamma_{i,j} := \{\text{the } 2i+1 \text{ domain points in } \mathcal{R}_{r+k-m+i}(v) \cap T^{[j-1]} \text{ closest to } \langle v, v_j \rangle\},$$

for $j = 2, \ldots, 6$. Define

$$\widetilde{\Gamma} := \Gamma \setminus \bigcup_{i=0}^{m} \bigcup_{j=1}^{6} \Gamma_{i,j}.$$

If $s \in \mathcal{V}_d^r(\triangle_H)$ and $\lambda_P s = 0$ for all $P \in \widetilde{\Gamma}$, then all coefficients corresponding to points on the rings $\mathcal{R}_0(v), \ldots, \mathcal{R}_{r+k-m-1}(v)$ and $\mathcal{R}_{r+k+1}(v), \ldots, \mathcal{R}_d(v)$ are zero. Then Lemma 2.2 implies that all coefficients corresponding to the remaining points in $\Gamma \setminus \widetilde{\Gamma}$ are zero. But then $s \equiv 0$, and we have shown that $\widetilde{\Gamma}$ is a determining set.

To compute the cardinality of $\widetilde{\Gamma}$, first we note that κ_1 is the same as in Case 1. Now

$$\kappa_2 := 3 \sum_{i=r+1}^{m+k} (i - r) = \frac{3(k-m-1)(k-m)_+}{2},$$

$$\kappa_3 := \sum_{i=0}^{m} [m+k+i+2-2(2i+1)]_+ = \sum_{i=0}^{m} [m+k-3i]_+ \le \frac{(k+m)^2 + 3(k+m) + 2}{6},$$

and

$$\kappa_4 := 3 \sum_{i=0}^{m} [m+k+i-r+1-(2i+1)]_+ = 3 \sum_{i=0}^{m} [k-m-i]_+ = \frac{3(k-m)(k-m-1)_+}{2}.$$

This leads to

$$n_{r,d} \le \begin{cases} (2k^2 + 4km + 6k + 2m^2 + 6m + 4)/3, & \text{if } 0 \le k \le m, \\ (11k^2 - 14km - 3k + 11m^2 + 15m + 4)/3, & \text{if } m \le k \le 2m+1. \end{cases}$$

As in the first case, the difference $\delta(k) := N_{r,d} - n_{r,d}$ is a positive quadratic polynomial on each of the intervals $[0, m]$ and $[m, 2m+1]$. This completes the proof.

Figure 4 illustrates the choice of $\widetilde{\Gamma}$ for the spaces $\mathcal{V}_7^2(\triangle_H)$ and $\mathcal{V}_{10}^3(\triangle_H)$. The boxes represent points in the set Γ, and the boxes containing dots represent points in the set $\Gamma \cap \widetilde{\Gamma}$. The numbers of linearly independent splines in $\mathcal{V}_7^2(\triangle_H)$ and $\mathcal{V}_{10}^3(\triangle_H)$, respectively, are bounded by the numbers of empty boxes. The numbers are 14 and 26, respectively (see also Table 2 below).

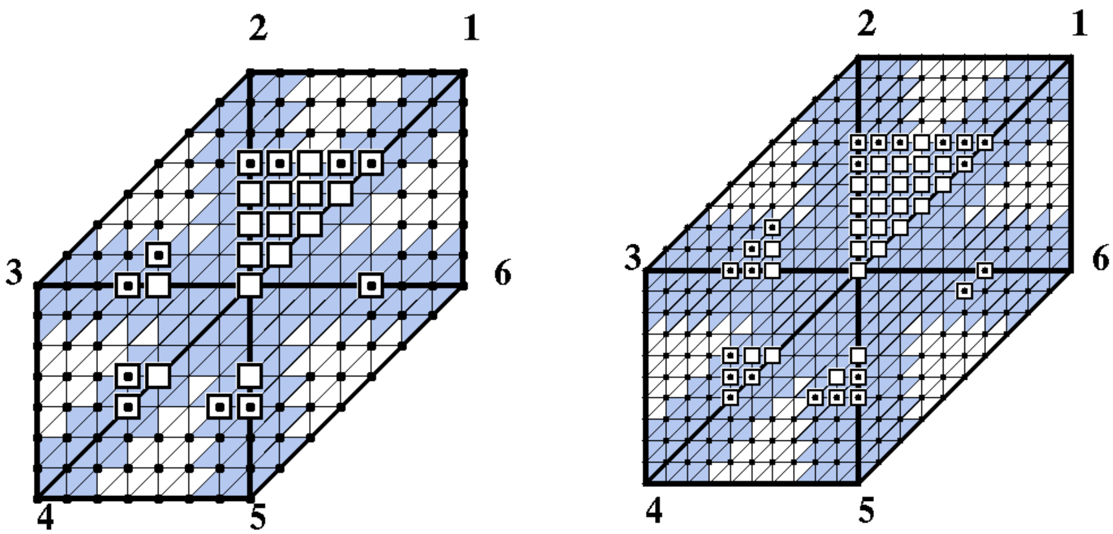

Fig. 4. Determining sets for $\mathcal{V}_7^2(\triangle_H)$ and $\mathcal{V}_{10}^3(\triangle_H)$.

§5. Remarks

Remark 5.1. To give an idea of the tightness of our upper bounds on the dimensions of the spaces $\mathcal{V}_d^r(\triangle_H)$, we have used the algebra package REDUCE to compute the dimensions for the case $d = 3r + 1$ for $r = 1, \ldots, 10$. The results are displayed in Table 2 which lists the true dimension $D_r := \dim\ \mathcal{V}_{3r+1}^r(\triangle_H)$, the value of our upper bound on $n_r := n_{r,3r+1}$, and the value of the coefficient $N_r := N_{r,3r+1}$ defined in (1.4).

r	D_r	n_r	N_r
1	4	4	6
2	14	14	16
3	25	26	30
4	44	45	49
5	64	66	72
6	92	94	100
7	121	124	132
8	158	161	169
9	196	200	210
10	242	246	256

Table 2. Computed values of D_r, n_r and N_r.

Remark 5.2. As explained in the introduction, to simplify the analysis, we have worked on uniform type-I triangulations generated by $L - 1$ interior lines in each direction on a unit square, and we have ignored the number of star-supported splines supported on the stars of boundary vertices. It is not difficult to include such splines in the counts. For example, it is easy to see that for S_4^1, the total number of star-supported splines is bounded by $4L^2 + 16L + 4$, while the dimension of the space is $6L^2 + 12L + 3$. Thus, we see that S_4^1 does not admit a star-supported spline basis for all $L \geq 3$. Similarly, for S_7^2, the total number of star-supported splines is bounded by $14L^2 + 40L + 8$, while the dimension of the space is $16L^2 + 28L + 7$. Thus, we see that S_7^2 does not admit a star-supported spline basis for all $L \geq 7$.

Remark 5.3. The problem of computing the exact number of star-supported splines in $\mathcal{S}_d^r(\triangle)$ which are associated with an interior vertex v of Δ is currently under study. The case $r = 1$ is considered in [5].

Remark 5.4. The figures for this paper were generated using a Java applet which can be found at http://www.math.utah.edu/~alfeld/MDS/.

References

1. Alfeld, P. and L. L. Schumaker, The dimension of bivariate spline spaces of smoothness r for degree $d \geq 4r + 1$, Constr. Approx. **3** (1987), 189–197.
2. Alfeld, P., B. Piper, Schumaker, and L. L., Minimally supported bases for spaces of bivariate piecewise polynomials of smoothness r and degree $d \geq 4r+1$, Comput. Aided Geom. Design **4** (1987), 105–123.
3. Alfeld, P., B. Piper, and L. L. Schumaker, An explicit basis for C^1 quartic bivariate splines, SIAM J. Numer. Anal. **24** (1987), 891–911.
4. Alfeld, P. and L. L. Schumaker, On the dimension of bivariate splines spaces of smoothness r and degree $d = 3r + 1$, Numer. Math. **57** (1990), 651–661.
5. Chui, C. K. and M. J. Lai, On bivariate vertex splines, in *Multivariate Approximation Theory III, ISNM 73,* W. Schempp and K. Zeller (eds), Birkhäuser, Basel, 1985, 84–115.
6. Hong, Dong, Spaces of bivariate spline functions over triangulation, Approx. Theory Appl. **7** (1991), 56–75.
7. Ibrahim, A. and L. L. Schumaker, Super spline spaces of smoothness r and degree $d \geq 3r + 2$, Constr. Approx. **7** (1991), 401–423.
8. Schumaker, L. L., On the dimension of spaces of piecewise polynomials in two variables, in *Multivariate Approximation Theory,* W. Schempp and K. Zeller (eds), Birkhäuser, Basel, 1979, 396–412.
9. Schumaker, L. L., Bounds on the dimension of spaces of multivariate piecewise polynomials, Rocky Mountain J. Math. **14** (1984), 251–264.
10. Schumaker, L. L., Dual bases for spline spaces on cells, Comput. Aided Geom. Design **5** (1988), 277–284.