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Control Curves and Knot Insertion for Trigonometric Splines

by

Per Erik Koch [1], Tom Lyche [2]

Marian Neamtu [3], and Larry L. Schumaker [4]

Abstract. We introduce control curves for trigonometric splines and show that they have properties similar to those for classical polynomial splines. In particular, we discuss knot insertion algorithms, and show that as more and more knots are inserted into a trigonometric spline, the associated control curves converge to the spline. In addition, we establish a convex-hull property and a variation-diminishing result.

1. Introduction

Since their introduction in [Schoenberg64], trigonometric splines have been studied in a number of papers. They turn out to have many properties in common with the classical polynomial splines. For example, they are linear combinations of locally supported functions (called trigonometric B-splines) which satisfy a three-term recurrence relation [LycheWinther79]. Approximation properties of trigonometric

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[2] Institutt for Informatikk, P. O. Box 1080, Blindern, 0316 Oslo, Norway, tom@ifi.uio.no. Supported by the Research Council of Norway through project STP28402 and SINTEF SI.

[3] Department of Mathematics, Vanderbilt University, Nashville, TN 37240, neamtu@orpheus.cas.vanderbilt.edu.

[4] Department of Mathematics, Vanderbilt University, Nashville, TN 37240, s@mars.cas.vanderbilt.edu. Supported by the National Science Foundation under grant DMS-9208413.

splines are well understood, and closely resemble the polynomial situation [Koch92, KochLycheSchumaker94]. Recently, wavelets associated with trigonometric splines have been constructed [LycheSchumaker94].

The objective of this paper is to derive a number of additional properties of trigonometric splines similar to familiar properties for the polynomial case. In particular, in Section 2 we introduce the notion of a control curve for trigonometric splines. This generalization is based on an analog of the classical Schoenberg operator. We also give a geometric interpretation of the evaluation algorithm for trigonometric splines, and prove a convex-hull property. In Section 3 we consider knot insertion, and in Section 4 we prove that as the number of inserted knots into the spline increases, the control points of the refined spline converge quadratically to this spline. Finally, in Section 5 we establish a variation-diminishing property of trigonometric splines, and in Section 6 we conclude the paper with a collection of remarks.

In the remainder of this section we introduce some basic definitions and notation. For any nonnegative integer k, we write

$$\sigma_k(x) := \sigma(kx), \quad \gamma_k(x) := \gamma(kx),$$

where $\sigma(x) := \sin \alpha x$ and $\gamma(x) := \cos \alpha x$, and α is a nonzero real constant (see Remark 2). For $m \geq 1$, let

$$\mathcal{T}_m := \begin{cases} \operatorname{span}\{1, \sigma_2(x), \gamma_2(x), \sigma_4(x), \gamma_4(x), \dots, \sigma_{m-1}(x), \gamma_{m-1}(x)\}, & m \text{ odd} \\ \operatorname{span}\{\sigma_1(x), \gamma_1(x), \sigma_3(x), \gamma_3(x), \dots, \sigma_{m-1}(x), \gamma_{m-1}(x)\}, & m \text{ even,} \end{cases}$$

be the space of *trigonometric polynomials of order* m. It is well known (see e.g. [LycheWinther79]) that $\mathcal{T}_m$ is the null space of the differential operator

$$D_m := \begin{cases} D(D^2 + 2^2\alpha^2)(D^2 + 4^2\alpha^2)\cdots(D^2 + (m-1)^2\alpha^2), & m \text{ odd} \\ (D^2 + 1^2\alpha^2)(D^2 + 3^2\alpha^2)\cdots(D^2 + (m-1)^2\alpha^2), & m \text{ even,} \end{cases}$$

where $D := d/dx$. An equivalent way of defining the spaces $\mathcal{T}_m$ is by

$$\mathcal{T}_m = \operatorname{span}\{\sigma^{m-i-1}(x)\gamma^i(x)\}_{i=0}^{m-1}.$$

In order to introduce spaces of piecewise trigonometric polynomials, let $I := [a, b]$ be a closed subinterval of the real line $\mathbb{R}$, and let

$$\Delta := \{a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b\}$$

be a partition of I into $k+1$ subintervals. Let $\mathcal{M} = (m_1, \dots, m_k)$ be a vector of integers satisfying $1 \leq m_i \leq m$, $i = 1, \dots, k$. Then the associated *space of trigonometric splines* (see e.g. [Schumaker81]) is defined by

$$\mathcal{S}(\mathcal{T}_m; \mathcal{M}; \Delta) := \{s \,:\, s|_{(x_i, x_{i+1})} \in \mathcal{T}_m, \quad i = 0, \dots, k, \text{ and}$$
$$D_-^{j-1}s(x_i) = D_+^{j-1}s(x_i), \quad j = 1, \dots, m - m_i, \quad i = 1, \dots, k\}.$$

It is well known that

$$\dim \mathcal{S}(\mathcal{T}_m; \mathcal{M}; \Delta) = n := m + \sum_{i=1}^{k} m_i.$$

To construct a basis of locally supported splines spanning $\mathcal{S}(\mathcal{T}_m; \mathcal{M}; \Delta)$, it is convenient to define the *extended knot sequence*

$$t = \{t_1 \le t_2 \le \cdots \le t_{n+m}\},$$

where

$$a = t_1 = \cdots = t_m, \qquad t_{n+1} = \cdots = t_{n+m} = b,$$

and where

$$\{t_{m+1} \le \cdots \le t_n\}$$

is the set obtained by repeating each x_i a total of m_i times, $i = 1, \ldots, k$. Throughout the paper we will assume that the knots t are such that

$$t_{j+m} - t_j < \pi/\alpha, \qquad j = 1, \ldots, n. \tag{1.1}$$

Since all information about $\mathcal{M}$ and Δ is contained in the sequence t, for the sake of brevity, we will write $\mathcal{S}_{m,t}$ instead of $\mathcal{S}(\mathcal{T}_m; \mathcal{M}; \Delta)$.

We now introduce the *normalized trigonometric B-splines* T_j^m associated with the knot sequence t by recursion. The first-order normalized trigonometric B-spline $T_j^1(x)$ is given by

$$T_j^1(x) := \begin{cases} 1, & t_j \le x < t_{j+1} \\ 0, & \text{otherwise,} \end{cases}$$

while the normalized trigonometric B-spline T_j^r of order $r = 2, \ldots, m$ are defined by the recursion [LycheWinther79]

$$T_j^r(x) := \sigma(x - t_j) Q_j^{r-1}(x) + \sigma(t_{j+r} - x) Q_{j+1}^{r-1}(x), \tag{1.2}$$

where

$$Q_j^r(x) := \begin{cases} T_j^r(x)/\sigma(t_{j+r} - t_j), & t_j < t_{j+r} \\ 0, & \text{otherwise.} \end{cases}$$

The trigonometric B-splines share many of the properties of the classical polynomial B-splines [Schumaker81]. For example, the B-spline T_j^m is finitely supported on $[t_j, t_{j+m}]$ and it is positive in the interior of its support. Moreover, the $\{T_j^m\}_{j=1}^n$ are linearly independent and span $\mathcal{S}_{m,t}$. Hence every element $s \in \mathcal{S}_{m,t}$ has a unique representation of the form

$$s(x) = \sum_{j=1}^{n} c_j T_j^m(x), \quad c_j \in \mathbb{R}, \quad j = 1, \ldots, n. \tag{1.3}$$

As an immediate consequence of the recurrence relation (1.2), we have the following *algorithm for the evaluation* of the spline (1.3), see e.g. [LycheWinther79].

Algorithm 1.1. *Let $x \in I$ and let μ be such that $x \in [t_\mu, t_{\mu+1})$.*

Set $c_j^0 := c_j$, $j = \mu - m + 1, \ldots, \mu$.
For $r = 1$ to $m - 1$,
For $j = \mu - m + r + 1$ to μ,

$$c_j^r := \frac{\sigma(x - t_j)}{\sigma(t_{j+m-r} - t_j)} c_j^{r-1} + \frac{\sigma(t_{j+m-r} - x)}{\sigma(t_{j+m-r} - t_j)} c_{j-1}^{r-1}. \tag{1.4}$$

Then $s(x) = c_\mu^{m-1}$.

2. Control Curves for Trigonometric Splines

Control points and control polygons of polynomial splines play an important role in CAGD (see *e.g.* [Farin88, HoschekLasser93]). It is therefore natural to ask whether these notions can also be defined for trigonometric splines. In this section we interpret the spline coefficients in (1.3) geometrically as control points, and we define an analog of a control polygon for trigonometric splines. For $m > 1$, let

$$\mathcal{L}_m := \operatorname{span}\{\sigma_{m-1}(x), \gamma_{m-1}(x)\}. \tag{2.1}$$

Definition 2.1. *Let $m > 1$. Suppose s is a trigonometric spline function of the form (1.3), and let t_j^* be the knot averages given by*

$$t_j^* := \frac{1}{m-1} \sum_{i=j+1}^{j+m-1} t_i. \tag{2.2}$$

We define the points $C_j := (t_j^*, c_j)$, $j = 1, \ldots, n$, to be the control points of the spline s. The function c which interpolates the values c_j at the points t_j^*, $j = 1, \ldots, n$, and which is such that $c|_{(t_j^*, t_{j+1}^*)} \in \mathcal{L}_m$, $j = 1, \ldots, n-1$, will be called the control curve of the spline s.

We note that the t_j^*'s defined in (2.2) satisfy $(m-1)(t_{j+1}^* - t_j^*) < \pi/\alpha$ in view of (1.1). The above definition is motivated by the following result derived in [Koch92].

Theorem 2.2. *Given an integer $m > 1$, let $V_m : C(I) \to \mathcal{S}_{m,t}$ be the linear operator defined by*

$$V_m g(x) := \sum_{j=1}^{n} g(t_j^*) T_j^m(x), \quad g \in C(I), \quad x \in I.$$

Then V_m reproduces the space $\mathcal{L}_m$, i.e.,

$$V_m g \equiv g, \qquad \text{for all } g \in \mathcal{L}_m.$$

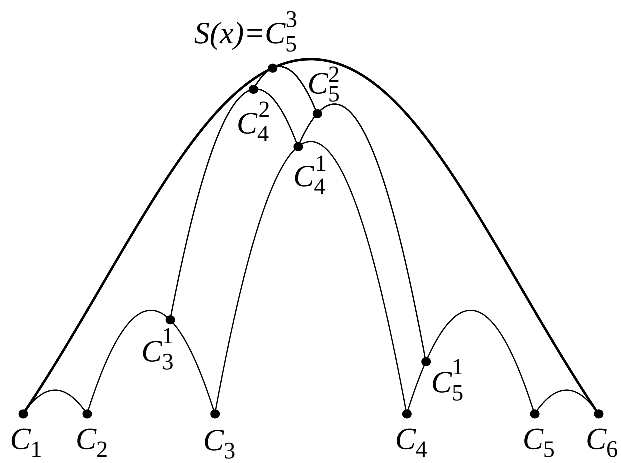

Figure 1. Algorithm 1.1 applied to evaluate a cubic spline with knots $t = \{0,0,0,0,1,2,3,3,3,3\}$ at $x = 1.25$. Here $\alpha = 1$, $m = 4$, $n = 6$, and $\mu = 5$.

It is quite surprising that the points t_j^* in this theorem are at the same locations as those in the Schoenberg variation-diminishing polynomial spline operator (see *e.g.* [Schumaker81]). V_m can be viewed as a trigonometric analog of the Schoenberg operator. Moreover, the theorem also suggests that the space $\mathcal{L}_m$ can be considered as a natural substitute for the space of linear functions appearing in the standard spline theory. Also note that the above definition of a control curve for trigonometric splines is reminiscent of the control polygon for polynomial splines in the sense that if all the control points C_j lie on a curve g, where $g \in \mathcal{L}_m$, then the associated control curve c will be identical with the spline s. Figure 2a shows an example of a trigonometric spline together with its associated control curve.

It is now possible to give a geometric interpretation of Algorithm 1.1 for evaluating a trigonometric spline of the form (1.3) at a point $x \in I$. Suppose $\{c_j^r\}$ are the numbers produced by Algorithm 1.1. For each $r = 0, \ldots, m-1$, and $j = \mu - m + r + 1, \ldots, \mu$, these numbers can be associated with the points

$$C_j^r := (t_{j,r}^*, c_j^r),$$

where

$$t_{j,r}^* := \frac{1}{m-1} \sum_{i=j+1}^{j+m-r-1} t_i + \frac{rx}{m-1}.$$

Note that $t_{\mu,m-1}^* = x$, and that the $t_{j,r}^*$'s depend on the variable x except when $r = 0$, in which case $t_{j,0}^* = t_j^*$.

Proposition 2.3. *For each $r = 1, \dots, m-1$, and $j = \mu - m + r + 1, \dots, \mu$, the point C_j^r lies on the curve*

$$G_j^r := \{(\xi, g_j^r(\xi)),\ \xi \in [t^*_{j-1,r-1}, t^*_{j,r-1}]\},$$

where g_j^r is the unique function in $\mathcal{L}_m$ which interpolates c_{j-1}^{r-1} and c_j^{r-1} at $t^*_{j-1,r-1}$ and $t^*_{j,r-1}$, respectively.

Proof: The function g_j^r is given by

$$g_j^r(\xi) = \frac{\sigma_{m-1}(\xi - t^*_{j-1,r-1})}{\sigma_{m-1}(t^*_{j,r-1} - t^*_{j-1,r-1})} c_j^{r-1} + \frac{\sigma_{m-1}(t^*_{j,r-1} - \xi)}{\sigma_{m-1}(t^*_{j,r-1} - t^*_{j-1,r-1})} c_{j-1}^{r-1}.$$

With $\xi = t^*_{j,r}$ this reduces to formula (1.4), and we have $g_j^r(t^*_{j,r}) = c_j^r$. ■

Figure 1 illustrates the steps of Algorithm 1.1. We now establish an analog of the *convex-hull property* of the classical splines. First we need a definition.

Definition 2.4. *Let B be a set in $\mathbb{R}^2$. We call B trigonometrically convex of order m (with $m \geq 2$) if for any two points $(\xi_i, c_i) \in B, i = 1, 2$, with $0 < \xi_2 - \xi_1 < \pi/((m-1)\alpha)$, the curve of the form $\{(\xi, g(\xi)), g \in \mathcal{L}_m\}$ connecting these two points lies entirely in B, that is*

$$\left(\xi, \frac{\sigma_{m-1}(\xi_2 - \xi)}{\sigma_{m-1}(\xi_2 - \xi_1)} c_1 + \frac{\sigma_{m-1}(\xi - \xi_1)}{\sigma_{m-1}(\xi_2 - \xi_1)} c_2\right) \in B, \quad \xi \in (\xi_1, \xi_2).$$

The trigonometric convex hull of order m of a set B, denoted by $TCH_m(B)$, is the smallest trigonometrically convex set of order m containing B.

Theorem 2.5. *Let $S := \{S(x),\ x \in I\} := \{(x, s(x)),\ x \in I\}$ be a trigonometric spline of order m on I, and let $\mathcal{C} := \{C_j\}_{j=1}^n$ be the set of its associated control points. Then S lies in the trigonometric convex hull of order m of $\mathcal{C}$, i.e.,*

$$S(x) \in TCH_m\{\mathcal{C}\}, \quad x \in I.$$

Proof: In computing $S(x) = C_\mu^{m-1}$ by Algorithm 1.1, it is clear by definition of a trigonometrically convex set that all of the points C_j^r arising in the steps of the algorithm belong to $\mathrm{TCH}_m\{\mathcal{C}\}$. ■

Note that if the control points of S all lie on a curve G of the form $\{(x, g(x)),\ g \in \mathcal{L}_m,\ x \in I\}$, then the trigonometric convex hull of $\mathcal{C}$ degenerates to the curve G itself. Theorem 2.5 is a generalization of a result for circular Bernstein-Bézier polynomials established in [AlfeldNeamtuSchumaker94].

3. Knot Insertion

In order to distinguish between B-splines associated with different knot vectors, in this section we will use the notation $T^m_{j,t}$ for the B-splines defined on the knot sequence t.

Let τ, t be two knot sequences with τ a subsequence of t. The problem of knot insertion can be viewed as a problem of converting a spline function from one basis to another refined basis. We first consider inserting one knot into the spline curve.

Theorem 3.1. *Suppose the refined knot sequence is* $t = \tau \cup \{\theta\}$, *where* $\theta \in [\tau_\mu, \tau_{\mu+1})$. *Then the trigonometric B-splines* $T^m_{j,\tau}$ *can be expressed in terms of the B–splines* $T^m_{j,t}$ *as*

$$T^m_{j,\tau} = d_j T^m_{j,t} + e_j T^m_{j+1,t}, \qquad j = 1, \ldots, n, \tag{3.1}$$

where the coefficients d_j, e_j *are given by*

$$d_j = \begin{cases} 1, & j \le \mu - m + 1 \\ \frac{\sigma(\theta - \tau_j)}{\sigma(\tau_{j+m-1} - \tau_j)}, & \mu - m + 1 < j \le \mu \\ 0, & \mu < j \end{cases} \tag{3.2}$$

and

$$e_j = \begin{cases} 0, & j \le \mu - m \\ \frac{\sigma(\tau_{j+m} - \theta)}{\sigma(\tau_{j+m} - \tau_{j+1})}, & \mu - m < j \le \mu - 1 \\ 1, & \mu - 1 < j. \end{cases} \tag{3.3}$$

Moreover,

$$\sum_{j=1}^{n} c_j T^m_{j,\tau} = \sum_{j=1}^{n+1} b_j T^m_{j,t}$$

if and only if

$$b_j = \begin{cases} c_j, & j \le \mu - m + 1 \\ \frac{\sigma(\theta - \tau_j)}{\sigma(\tau_{j+m-1} - \tau_j)} c_j + \frac{\sigma(\tau_{j+m-1} - \theta)}{\sigma(\tau_{j+m-1} - \tau_j)} c_{j-1}, & \mu - m + 1 < j \le \mu \\ c_{j-1}, & \mu < j, \end{cases} \tag{3.4}$$

for $j = 1, \ldots, n + 1$.

Proof: By Theorem 2.2,

$$g = \sum_{j=1}^{n} g(\tau^*_j) T^m_{j,\tau} = \sum_{j=1}^{n+1} g(t^*_j) T^m_{j,t},$$

for all $g \in \mathcal{L}_m$. Choosing alternately, $g(x) = \sigma_{m-1}(x)$ and $g(x) = \gamma_{m-1}(x)$, and inserting (3.1) leads to the linear system

$$\sigma_{m-1}(t^*_j) = d_j \sigma_{m-1}(\tau^*_j) + e_{j-1} \sigma_{m-1}(\tau^*_{j-1})$$
$$\gamma_{m-1}(t^*_j) = d_j \gamma_{m-1}(\tau^*_j) + e_{j-1} \gamma_{m-1}(\tau^*_{j-1}),$$

which gives

$$d_j = \frac{\sigma_{m-1}(t_j^* - \tau_{j-1}^*)}{\sigma_{m-1}(\tau_j^* - \tau_{j-1}^*)}, \qquad e_{j-1} = \frac{\sigma_{m-1}(\tau_j^* - t_j^*)}{\sigma_{m-1}(\tau_j^* - \tau_{j-1}^*)}.$$

The formulae (3.2) and (3.3) for d_j and e_j can now be obtained by considering each of the three cases for j separately. For example, if j is such that $\mu - m + 1 < j \le \mu$ then

$$\begin{aligned} \tau_{j-1}^* &= (\tau_j + \ldots + \tau_{j+m-2})/(m-1) \\ \tau_j^* &= (\tau_{j+1} + \ldots + \tau_{j+m-1})/(m-1) \\ t_j^* &= (\tau_{j+1} + \ldots + \tau_{j+m-2} + \theta)/(m-1), \end{aligned}$$

from which the formulae follow. The other two cases can be handled analogously.

The second part of the theorem follows immediately from (3.1) and the support properties of the B-splines. ■

Theorem 3.1 is an exact analog of the corresponding result for polynomial splines, see [Boehm80]. We now develop an analog of the so-called *Oslo Algorithm* of [CohenLycheRiesenfeld80] which allows the insertion of several new knots *simultaneously.*

Let p be the number of new knots inserted in τ. In analogy with the polynomial case, we introduce *discrete trigonometric B-splines* α_j^m recursively by

$$\alpha_j^1(i) := \begin{cases} 1, & \tau_j \le t_i < \tau_{j+1} \\ 0, & \text{otherwise}, \end{cases} \tag{3.5}$$

and

$$\alpha_j^r(i) := \sigma(t_{i+r-1} - \tau_j)\beta_j^{r-1}(i) + \sigma(\tau_{j+r} - t_{i+r-1})\beta_{j+1}^{r-1}(i), \tag{3.6}$$

for $i = 1, \ldots, n+p$, where

$$\beta_j^r(i) := \begin{cases} \alpha_j^r(i)/\sigma(\tau_{j+r} - \tau_j), & \tau_j < \tau_{j+r} \\ 0, & \text{otherwise}, \end{cases} \tag{3.7}$$

for $j = 1, \ldots, n$ and $r = 2, \ldots, m$. It follows directly from the recursion that

$$\alpha_j^m(i) = 0, \qquad \text{for all } i \text{ with } t_i \notin [\tau_j, \tau_{j+m}].$$

We are now ready to prove an analog of the Oslo Algorithm.

Theorem 3.2. *For all* $j = 1, \ldots, n$,

$$T_{j,\tau}^m = \sum_{i=1}^{n+p} \alpha_j^m(i) T_{i,t}^m. \tag{3.8}$$

Moreover,

$$\sum_{j=1}^{n} c_j T_{j,\tau}^m = \sum_{i=1}^{n+p} b_i T_{i,t}^m \tag{3.9}$$

if and only if

$$b_i = \sum_{j=\mu-m+1}^{\mu} \alpha_j^m(i) c_j,$$

for $i = 1, \ldots, n+p$, where μ is such that $t_i \in [\tau_\mu, \tau_{\mu+1})$.

Proof: The existence of coefficients $\alpha_j^m(i)$ such that (3.8) holds follows immediately from the fact that t is a refinement of τ. Substituting (3.8) in (3.9) implies

$$b_i = \sum_{j=1}^{n+p} \alpha_j^m(i) c_j,$$

It remains to show that the $\alpha_j^m(i)$ satisfy formulae (3.5)–(3.7). We proceed by induction on m. The claim is trivial for $m = 1$. We now assume that it holds for $m-1$, and prove it for m.

We recall [LycheWinther79, Schumaker81] the following *Marsden identity for trigonometric splines*: for all $k = 1, \ldots, m$,

$$\sigma^{k-1}(y-x) = \sum_{j=1}^{n} \psi_{j,\tau}^k(y) T_{j,\tau}^k(x) = \sum_{j=1}^{n+p} \psi_{j,t}^k(y) T_{j,t}^k(x),$$

where

$$\psi_{j,\tau}^k(y) := \prod_{i=j+1}^{j+k-1} \sigma(y-\tau_i), \qquad \psi_{j,t}^k(y) := \prod_{i=j+1}^{j+k-1} \sigma(y-t_i).$$

Setting $b_i = \psi_{i,t}^k(y)$ and $c_j = \psi_{j,\tau}^k(y)$ in (3.9), we conclude that

$$\psi_{i,t}^k(y) = \sum_{j=1}^{n+p} \alpha_j^k(i) \psi_{j,\tau}^k(y) \tag{3.10}$$

for $k = 1, \ldots, m$. By the inductive hypothesis, we know that the $\alpha_j^k(i)$ satisfy formulae (3.5)–(3.7) for all $1 \leq k \leq m-1$. We now show that this is also the case for $k = m$. Consider

$$s := \sum_{j=\mu-m+1}^{\mu} \left[\sigma(t_{i+m-1} - \tau_j)\beta_j^{m-1}(i) + \sigma(\tau_{j+m} - t_{i+m-1})\beta_{j+1}^{m-1}(i)\right] \psi_{j,\tau}^m(y)$$

$$= \sum_{j=\mu-m+2}^{\mu} \left[\sigma(t_{i+m-1} - \tau_j)\psi_{j,\tau}^m(y) + \sigma(\tau_{j+m-1} - t_{i+m-1})\psi_{j-1,\tau}^m(y)\right] \beta_j^{m-1}(i)$$

$$= \sum_{j=\mu-m+2}^{\mu} \Big[\sigma(t_{i+m-1} - \tau_j)\sigma(y - \tau_{j+m-1}) +$$

$$\sigma(\tau_{j+m-1} - t_{i+m-1})\sigma(y - \tau_j)\Big] \psi_{j,\tau}^{m-1}(y)\beta_j^{m-1}(i).$$

Using the relation

$$e^{2i\alpha a} - e^{2i\alpha b} = 2ie^{i\alpha(a+b)}\sigma(a-b), \qquad i = \sqrt{-1},$$

the term in square brackets simplifies to

$$\sigma(y - t_{i+m-1})\sigma(\tau_{j+m-1} - \tau_j).$$

Thus, using (3.10) for $m-1$, we have

$$s = \sigma(y - t_{i+m-1}) \sum_{j=\mu-m+2}^{\mu} \psi_{j,\tau}^{m-1}(y)\sigma(\tau_{j+m-1} - \tau_j)\beta_j^{m-1}(i)$$

$$= \sigma(y - t_{i+m-1}) \sum_{j=\mu-m+2}^{\mu} \psi_{j,\tau}^{m-1}(y)\alpha_j^{m-1}(i).$$

$$= \sigma(y - t_{i+m-1})\psi_{i,t}^{m-1}(y) = \psi_{i,t}^m(y).$$

Comparing (3.10) for $k = m$ with the original definition of s, and using the linear independence of the $\psi_{j,\tau}^m$'s, it follows that the $\alpha_j^m(i)$ satisfy (3.6). ■

The identity (3.6) leads to the following recursive algorithm for the b_i.

Algorithm 3.3. *Fix* $1 \le i \le n+p$, *and let* μ *be such that* $t_i \in [\tau_\mu, \tau_{\mu+1})$.

Set $c_{j,i}^0 := c_j$, $j = \mu - m + 1, \ldots, \mu$.

For $r = 1$ *to* $m-1$,

For $j = \mu - m + r + 1$ *to* μ,

$$c_{j,i}^r := \frac{\sigma(t_{i+m-r} - \tau_j)}{\sigma(\tau_{j+m-r} - \tau_j)} c_{j,i}^{r-1} + \frac{\sigma(\tau_{j+m-r} - t_{i+m-r})}{\sigma(\tau_{j+m-r} - \tau_j)} c_{j-1,i}^{r-1}.$$

Then $b_i = c_{\mu,i}^{m-1}$.

Figure 2 illustrates repeated knot insertion on the spline in Figure 1.

We conclude this section by noting that, as in the polynomial case, Algorithm 1.1 can be viewed as a special case of the knot insertion Algorithm 3.3, since

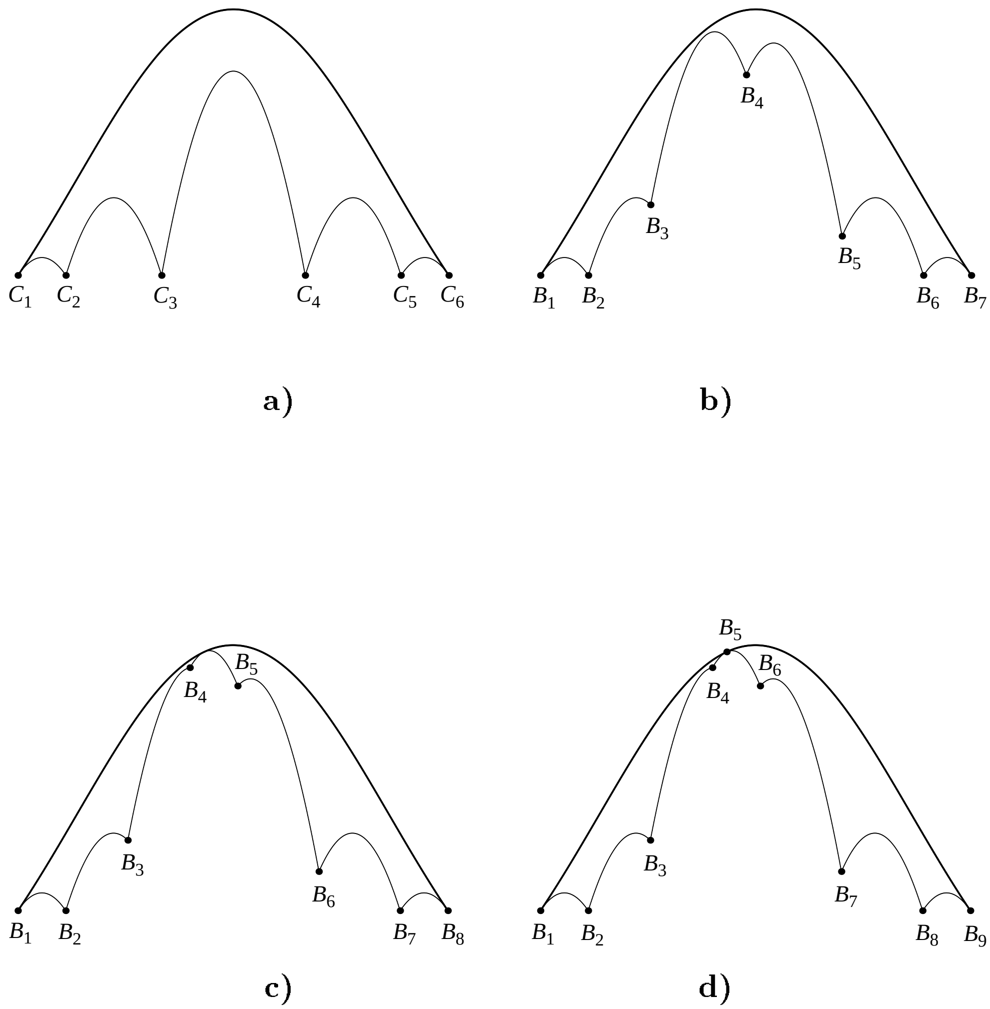

Figure 2. The spline of Figure 1, together with the control curves obtained after inserting the knot $\theta = 1.25$ one, two, and three times.

evaluation of a trigonometric spline at a point x can be viewed as knot insertion, where the new knot x is inserted into the spline curve a total of $m-1$ times. Indeed, if $t_{i+1} = \cdots = t_{i+m-1} = x$, for some i, then the two algorithms are identical in the sense that all values c_j^r produced by the first algorithm are the same as the numbers $c_{j,i}^r$ produced by the second one (cf. Figures 1 and 2). Also, by inserting multiple knots into the spline curve such that every knot has multiplicity m, the spline can be converted into a piecewise trigonometric curve whose individual pieces are represented in *trigonometric Bernstein–Bézier form* (see Remark 1).

4. Convergence of Subdivision of Control Curves

In this section we will show that if more and more knots are inserted into a trigonometric spline, the corresponding control curves will converge pointwise to the trigonometric spline with quadratic rate of convergence. The process of successive refinement of control curves, called *subdivision*, and its convergence properties are well understood in the polynomial setting (see [CohenSchumaker85, Dahmen86]). As for classical splines, the key ingredient in the proof of convergence will be the following stability result derived in [LycheSchumaker94].

Lemma 4.1. *Let*

$$s(x) = \sum_{j=1}^{n} c_j T_{j,t}^{m}(x). \tag{4.1}$$

Then there exists a constant K, depending only on m and not on the knot sequence t, such that

$$|c_j| \le K \|s\|_{J_j}, \quad j = 1, \dots, n,$$

where $\|\cdot\|_J$ denotes the usual supremum norm on the interval J, and $J_j := [t_{j+1}, t_{j+m-1}]$.

Next we study how well smooth functions can be approximated by functions in the spaces defined in (2.1). For related results see [KochLyche80].

Lemma 4.2. *Suppose f is in the usual Sobolev space $L^2_\infty[I]$ for some interval I, and suppose k is a positive integer. Then for any $x, x_0 \in I$, f has a trigonometric Taylor expansion of the form*

$$f(x) = \gamma_k(x - x_0) f(x_0) + \frac{1}{\alpha k} \sigma_k(x - x_0) f'(x_0) + \frac{1}{\alpha k} \int_{x_0}^{x} \sigma_k(x - y) L f(y) dy, \tag{4.2}$$

where $L := D^2 + k^2\alpha^2$. Moreover, for any $0 < x_1 - x_0 < \pi/(k\alpha)$ in I, we have the linear trigonometric interpolation formula

$$f(x) = Qf(x) + Rf(x), \tag{4.3}$$

where

$$Qf(x) := \frac{\sigma_k(x_1 - x) f(x_0) + \sigma_k(x - x_0) f(x_1)}{\sigma_k(x_1 - x_0)},$$

$$Rf(x) := \begin{cases} \sigma_k(x - x_0)\sigma_k(x - x_1)[x_0, x_1, x]f, & x_0 < x < x_1 \\ 0, & x = x_0, x_1, \end{cases}$$

and

$$[x_0, x_1, x]f := \frac{f(x_0)}{\sigma_k(x_0 - x)\sigma_k(x_0 - x_1)} + \frac{f(x_1)}{\sigma_k(x_1 - x_0)\sigma_k(x_1 - x)} + \frac{f(x)}{\sigma_k(x - x_0)\sigma_k(x - x_1)}.$$

Finally, for all $x \in [x_0, x_1]$,

$$|Rf(x)| \leq \frac{1}{8}(x_1 - x_0)^2 \|Lf\|_{[x_0,x_1]}/\gamma_k(\frac{x_1 - x_0}{2}). \tag{4.4}$$

Proof: The Taylor formula (4.2) follows easily by integrating the remainder term by parts, while (4.3) is immediate from the definition of Qf and Rf. We now show (4.4). We first observe that $Qf = f$ for all $f \in \mathcal{L}_{k+1}$, where $\mathcal{L}_{k+1}$ is the two-dimensional linear space defined in (2.1). Then (4.3) implies $[x_0, x_1, x]f = 0$ for all $f \in \mathcal{L}_{k+1}$. Writing the remainder term in (4.2) in the form

$$\frac{1}{\alpha k} \int_{x_0}^{x_1} \sigma_k(x - y)_+ Lf(y)\, dy,$$

with

$$\sigma_k(x - y)_+ := \begin{cases} \sigma_k(x - y), & \text{if } x \geq y \\ 0, & \text{otherwise,} \end{cases}$$

and applying $[x_0, x_1, x]f$ to both sides of (4.2), we see that

$$[x_0, x_1, x]f = \frac{1}{\alpha k} \int_{x_0}^{x_1} T(x; y) Lf(y)\, dy,$$

where

$$T(x; y) := [x_0, x_1, x]\sigma_k(\cdot - y)_+.$$

Now since T is nonnegative and

$$\int_{x_0}^{x_1} T(x; y)\, dy = \Big[2\alpha k \gamma_k(\frac{x - x_0}{2})\gamma_k(\frac{x - x_1}{2})\gamma_k(\frac{x_1 - x_0}{2})\Big]^{-1},$$

we get

$$\begin{aligned} |Rf(x)| &\leq \big|\sigma_k(x - x_0)\sigma_k(x - x_1) \int_{x_0}^{x_1} T(x; y)\, dy\big| \|Lf\|_{[x_0,x_1]}/(\alpha k) \\ &\leq \big|\frac{2}{\alpha^2 k^2}\sigma_k(\frac{x - x_0}{2})\sigma_k(\frac{x - x_1}{2})/\gamma_k(\frac{x_1 - x_0}{2})\big| \|Lf\|_{[x_0,x_1]}. \end{aligned}$$

Using the formula $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$ and the fact that $|\sin x| \leq |x|$, we get (4.4). ■

We can now prove that the control points of the refined control curve converge quadratically to the associated spline curve.

Theorem 4.3. *Let s be a spline series given by (4.1), and let $L := D^2 + (m-1)^2\alpha^2$. Then*

$$|c_j - s(t_j^*)| \le \frac{K(t_{j+m-1} - t_{j+1})^2}{2} \|Ls\|_{J_j}, \quad j = 1, \ldots, n, \tag{4.5}$$

where the constant K and the interval J_j are the same as in Lemma 4.1.

Proof: If $t_{j+1} = \cdots = t_{j+m-1}$, then $c_j = s(t_j^*)$ and there is nothing to prove. Suppose in the rest of the proof that $m > 2$ and $t_{j+1} < t_{j+m-1}$. This assures $s \in \mathcal{S}_{m,t} \cap C^1[x_0, x_1] \subset L^2_\infty[x_0, x_1]$, and we can now apply (4.2) with $k = m-1$ and $f := s = \sum_{j=1}^n c_j T_{j,t}^m$. We will choose x_0 later. Let g be the corresponding error term in this Taylor expansion. We observe that $g \in \mathcal{S}_{m,t}$, *i.e.* for some d_j we have $g(x) = \sum_{j=1}^n d_j T_{j,t}^m(x)$. These coefficients can be found from Theorem 2.2. Indeed,

$$\sigma_{m-1}(x - x_0) = \sum_{j=1}^n \sigma_{m-1}(t_j^* - x_0) T_{j,t}^m(x)$$

$$\gamma_{m-1}(x - x_0) = \sum_{j=1}^n \gamma_{m-1}(t_j^* - x_0) T_{j,t}^m(x).$$

Hence, $d_j = c_j - \gamma_{m-1}(t_j^* - x_0)s(x_0) - \sigma_{m-1}(t_j^* - x_0)s'(x_0)/((m-1)\alpha)$. Choosing $x_0 := t_j^*$, we obtain $d_j = c_j - s(t_j^*)$. Appealing to the stability result in Lemma 4.1, we obtain

$$\begin{aligned} |c_j - s(t_j^*)| = |d_j| \le K\|g\|_{J_j} &= K\left\| \int_{t_j^*}^x \sigma_{m-1}(x-y) Ls(y)\, dy \right\|_{J_j} /((m-1)\alpha) \\ &\le K \max_{x \in J_j} \left| \int_{t_j^*}^x (x-y)\, dy \right| \|Ls\|_{J_j} = K \max_{x \in J_j} (x - t_j^*)^2 \, \|Ls\|_{J_j}/2. \end{aligned}$$

Since $(x - t_j^*)^2 \le (t_{j+m-1} - t_{j+1})^2$ for $x \in J_j$, (4.5) follows. ■

As a consequence of this result we can prove the stronger fact that the quadratic convergence holds for the entire control curve, and not just for the individual control points.

Theorem 4.4. *Suppose c is the control curve of $s \in \mathcal{S}_{m,t}$. For any j such that $(t_{j+1} + \cdots + t_{j+m-1})/(m-1) = t_j^* < t_{j+1}^*$ and $x \in [t_j^*, t_{j+1}^*]$,*

$$|s(x) - c(x)| \le \frac{Ch_j^2}{\gamma(h_j/2)} ||Ls||_{I_j}, \tag{4.6}$$

where $h_j := t_{j+m} - t_{j+1}$, $I_j := [t_{j+1}, t_{j+m}]$, and the constant C only depends on m. Moreover, on $I := [t_1, t_{n+m}]$ we have

$$||s - c||_I \le \frac{Ch^2}{\gamma(h/2)} \|Ls\|_I, \tag{4.7}$$

where $h := \max_{1\le j\le n} h_j$.

Proof: We first observe that s is always a C^1 function in (t_j^*, t_{j+1}^*). To see this, recall that a spline s of order m is C^1 at a knot provided the multiplicity of the knot is at most $m-2$. Now if $t_j^* < t_{j+1}^*$, then we must have $t_{j+1} < t_{j+m}$. Therefore, the highest multiplicity of an interior knot in (t_j^*, t_{j+1}^*) is $m-2$, and this happens if and only if $t_{j+1} < t_{j+2} = \cdots = t_{j+m-1} < t_{j+m}$. This means that we can apply the error estimates in Lemma 4.2 on $[t_j^*, t_{j+1}^*]$. With Qs the linear interpolant to s on this interval, we can write

$$|s(x) - c(x)| \le |s(x) - Qs(x)| + |Qs(x) - c(x)|.$$

For $x \in [t_j^*, t_{j+1}^*]$ we obtain from (4.4)

$$\begin{aligned} |s(x) - Qs(x)| \le& \frac{(t_{j+1}^* - t_j^*)^2}{8\gamma_{m-1}((t_{j+1}^* - t_j^*)/2)} ||Lf||_{I_j} \\ \le& \frac{h_j^2}{8(m-1)^2\gamma(h_j/2)} ||Lf||_{I_j}. \end{aligned} \tag{4.8}$$

For the second term we find

$$|Qs(x) - c(x)| = |Q(s-c)(x)| \le C_1 \max\{|s(t_j^*) - c_j|, |s(t_{j+1}^*) - c_{j+1}|\},$$

where

$$C_1 = \max_{x\in I_j} [\sigma_{m-1}(t_{j+1}^* - x) + \sigma_{m-1}(x - t_j^*)]/\sigma_{m-1}(t_{j+1}^* - t_j^*) = 1/\gamma_{m-1}((t_{j+1}^* - t_j^*)/2).$$

By Theorem 4.3,

$$\begin{aligned} |Qs(x) - c(x)| \le& \frac{K \max_{i=j,j+1}\{(t_{i+m-1} - t_{i+1})^2 \|Ls\|_{J_i}\}}{2\gamma_{m-1}((t_{j+1}^* - t_j^*)/2)} \\ \le& \frac{K h_j^2}{2\gamma(h_j/2)} ||Ls||_{I_j}. \end{aligned} \tag{4.9}$$

Combining (4.8) and (4.9) we obtain (4.6) with $C = 1/(8(m-1)^2) + K/2$. Clearly (4.7) immediately follows from (4.6). ■

Figure 3 illustrates the convergence results of this section for a typical trigonometric spline with $m = 4$.

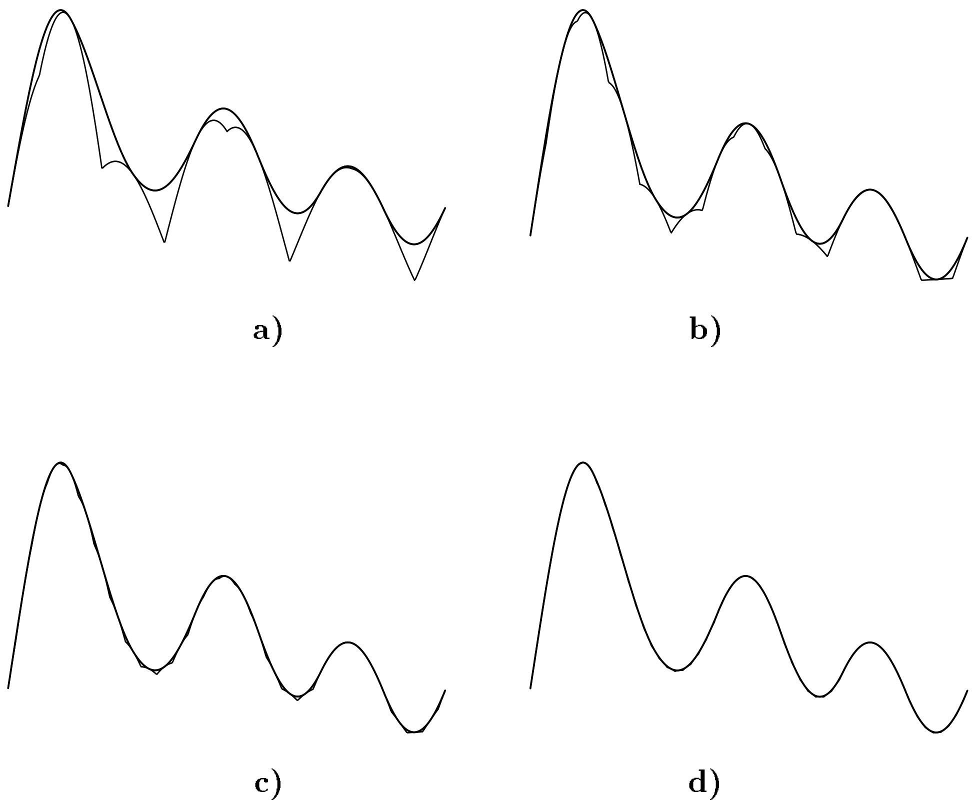

Figure 3. Subdivision of a cubic trigonometric spline by simultaneously inserting knots halfway between each pair of old knots.

5. A Variation Diminishing Property for Trigonometric Splines

The control curve of a trigonometric spline as defined in Section 2 also gives rise to a variation-diminishing property for trigonometric splines familiar in the polynomial case. For trigonometric splines this means, roughly speaking, that a function $g \in \mathcal{L}_m$ has no more intersections with the spline s than with the control curve c of s. In order to formulate this property more precisely, we need some notation. We will restrict ourselves to splines s which are continuous on the entire interval I. This is not a serious restriction since a discontinuous spline contains knots with multiplicity m, and thus it can be viewed as a collection of separate spline pieces which are continuous on subintervals of I. Thus, the analysis for continuous splines carries over to discontinuous splines with only minor additional work.

Definition 5.1. *We define the number of strong sign changes $S^-(a_1, \ldots, a_r)$ of a finite sequence of real numbers $a_1, \ldots, a_r$ to be the number of sign changes in this sequence, where zeros are ignored. For convenience, let $S^-(0, \ldots, 0) := 0$. The number of strong changes of a continuous function f, $S^-(f)$ is the supremum of all* *numbers* $S^-(f(\theta_1), \ldots, f(\theta_r))$ *for an arbitrary* r *and arbitrary* $\theta_1 < \cdots < \theta_r$.

We first prove the following auxiliary lemma.

Lemma 5.2. *Let* $s = \sum_{j=1}^n c_j T_{j,\tau}^m = \sum_{i=1}^{n+p} b_i T_{i,t}^m$ *be a trigonometric spline expressed on both a coarse knot sequence* τ *and a fine knot sequence* $t \supset \tau$. *Furthermore, let* c *and* b *be the control curves of* s *corresponding to* τ *and* t*, respectively. Then*

$$S^-(b) \leq S^-(c). \tag{5.1}$$

Proof: It will be sufficient to prove the assertion for the case $t = \tau \cup \{\theta\}$, *i.e.*, the case where the refined control curve is obtained by inserting one knot into the spline curve. The general result then follows by induction on the number p of inserted knots. Obviously, by the definition of control curves, we have $S^-(b) = S^-(b_1, \ldots, b_{n+1})$. Moreover, by an argument similar to the one in the proof of Proposition 2.3, it follows from (3.4) that the control points associated with the coefficients $b_1, \ldots, b_{n+1}$ lie on the control curve c. Therefore,

$$S^-(b_1, \ldots, b_{n+1}) \leq S^-(c),$$

which completes the proof of (5.1). ■

Theorem 5.3. *Let* $s \in \mathcal{S}_{m,\tau}$ *be a continuous spline of the form*

$$s = \sum_{j=1}^n c_j T_{j,\tau}^m,$$

with corresponding control curve c. *Then for any* $g \in \mathcal{L}_m$,

$$S^-(s - g) \leq S^-(c - g). \tag{5.2}$$

Proof: We first observe that, without loss of generality, it suffices to prove the theorem for $g = 0$, since $g \in \mathcal{L}_m \subset \mathcal{S}_{m,t}$ implies $s - g \in \mathcal{S}_{m,t}$, and by Theorem 2.2, the control curve of the spline $s - g$ equals $c - g$. Let g be the zero function, and let $\theta_1, \ldots, \theta_r$ be an arbitrary increasing sequence of knots in I. Suppose t is a new knot sequence obtained by inserting the knots $\theta_1, \ldots, \theta_r$ into τ so that each of these knots has multiplicity $m - 1$. Let b be the corresponding refined control curve of the spline s. As observed in Section 3, on account of the knot multiplicities, the sequences $s(\theta_1), \ldots, s(\theta_r)$ and $b(\theta_1), \ldots, b(\theta_r)$ are identical. Therefore, by Lemma 5.2 we have

$$S^-(s(\theta_1), \ldots, s(\theta_r)) \leq S^-(c).$$

Since this is true for every r and every sequence $\theta_1, \ldots, \theta_r$, it follows that

$$S^-(s) \leq S^-(c). \quad ■$$

The inequality (5.2) resembles the traditional formulation of the variation-diminishing property for polynomial splines (see [Schumaker81, Theorem 4.76], [deBoor78, Corollary XI.4]). The above idea of proving (5.2) by knot insertion has been utilized in [LaneRiesenfeld83] for polynomial splines.

We would like to thank an anonymous referee for pointing out that a version of Theorem 5.3 (without reference to control curves) was established in [GoodmanLee84]. In particular, they showed that

$$S^-(s) \leq S^-(c_1, \ldots, c_n).$$

6. Remarks

Remark 1. In the case where there are no interior knots on an interval I, the trigonometric splines discussed here reduce to trigonometric Bernstein basis polynomials. For a detailed treatment, see [AlfeldNeamtuSchumaker94].

Remark 2. We have defined the trigonometric polynomial spaces $\mathcal{T}_m$ in terms of a scaling parameter α. Typical values are $\alpha = 1/2, 1, \pi/2$. The value $\alpha = 1/2$ is used in [LycheWinther79], while choosing $\alpha = 1$ makes it is possible to interpret trigonometric splines as circular analogs of the classical polynomial splines, see [AlfeldNeamtuSchumaker94].

Remark 3. If we choose $\sigma(x) = x$ and $\gamma(x) = 1$, we can recover the standard results for polynomial splines. If $\sigma(x) := \sinh \alpha x$ and $\gamma(x) := \cosh \alpha x$, we get analogous results for hyperbolic splines (see *e.g.* [Schumaker83]).

Remark 4. As in the polynomial case, it is possible to formulate most of the results of this paper in a framework of trigonometric polar forms, introduced in [GonsorNeamtu94].

Remark 5. For large intervals I, the control curve for trigonometric splines may not reflect the shape of the underlying spline as well as in the polynomial case, see *e.g.* Figure 1 which corresponds to $I = [0, 3]$. This seems to be a consequence of the fact that for $m > 1$, the spaces $\mathcal{L}_m$ do not contain constants. The situation is much better for intervals which are small compared to π/α.

Remark 6. An analog of Theorem 3.1 on knot insertion has been established for Tchebycheffian splines in [Lyche85].

Remark 7. In view of Remark 3, Theorems 4.3 and 4.4 apply to polynomial splines. Since these theorems do not require any smoothness assumptions on the spline, they constitute extensions of Theorem 3.3 in [CohenSchumaker85] and Theorem 2.1 in [Dahmen86].

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