Trivariate C^r Polynomial Macro-Elements

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Abstract. Trivariate C^r macro-elements defined in terms of polynomials of degree $8r+1$ on tetrahedra are analyzed. For $r=1,2$, these spaces reduce to well-known macro-element spaces used in data fitting and in the finite-element method. We determine the dimension of these spaces, and describe stable local minimal determining sets and nodal minimal determining sets. We also show that the spaces approximate smooth functions to optimal order.

§1. Introduction

Let $\triangle$ be a tetrahedral partition of a set $\Omega \in \mathbb{R}^3$. We denote the sets of vertices, edges, and faces of $\triangle$, by $\mathcal{V}$, $\mathcal{E}$, and $\mathcal{F}$, respectively. In this paper we study the superspline space

$$\begin{aligned} \mathcal{S}_r(\triangle) := \{s \in C^r(\Omega) \ : & s|_T \in \mathcal{P}_{8r+1}, \text{ all tetrahedra } T \in \triangle, \\ & s \in C^{4r}(v), \text{ all } v \in \mathcal{V}, \\ & s \in C^{2r}(e), \text{ all } e \in \mathcal{E}\}, \end{aligned} \tag{1.1}$$

where in general we write $\mathcal{P}_d$ for the $\binom{d+3}{3}$ dimensional space of trivariate polynomials of degree d. Here $s \in C^\rho(v)$ means that all polynomial pieces $s|_T$ associated with tetrahedra T sharing the vertex v have common derivatives up to order ρ at v. Similarly, $s \in C^\mu(e)$ means that all polynomial pieces $s|_T$ associated with tetrahedra T sharing the edge e have common derivatives up to order μ at all points along the edge e.

For $r=1$ this space corresponds to a macro-element space first introduced in the finite-element literature in [23]. The analogous C^2 macro-element was developed in [16]. Both authors described their elements in terms of Hermite interpolation. It is well known, see Remark 1, that in order to construct similar macro-element spaces for higher values of r, we must work with splines of degree $8r+1$, and we must enforce C^{4r} supersmoothness at the vertices and C^{2r} supersmoothness around the edges of $\triangle$. This suggest a natural set of Hermite data to associate with the element. But it is a nontrivial problem to describe what additional data

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is needed to uniquely determine $s \in \mathcal{S}_r(\triangle)$. To solve this problem, we first analyze $\mathcal{S}_r(\triangle)$ in terms of its Bernstein–Bézier representation. This allows us to identify the dimension of $\mathcal{S}_r(\triangle)$, and to show that the space has full approximation power in all q-norms. We then go on to show how to parametrize $\mathcal{S}_r(\triangle)$ nodally, i.e., in terms of Hermite interpolation conditions.

The paper is organized as follows. In Sect. 2 we review Bernstein–Bézier methods and notation, and prove two useful lemmas about polynomial interpolation. In the next section we compute the dimension of our macro-element space, and construct a stable local minimal determining set for it. The approximation power of the space is dealt with in Sect. 4, while the construction of a stable nodal basis for it can be found in Sect. 5. In Sect. 6 we discuss a corresponding Hermite interpolation method, and give an error bound for it. We conclude the paper with remarks and references.

§2. Preliminaries

We will make extensive use of well-known Bernstein–Bézier methods, see e.g. [1–8,11–15,17,18,21,22]. For convenience we review the notation and basic concepts. As usual, given a tetrahedron $T := \langle v_1, v_2, v_3, v_4 \rangle$ and a polynomial p of degree d, we denote the B-coefficients of p by $c^{T,d}_{ijkl}$ and associate them with the domain points $\xi^{T,d}_{ijkl} := \frac{(iv_1+jv_2+kv_3+lv_4)}{d}$, where $i+j+k+l=d$. We write $\mathcal{D}_{d,T}$ for the set of all domain points associated with T. We say that the domain point $\xi^{T,d}_{ijkl}$ has distance $d-i$ from the vertex v_1, with similar definitions for the other vertices. We say that $\xi^{T,d}_{ijkl}$ is at a distance $i+j$ from the edge $e := \langle v_3, v_4 \rangle$, with similar definitions for the other edges of T. If $\triangle$ is a tetrahedral partition of a set Ω, we write $\mathcal{D}_{d,\triangle}$ for the collection of all domain points associated with tetrahedra in $\triangle$, where common points in neighboring tetrahedra are not repeated. Given $\xi \in \mathcal{D}_{d,T}$, we denote the associated Bernstein basis polynomial by $B^{T,d}_{\xi}$.

Given $\rho > 0$, we refer to the set $D_\rho(v)$ of all domain points which are within a distance ρ from v as the ball of radius ρ around v. Similarly, we refer to the set $R_\rho(v)$ of all domain points which are at a distance ρ from v as the shell of radius ρ around v. If e is an edge of $\triangle$, we define the tube of radius μ around e to be the set of domain points whose distance to e is at most μ.

For any tetrahedron T in a partition $\triangle$, let star(T) be the set of all tetrahedra in $\triangle$ that touch T. For any vertex v of $\triangle$, we define star(v) similarly. We write $\phi_\triangle$ for the smallest face angle of $\triangle$, i.e., the smallest angle in the triangular faces of $\triangle$. Similarly, we write $\theta_\triangle$ for the smallest spherical angle (solid angle) in any tetrahedron $T \in \triangle$, where the spherical angle at a vertex v_1 of a tetrahedron $T := \langle v_1, v_2, v_3, v_4 \rangle$ is the area of the intersection of a unit sphere centered at v_1 with the extensions of the three faces of T meeting at v_1.

Suppose $\mathcal{S}$ is a linear subspace of $\mathcal{S}^0_d(\triangle)$, and suppose $\mathcal{M}$ is a subset of $\mathcal{D}_{d,\triangle}$. Then $\mathcal{M}$ is said to be a determining set for $\mathcal{S}$ provided that if $s \in \mathcal{S}$ and its B-coefficients satisfy $c_\xi = 0$ for all $\xi \in \mathcal{M}$, then $s \equiv 0$. It is called a minimal determining set (MDS) for $\mathcal{S}$ provided there is no smaller determining set. It is well known that $\mathcal{M}$ is a MDS for $\mathcal{S}$ if and only if setting the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$ of a spline in $\mathcal{S}$ uniquely determines all coefficients of s. It is also known that the cardinality of any minimal determining set for $\mathcal{S}$ equals the dimension of $\mathcal{S}$.

A minimal determining set $\mathcal{M}$ is called 1-local provided that for all $\xi \in \mathcal{D}_{d,\triangle} \setminus \mathcal{M}$, c_ξ depends only on coefficients corresponding to domain points in a set Γ_ξ contained in $\mathrm{star}(T_\xi)$, where T_ξ is the tetrahedron containing ξ. Throughout the paper we shall simply say local instead of 1-local. We say that $\mathcal{M}$ is stable provided that there exists a constant K depending on the smallest angles $\theta_\triangle$ and $\phi_\triangle$ such that

$$|c_\xi| \le K \max_{\eta\in\Gamma_\xi} |c_\eta|. \tag{2.1}$$

Suppose $\mathcal{N}$ is a collection of linear functionals λ, where λs is defined by a linear combination of values or derivatives of s at a point η_λ in Ω. Then $\mathcal{N}$ is said to be a nodal determining set (NDS) for $\mathcal{S}$ provided that if $s \in \mathcal{S}$ and $\lambda s = 0$ for all $\lambda \in \mathcal{N}$, then $s \equiv 0$. It is called a nodal minimal determining set (NMDS) for $\mathcal{S}$ provided that there is no smaller NDS, or equivalently, for each set of real numbers $\{z_\lambda\}_{\lambda\in\mathcal{N}}$, there exists a unique $s \in \mathcal{S}$ such that $\lambda s = z_\lambda$ for all $\lambda \in \mathcal{N}$. We say that $\mathcal{N}$ is m-stable provided there exists a constant K depending on the smallest angles $\theta_\triangle$ and $\phi_\triangle$ such that for every $s \in \mathcal{S}$ and every $\xi \in \mathcal{D}_{d,\triangle}$,

$$|c_\xi| \le K \sum_{\nu=0}^{m} |T_\xi|^\nu |s|_{\nu,\infty,T_\xi}, \tag{2.2}$$

where T_ξ is a tetrahedron containing ξ. Here

$$|f|_{\nu,q,B} := \begin{cases} \Big[\sum_{|\alpha|=\nu} \|D^\alpha f\|_{q,B}^q\Big]^{1/q}, & \text{if } 1 \le q < \infty, \\ \max_{|\alpha|=\nu} \|D^\alpha f\|_{\nu,B}, & \text{if } q = \infty, \end{cases} \tag{2.3}$$

is the usual semi-norm defined on any subset $B \subseteq \Omega$.

Lemma 2.1. *Let v_c be an arbitrary point in the interior of a tetrahedron T, and let $d \ge 4r+4$. Suppose that the B-coefficients of a polynomial p of degree d are known except for those corresponding to the domain points*

$$\Gamma := \{\xi_{ijkl}^{T,d} \,:\, i,j,k,l > r\}.$$

Then the coefficients $\{c_\xi\}_{\xi\in\Gamma}$ of p are uniquely determined from the values $\{D^\alpha p(v_c)\}_{|\alpha|\le d-4r-4}$.

Proof: The known coefficients are associated with domain points that lie on the outer faces of T and on the r layers next to those outer faces. We are left with $N := \binom{d-4r-1}{3}$ coefficients which are to be determined from the same number of Hermite interpolation conditions. Enforcing these conditions leads to a $N\times N$ linear system of equations. We need to show that the associated matrix M is nonsingular.

It suffices to show that if the coefficients corresponding to $\mathcal{D}_{d,T} \setminus \Gamma$ are all zero and we set $D^\alpha p(v_c) = 0$ for $|\alpha| \le d - 4r - 4$, then $p \equiv 0$. Now by Bezout's theorem, we can write

$$p = \ell_1^{r+1}\, \ell_2^{r+1}\, \ell_3^{r+1}\, \ell_4^{r+1}\, q,$$

where ℓ_i is a linear polynomial which vanishes on the the i-th face of T, and q is a polynomial of degree $d - 4r - 4$. Since v_c is inside of T, setting $D^\alpha p(v_c) = 0$ for $|\alpha| \le d - 4r - 4$ is equivalent to setting $D^\alpha q(v_c) = 0$ for $|\alpha| \le d - 4r - 4$. But this implies $q \equiv 0$, and it follows that $p \equiv 0$. We conclude that M is nonsingular and the proof is complete. □

If $F := \langle v_1, v_2, v_3 \rangle$ is a triangular face of a tetrahedron T, and p is a trivariate polynomial of degree d, then the restriction of p to F is a bivariate polynomial of degree d which can also be written in B-form. We write $\mathcal{D}_{d,F} := \{\xi_{ijk}^{F,d}\}_{i+j+k=d}$ for the set of domain points that lie on F, and $\{B_\xi^{F,d}\}_{\xi \in \mathcal{D}_{d,F}}$ for the associated bivariate Bernstein–Bézier basis polynomials. As usual, we call the set $D_\rho(v_1)$ of points in $\mathcal{D}_{d,F}$ within a distance ρ from v_1 the disk of radius ρ around v_1. Similarly, the set $R_\rho(v_1)$ of points in $\mathcal{D}_{d,F}$ at a distance ρ from v_1 is called the ring of radius ρ around v_1. Although we use the same notation for disks/balls and shells/rings, the meaning should be clear from the context.

In the analysis of the macro-elements in this paper, we need to solve certain bivariate interpolation problems involving subsets of the Bernstein–Bézier basis polynomials associated with a triangular face F. The following conjecture is due to the second author, see e.g. [2].

Conjecture 2.2. *The matrix*

$$M := \left[B_\xi^{F,d}(\eta)\right]_{\xi,\eta \in \Gamma} \tag{2.4}$$

is nonsingular for every nonempty subset Γ *of* $\mathcal{D}_{d,F}$.

At this time, the full conjecture is still open, but several special cases have been settled. The following special case is needed below.

Lemma 2.3. *Let* $\Gamma := \{\xi_{ijk}^{F,d} \,:\, i \ge m_1,\ j \ge m_2,\ k \ge m_3\} \subseteq \mathcal{D}_{d,F}$ *for some* $m_1, m_2, m_3 > 0$ *with* $m := m_1 + m_2 + m_3 < d$. *Then the matrix (2.4) is nonsingular.*

Proof: In this case the set Γ is just the set of domain points such that for each $i = 1, 2, 3$, their distance to the edge $\langle v_i, v_{i+1} \rangle$ of F is at least m_i. This set has cardinality $n := \binom{d-m+2}{2}$. After multiplying the columns of M by appropriate ratios of factorials, and removing common factors of the form $(\frac{\nu}{d})^{m_1} (\frac{\mu}{d})^{m_2} (\frac{\kappa}{d})^{m_3}$ from each row of M, we find that $M = a\widetilde{M}$, where a is a nonzero constant depending on m and d, and $\widetilde{M}$ is the $n \times n$ matrix with entries $B_{ijk}^{F,d-m}(\xi_{\nu\mu\kappa}^{F,d})$ where $i+j+k = d-m$ and (ν, μ, κ) run over Γ. But this set of points satisfies the conditions of Chung-Yao [10], insuring that $\widetilde{M}$ and thus M is nonsingular. □

§3. A Stable Local MDS for $\mathcal{S}_r(\triangle)$

For ease of notation, in the remainder of the paper we define $d := 8r+1$, $\rho := 4r$, and $\mu := 2r$. To describe a stable local minimal determining set for $\mathcal{S}_r(\triangle)$, we introduce some notation for subsets of the set of domain points $\mathcal{D}_{d,\triangle}$ associated with $\triangle$:

1) For each vertex v of $\triangle$, let T_v be some tetrahedron containing v, and let $\mathcal{M}_v := D_\rho(v) \cap T_v$. This set has cardinality $\binom{\rho+3}{3} = \binom{4r+3}{3} = (32r^3+48r^2+22r+3)/3$.
2) For each edge $e := \langle u, v \rangle$ of $\triangle$, we write $E_\mu(e)$ for the set of all domain points which lie in the tube of radius μ around e, but which do not lie in either of the balls $D_\rho(u)$ or $D_\rho(v)$. Since $\rho \geq 2\mu$, the sets $E_\mu(e)$ are disjoint. For each edge e of $\triangle$, let T_e be some tetrahedron containing e, and let $\mathcal{M}_e := E_\mu(e) \cap T_e$. This set has cardinality $\frac{(r+1)(2r+1)(4r)}{3} = (8r^3 + 12r^2 + 4r)/3$.
3) For each face $F := \langle v_1, v_2, v_3 \rangle$ of $\triangle$, let $G_r(F)$ be the set of domain points which are at a distance at most r from F, but which do not lie in any of the sets $D_\rho(v)$ or $E_\mu(e)$. Suppose T_F is a tetrahedron in $\triangle$ which contains F, and let $\mathcal{M}_F := G_r(F) \cap T_F$. The cardinality of this set is $(25r^3 + 21r^2 - 4r)/6$.
4) For each tetrahedron T, let $\mathcal{M}_T$ be the set of domain points in $\mathcal{D}_{d,T}$ which do not lie in any of the previous sets. The cardinality of this set is $\binom{4r}{3} - 4\binom{r}{3} = 10r^3 - 6r^2$.

Let n_V, n_E, n_F, n_T be the number of vertices, edges, faces, and tetrahedra in $\triangle$, respectively.

Theorem 3.1. *The set*

$$\mathcal{M} := \bigcup_{v \in \mathcal{V}} \mathcal{M}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{M}_e \cup \bigcup_{F \in \mathcal{F}} \mathcal{M}_F \cup \bigcup_{T \in \triangle} \mathcal{M}_T \tag{3.1}$$

is a stable local minimal determining set for $\mathcal{S}_r(\triangle)$, and

$$\dim \mathcal{S}_r(\triangle) = \frac{(32r^3+48r^2+22r+3)}{3} n_V + \frac{(8r^3+12r^2+4r)}{3} n_E + \frac{(25r^3+21r^2-4r)}{6} n_F + (10r^3 - 6r^2) n_T. \tag{3.2}$$

Proof: To show that $\mathcal{M}$ is a minimal determining set for $\mathcal{S}_r(\triangle)$, we need to show that the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}}$ of a spline $s \in \mathcal{S}_d^0(\triangle)$ can be set to arbitrary values, and that all other coefficients of s are determined in such a way that s satisfies all smoothness conditions that are required for s to belong to $\mathcal{S}_r(\triangle)$.

First for each vertex $v \in \mathcal{V}$, we set the coefficients of s corresponding to domain points in the set $\mathcal{M}_v$ to arbitrary values. Then using the C^ρ smoothness at v, we can compute all remaining coefficients corresponding to domain points in the ball $D_\rho(v)$ from smoothness conditions. This is a stable local process, and in particular for each $\xi \in D_\rho(v)$, (2.1) holds with $\Gamma_\xi = \mathcal{M}_v$. Since the balls $D_\rho(v)$ do not overlap, there are no smoothness conditions connecting coefficients associated with domain points in two different balls.

For each edge $e := \langle u, v\rangle$ of $\triangle$, we now set the coefficients of s corresponding to domain points in $\mathcal{M}_e$. We can then use the C^μ supersmoothness around e to determine all of the coefficients of s corresponding to the domain points in $E_\mu(e)$. The computation of these coefficients is a stable local process, and (2.1) holds with $\Gamma_\xi := \mathcal{M}_e \cup \mathcal{M}_u \cup \mathcal{M}_v$. The sets $E_\mu(e)$ are disjoint from each other and from all balls $D_\rho(v)$, and thus we can be sure that none of the smoothness conditions defining $\mathcal{S}_r(\triangle)$ have been violated.

Since the sets $G_r(F)$ are disjoint from each other, there are no smoothness conditions connecting coefficients associated with domain points in two different such sets. Thus, for each face F of $\triangle$, we can set the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}_F}$, where $\mathcal{M}_F := G_r(F) \cap T_F$. If F is an interior face, then the coefficients corresponding to $G_r(F) \cap \widetilde{T}_F$ are uniquely determined from the C^r smoothness across F, where $\widetilde{T}_F$ is the other tetrahedron in $\triangle$ sharing the face F. This is a stable local process, and (2.1) holds with Γ_ξ equal to the union of $\mathcal{M}_F$ with all $\mathcal{M}_v$ and $\mathcal{M}_e$ such that v and e are vertices and edges of F.

We have now determined all coefficients of s except for those corresponding to domain points in the sets $\mathcal{M}_T$. These sets are disjoint from each other, and there are no smoothness conditions connecting coefficients associated with domain points in two such sets. Thus, for each T, the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}_T}$ can be set to arbitrary values. Since all coefficients of s either have been fixed, or have been stably and locally computed using smoothness conditions, we have shown that $\mathcal{M}$ is a stable local MDS.

To finish the proof, we note that the dimension of $\mathcal{S}_r(\triangle)$ is just the cardinality of $\mathcal{M}$, which is easily seen to be given by the formula (3.2). □

Example 3.2. *Let $\triangle$ consist of a single tetrahedron.*

Discussion: In this case $n_{_V} = n_{_F} = 4$, $n_{_E} = 6$, and $n_{_T} = 1$. Thus, (3.2) reduces to $\dim \mathcal{S}_r(\triangle) = (256r^3 + 288r^2 + 104r + 12)/3$. This is equal to $\dim \mathcal{P}_{8r+1} = \binom{8r+4}{3}$. □

For $r = 1, 2$, the space $\mathcal{S}_r(\triangle)$ is a classical finite-element space. In the literature, finite-element spaces have traditionally been paramerized in terms of nodal functionals. We construct a stable nodal basis for $\mathcal{S}_r(\triangle)$ for all $r \geq 1$ in Sect. 5.

§4. Approximation Power of $\mathcal{S}_r(\triangle)$

Based on the fact that $\mathcal{S}_r(\triangle)$ has a stable local MDS $\mathcal{M}$, we can now show that it has full approximation power. To this end, we now describe a quasi-interpolation operator mapping $L_1(\Omega)$ onto $\mathcal{S}_r(\triangle)$. For each $\xi \in \mathcal{M}$, let T_ξ be a tetrahedron that contains ξ. For any $f \in L_1(\Omega)$, let $A_\xi f$ be the averaged Taylor expansion of degree $8r+1$ associated with the largest ball contained in T_ξ, see [9]. Finally, let γ_ξ be the linear functional such that if p is a polynomial of degree d defined on T_ξ, then $\gamma_\xi p$ is the B-coefficient of p associated with the domain point ξ. Set $c_\xi(f) := \gamma_\xi A_\xi f$ for all

$\xi \in \mathcal{M}$. Since $\mathcal{M}$ is a MDS for $\mathcal{S}_r(\triangle)$, we can now determine all other coefficients of a spline Qf in $\mathcal{S}_r(\triangle)$ from smoothness conditions.

Clearly, Q is a linear operator mapping $L_1(\Omega)$ into $\mathcal{S}_r(\triangle)$. If $f \in \mathcal{S}_r(\triangle)$, then for each tetrahedron T_ξ, $f|_{T_\xi}$ is a polynomial of degree $8r+1$, and $A_\xi f = f$. This shows that Q is a projector onto $\mathcal{S}_r(\triangle)$, and since $\mathcal{P}_{8r+1} \in \mathcal{S}_r(\triangle)$, it follows that $Qf = f$ for all $f \in \mathcal{P}_{8r+1}$.

Lemma 4.1. *For every tetrahedron $T \in \triangle$ and all $1 \le q \le \infty$,*

$$\|Qf\|_{q,T} \le K \|f\|_{q,\Omega_T}, \qquad \text{all } f \in L_1(\Omega_T), \tag{4.1}$$

where $\Omega_T :=$ star(T). Here K depends only on r and the smallest angles $\theta_\triangle$ and $\phi_\triangle$ of $\triangle$.

Proof: We establish (4.1) in the case $1 \le q < \infty$. The case $q = \infty$ is similar and simpler. Given $\xi \in \mathcal{M}$, let T_ξ be a tetrahedron containing ξ. Then using the stability of the B-form, the equivalence of norms of polynomials of a fixed degree, and the fact that the averaged Taylor operator is bounded (see Corollary 4.1.15 in [9]), we have

$$\begin{aligned} |c_\xi| = |\gamma_\xi A_\xi f| &\le K_1 \, \|A_\xi f\|_{\infty,T_\xi} \\ &\le K_2 \, vol\,(T_\xi)^{-1/q} \, \|A_\xi f\|_{q,T_\xi} \\ &\le K_3 \, vol\,(T_\xi)^{-1/q} \, \|f\|_{q,T_\xi}, \end{aligned}$$

where $vol\,(T_\xi)$ is the volume of T_ξ. Here K_1 is a constant depending only on r, and the constants K_2 and K_3 depend only on r and $\theta_\triangle$ and $\phi_\triangle$. Now fix $T \in \triangle$. Since $\mathcal{M}$ is stable and 1-local, it follows that

$$|c_\eta| \le K_4 vol\,(\widetilde{T})^{-1/q} \|f\|_{q,\Omega_T}, \qquad \text{all } \eta \in \mathcal{D}_{d,T}, \tag{4.2}$$

where $\widetilde{T}$ is the tetrahedron in star(T) with smallest volume. Then using the fact that the Bernstein basis polynomials form a partition of unity,

$$\|Qf\|_{q,T} = \left[\int_T \Big| \sum_{\eta \in \mathcal{D}_{d,T}} c_\eta B_\eta^T \Big|^q \right]^{1/q} \le vol\,(T)^{1/q} \max_{\eta \in \mathcal{D}_{d,T}} |c_\eta|. \tag{4.3}$$

To complete the proof, we insert (4.2) in (4.3) and use the fact that $vol\,(T)/vol\,(\widetilde{T})$ is bounded by a constant also depending on the angles, see [15]. $\square$

We now give a local approximation result for Q. For the proof we need the following Markov inequality, see [15,20]. Suppose T is a tetrahedron, and let ρ_T be the radius of the largest ball that can be inscribed in T. Then for every polynomial $p \in \mathcal{P}_d$,

$$|p|_{k,q,T} \le \frac{K}{\rho_T^k} \, \|p\|_{q,T}, \tag{4.4}$$

for all $0 \le k \le d$ and all $1 \le q \le \infty$. Here K is a constant depending only on d.

Theorem 4.2. *Given a tetrahedron $T \in \triangle$, let $\Omega_T := star(T)$, and suppose $f \in W_q^{m+1}(\Omega_T)$ for some $0 \le m \le 8r+1$ and $1 \le q \le \infty$. Then*

$$\|D^\alpha(f - Qf)\|_{q,T} \le K\,|T|^{m+1-|\alpha|}\;|f|_{m+1,q,\Omega_T}, \tag{4.5}$$

for all $0 \le |\alpha| \le m$. If Ω_T is convex, the constant K depends only on r and the smallest angles $\theta_\triangle$ and $\phi_\triangle$ associated with $\triangle$. If Ω_T is not convex, K also depends on the Lipschitz constant of the boundary of Ω_T.

Proof: We recall that the linear operator Q reproduces polynomials of degree $d := 8r+1$, and that for any function g, the restriction of Qg to T is a polynomial of degree d. Thus, using (4.1) and the Markov inequality (4.4), it follows that for any $p \in \mathcal{P}_m$,

$$\begin{aligned}\|D^\alpha(f - Qf)\|_{q,T} &\le \|D^\alpha(f-p)\|_{q,T} + \|D^\alpha Q(f-p)\|_{q,T} \\ &\le \|D^\alpha(f-p)\|_{q,T} + \frac{K_1}{\rho_T^{|\alpha|}}\|Q(f-p)\|_{q,T} \\ &\le \|D^\alpha(f-p)\|_{q,T} + \frac{K_2}{\rho_T^{|\alpha|}}\|f-p\|_{q,\Omega_T}.\end{aligned} \tag{4.6}$$

Assume for the moment that Ω_T is convex. Then by Lemma 4.3.8 in [9], there exists a polynomial $p \in \mathcal{P}_m$ depending on f so that

$$\|D^\beta(f-p)\|_{q,\Omega_T} \le K_3|\Omega_T|^{m+1-|\beta|}|f|_{m+1,q,\Omega_T}, \tag{4.7}$$

for all $0 \le |\beta| \le d$, where the constant K_3 depends on r, $\theta_\triangle$, and $\phi_\triangle$. It is easy to see that $|\Omega_T| \le K_4|T|$ and $|T| \le K_5\rho_T$, where K_4, K_5 are constants depending on the smallest angles in Ω_T. Combining these facts with (4.6) and (4.7), we immediately get (4.5). If Ω_T is not convex, we first use the Stein extension theorem [19] to extend f to the convex hull of Ω_T, and then proceed as above. In this case the final constant also depends on the Lipschitz constant of the boundary of Ω_T. $\square$

We can now give the corresponding global approximation result which shows that $\mathcal{S}_r(\triangle)$ has full approximation power. Let $|\triangle|$ be the mesh size of $\triangle$, i.e., the length of the longest edge in $\triangle$.

Theorem 4.3. *There exists a constant K such that if $f \in W_q^{m+1}(\Omega)$ for some $0 \le m \le 8r+1$ and $1 \le q \le \infty$, then*

$$\|D^\alpha(f - Qf)\|_{q,\Omega} \le K\;|\triangle|^{m+1-|\alpha|}\;|f|_{m+1,q,\Omega}, \tag{4.8}$$

for all $0 \le |\alpha| \le m$. If Ω is convex, the constant K depends only on r, $\theta_\triangle$, and $\phi_\triangle$. If Ω is not convex, K also depends on the Lipschitz constant of the boundary of $\triangle$.

Proof: For $q = \infty$, (4.8) follows immediately from (4.5) by taking the maximum over all tetrahedra T in $\triangle$. To get the result for $q < \infty$, we take the q-th power of both sides of (4.5) and sum over all tetrahedra in $\triangle$. Since Ω_T contains other tetrahedra besides T, some tetrahedra appear more than once in the sum on the right. However, a given tetrahedron T_R appears on the right only if it is associated with a tetrahedron T_L on the left which lies in star(T_R). But it is not hard to see (cf. [15]) that there is a constant K_1 depending only on $\theta_\triangle$ such that T_R enters at most K_1 times on the right, and (4.8) follows. $\square$

§5. A Stable Nodal Basis for $\mathcal{S}_r(\triangle)$

To describe a nodal basis for $\mathcal{S}_r(\triangle)$, we need some notation for directional derivatives. Suppose $T := \langle v_1, v_2, v_3, v_4\rangle$ is a tetrahedron in $\triangle$. Then corresponding to the edge $e := \langle v_1, v_2\rangle$ we define $D_{T,e,1}$ to be the directional derivative associated with a unit vector perpendicular to e and lying in the face $\langle v_1, v_2, v_3\rangle$. Similarly, we define $D_{T,e,2}$ to be the directional derivative associated with a unit vector perpendicular to e and lying in the face $\langle v_1, v_2, v_4\rangle$. For each triangular face F of $\triangle$, we write D_F for the directional derivative associated with a unit vector that is perpendicular to F. For each e of F, let $D_{F,e}$ be the directional derivative associated with a unit vector that lies in F and is perpendicular to e.

We also need some notation for certain sets of points lying on faces and edges of tetrahedra in $\triangle$. Given any face $F := \langle v_1, v_2, v_3\rangle$ and integer $m > 0$, let

$$\mathcal{D}_{m,F} := \left\{ \xi^{F,m}_{ijk} := \frac{iv_1 + jv_2 + kv_3}{m} \right\}_{i+j+k=m}.$$

For any $\ell \ge 0$, let

$$A_{F,\ell} := \{\xi^{F,8r+1-\ell}_{ijk} \, : \, i,j,k \ge 2r+1-\ell+\lfloor \ell/2 \rfloor\}.$$

These sets depend on r, but for ease of notation we do not write this dependence explicitly. For $r = 3$, we have marked the points in the sets $A_{F,\ell}$ for $\ell = 0, \dots, 4$ with $\boxdot$ in Figs. 1–3. For any $i > 0$, we define equally spaced points in the interior of $e := \langle v_1, v_2\rangle$ as follows:

$$\eta^i_{e,j} := \frac{(i-j+1)v_1 + jv_2}{i+1}, \qquad j = 1, \dots, i. \tag{5.1}$$

For each tetrahedron T of $\triangle$, let v_T be its barycenter, $\mathcal{E}_T$ its set of edges, and $\mathcal{F}_T$ its set of faces. For each face F of $\triangle$, let $\mathcal{E}_F$ be its set of edges. For each edge e of $\triangle$, pick some tetrahedron T_e containing e. For any point $t \in \mathbb{R}^3$, let ε_t be the point-evaluation functional at t.

Theorem 5.1. *Given $r > 0$, let $n := \lfloor \frac{r}{3} \rfloor$. Then*

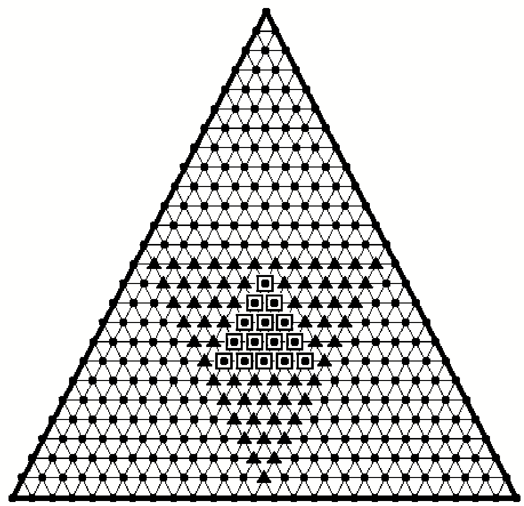
$$\mathcal{N} := \bigcup_{v\in\mathcal{V}} \mathcal{N}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{N}_e \cup \bigcup_{F\in\mathcal{F}} [\mathcal{N}^1_F \cup \mathcal{N}^2_F] \cup \bigcup_{T\in\triangle} [\mathcal{N}^1_T \cup \mathcal{N}^2_T \cup \mathcal{N}^3_T \cup \mathcal{N}^4_T],$$

is a stable nodal minimal determining set for $\mathcal{S}_r(\triangle)$, *where*

1) $\mathcal{N}_v := \{\varepsilon_v D^\alpha\}_{|\alpha|\le 4r}$,
2) $\mathcal{N}_e := \bigcup_{\ell=1}^{2r} \bigcup_{m=0}^{\ell} \{\varepsilon_{\eta^\ell_{e,k}} D^m_{T_e,e,1} D^{\ell-m}_{T_e,e,2}\}_{k=1}^{\ell}$,
3) $\mathcal{N}_F^1 := \bigcup_{\ell=0}^{r} \{\varepsilon_\xi D_F^\ell\}_{\xi\in A_{F,\ell}}$,
4) $\mathcal{N}_F^2 := \bigcup_{e\in\mathcal{E}_F} \bigcup_{\ell=2}^{r} \bigcup_{m=1}^{\lfloor \ell/2\rfloor} \{\varepsilon_{\eta^{2r+m}_{e,k}} D_F^\ell D_{F,e}^{2r-\ell+m}\}_{k=1}^{2r+m}$,
5) $\mathcal{N}_T^1 := \bigcup_{e\in\mathcal{E}_T} \bigcup_{\ell=r+1}^{r+n} \{\varepsilon_{\eta^{2\ell}_{e,k}} D^\ell_{T,e,1} D^\ell_{T,e,2}\}_{k=1}^{2\ell}$,
6) $\mathcal{N}_T^2 := \bigcup_{F\in\mathcal{F}_T} \bigcup_{e\in\mathcal{E}_F} \bigcup_{\ell=r+1}^{r+n} \bigcup_{m=1}^{2r-2\ell+\lfloor \ell/2\rfloor} \{\varepsilon_{\eta^{2\ell+m}_{e,k}} D_F^\ell D_{F,e}^{\ell+m}\}_{k=1}^{2\ell+m}$,
7) $\mathcal{N}_T^3 := \bigcup_{F\in\mathcal{F}_T} \bigcup_{\ell=r+1}^{r+n} \{\varepsilon_\xi D_F^\ell\}_{\xi\in A_{F,\ell}}$,
8) $\mathcal{N}_T^4 := \{\varepsilon_{v_T} D^\alpha\}_{|\alpha|\le 4r-4n-3}$.

Proof: First we show that $\mathcal{N}$ is a nodal determining set for $\mathcal{S}_r(\triangle)$. We show later that it is minimal and stable. Suppose $s \in \mathcal{S}_r(\triangle)$ and that we have assigned values for $\{\lambda s\}_{\lambda\in\mathcal{N}}$. We show how to compute all B-coefficients of s from this derivative data. For each $v \in \mathcal{V}$, we can immediately compute all of the coefficients $\{c_\xi\}_{\xi\in D_{4r}(v)}$ from $\{D^\alpha s(v)\}_{|\alpha|\le 4r}$, which corresponds to $\mathcal{N}_v$. Similarly, for each edge e of $\triangle$, using the derivative information associated with $\mathcal{N}_e$, we can compute all coefficients c_ξ associated with points $\xi \in E_{2r}(e)$.

Now fix a tetrahedron $T := \langle v_1, v_2, v_3, v_4\rangle$ in $\triangle$. We show how to compute the remaining coefficients of s associated with domain points in $\mathcal{D}_{d,T}$. We start with domain points on the outer faces of T and work our way inward. Let $F := \langle v_1, v_2, v_3\rangle$ be one of the faces of T. We already know the coefficients of $s|_F$ corresponding to domain points in the disks $D_{4r}(v_i)$ for $i = 1, 2, 3$. We also know the coefficients of $s|_F$ corresponding to domain points within a distance of $2r$ of any edge of F. This leaves only the coefficients associated with domain points in the set $\{\xi^{T,8r+1}_{ijk0} : i, j, k \ge 2r + 1\}$. The coefficients corresponding to these domain points can be computed from the values of s at the points of $A_{F,0}$, which is part of the data corresponding to $\mathcal{N}_F^1$. This leads to a linear system with matrix $M_0 := [B^{F,8r+1}_\xi(\eta)]_{\xi,\eta\in A_{F,0}}$. This matrix is independent of the size and shape of F since the entries depend only on barycentric coordinates. It is nonsingular by Lemma 2.3. Fig. 1 (top) shows the domain points of s on F for the case $r = 3$. Points corresponding to coefficients that are determined from the sets $\mathcal{N}_v$ (i.e.,



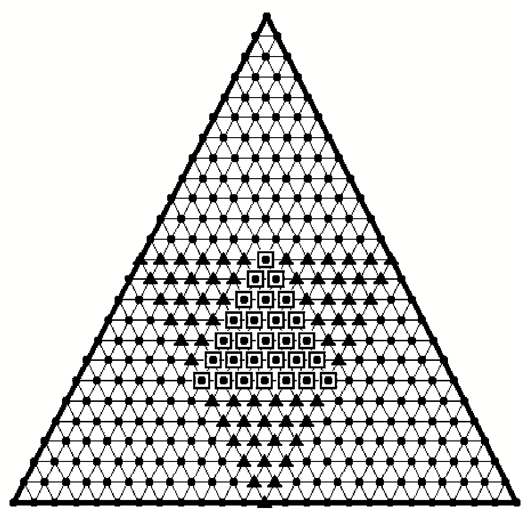


Fig. 1. Domain points on layers $\ell = 0$ (top) and $\ell = 1$ (bottom) for $r = 3$.

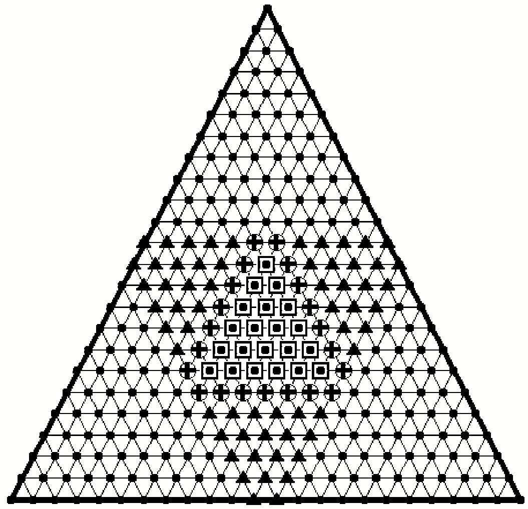
those in the disks $D_{12}(v_i)$) are marked with dots, while those corresponding to coefficients determined by the sets $\mathcal{N}_e$ (i.e. those within a distance 6 of edges) are marked with triangles. Points corresponding to coefficients determined by the set $\mathcal{N}_F^1$ are marked with a $\boxdot$.

We now compute the coefficients associated with domain points on the first layer inward from an outer face F of T. Let F_1 be a triangular face in the first layer next to F. We have determined the coefficients of s corresponding to the balls of radius $4r$ around the vertices of T which correspond to disks of radius $4r-1$ around the vertices of F_1. In addition, we know the coefficients of s corresponding to tubes of radius $2r$ around the edges of T which gives us the points within a distance

$2r-1$ of the edges of F_1. It remains to compute the coefficients corresponding to domain points on F_1 with indices $i,j,k \geq 2r$. We compute these coefficients from the values of $\{D_F s(\xi)\}_{\xi \in A_{F,1}}$, which is part of the data associated with $\mathcal{N}_F^1$. This involves solving a linear system with matrix $M_1 := [B_\xi^{F,8r}(\eta)]_{\xi,\eta \in A_{F,1}}$. This matrix is independent of the size and shape of F, and is nonsingular by Lemma 2.3. Fig. 1 (bottom) shows the domain points on this layer for the case $r=3$. Points corresponding to coefficients that are determined from the sets $\mathcal{N}_v$ (i.e., those in the disks of radius 11 around the vertices of F_1) are marked with small dots, while those corresponding to coefficients that are determined from the sets $\mathcal{N}_e$ (i.e., those within a distance 5 of edges of F_1) are marked with triangles. Points corresponding to coefficients that are determined from $\mathcal{N}_F^1$ are marked with a $\boxdot$.

We continue with layers that are a distance $\ell = 2,\ldots,r$ from the faces of T. The analysis of these layers is a little different from layers 0 and 1 since now we have to make use of the data associated with the functionals in the sets $\mathcal{N}_F^2$. Let F_ℓ be a triangular face of layer ℓ. We have determined the coefficients of s corresponding to the balls of radius $4r$ around the vertices of T which correspond to disks of radius $4r-\ell$ around the vertices of F_ℓ. In addition, we know the coefficients of s corresponding to tubes of radius $2r$ around the edges of T which gives us the points within a distance $2r-\ell$ of the edges of F_ℓ. To compute the remaining coefficients on F_ℓ, we first use the data associated with the sets $\mathcal{N}_F^2$ to compute the remaining unknown coefficients of s corresponding to domain points at a distance $2r-\ell+m$ from the edges of F_ℓ for $m = 1,\ldots,\lfloor \ell/2 \rfloor$. Then we use the values $\{D_F^\ell s(\xi)\}_{\xi \in A_{F,\ell}}$ (which correspond to functionals in $\mathcal{N}_F^1$) to solve for the coefficients of s corresponding to the domain points $\{\xi_{ijk\ell}^{T,8r+1} : i,j,k \geq 2r+1-\ell+\lfloor \ell/2 \rfloor\}$. This involves solving a linear system with matrix $M_\ell := [B_\xi^{F,8r-\ell+1}(\eta)]_{\xi,\eta \in A_{F,\ell}}$. This matrix is independent of the size and shape of F, and is nonsingular by Lemma 2.3. Fig. 2 shows the domain points on layers $\ell = 2,3$ for the case $r=3$. Points corresponding to coefficients that are determined from the sets $\mathcal{N}_v$ and $\mathcal{N}_e$ (i.e., those that lie in the disks of radius $12-\ell$ around vertices of F_ℓ or within a distance $6-\ell$ of an edge of F_ℓ are marked with dots and triangles, respectively. Points marked with $\oplus$ indicate coefficients that are computed from the sets $\mathcal{N}_F^2$. Points corresponding to coefficients that are determined from $\mathcal{N}_F^1$ are marked with a $\boxdot$.

We now proceed to compute unknown coefficients on layers $\ell = r+1,\ldots,r+n$. Let F_ℓ be a triangular face on layer ℓ. We already know the coefficients corresponding to domain points in disks of radius $4r-\ell$ around the vertices of F_ℓ. We also know the coefficient associated with all domain points within a distance $\ell-1$ of the edges of F. We now use the data associated with $\mathcal{N}_T^1$ to compute the remaining unknown coefficients of s corresponding to domain points at a distance ℓ from the edges of F_ℓ. Similarly, we use the data associated with $\mathcal{N}_T^2$ to compute the coefficients of s corresponding to domain points at a distance $\ell+1,\ldots,2r-\ell+\lfloor \ell/2 \rfloor$ from the edges of F_ℓ. Finally, we use the values $\{D_F^\ell s(\xi)\}_{\xi \in A_{F,\ell}}$ (which correspond to functionals in $\mathcal{N}_T^3$) to solve for the coefficients of s corresponding to the domain



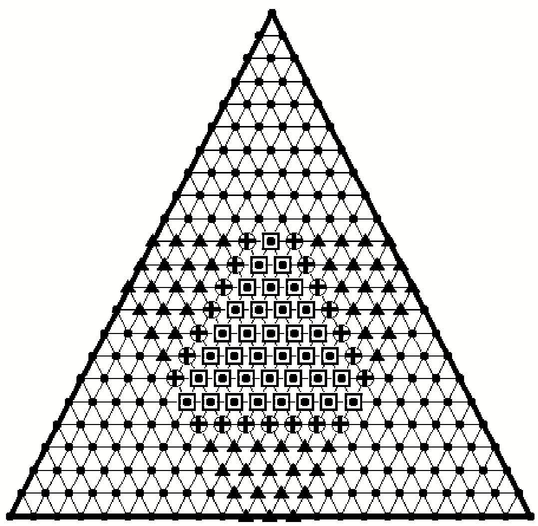


Fig. 2. Domain points on layers $\ell = 2$ (top) and $\ell = 3$ (bottom) for $r = 3$.

points $\{\xi^{T,8r+1}_{ijk\ell} : i, j, k \geq 2r + 1 - \ell + \lfloor \ell/2 \rfloor\}$. This involves solving a linear system with matrix $M_\ell := [B^{F,8r-\ell+1}_\xi(\eta)]_{\xi,\eta \in A_{F,\ell}}$. This matrix is independent of the size and shape of F, and is nonsingular by Lemma 2.3. Fig. 3 shows the domain points on layer $\ell = 4$ for the case $r = 3$. Points corresponding to coefficients that are determined from the sets $\mathcal{N}_v$ or $\mathcal{N}_e$ (i.e., those that lie in the disks of radius 8 around the vertices of F_4 or within a distance 2 of the edges of F_4) are marked with dots and triangles, respectively. Points marked with $*$ indicate coefficients that were computed in previous steps, while those marked with squares correspond to coefficients that are determined from the data of $\mathcal{N}^1_T$. $\mathcal{N}^2_T$ is empty for $r = 3$. Points on F_4 corresponding to coefficients that are determined from $\mathcal{N}^3_T$ are marked

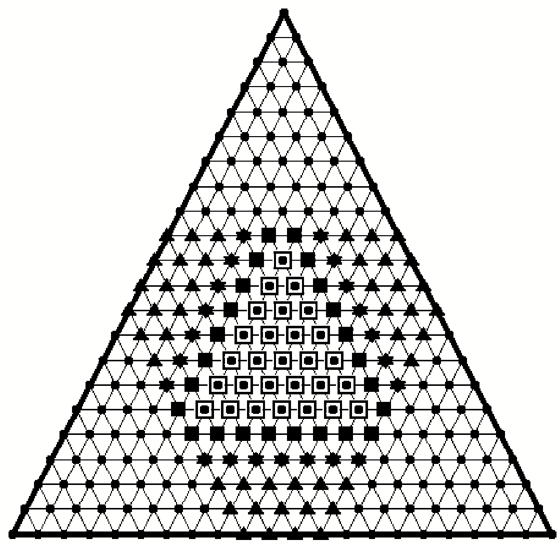


Fig. 3. Domain points on layer $\ell = 4$ for $r = 3$.

with a $\boxdot$.

After completing the above steps for layers $0, \ldots, r+n$, it remains to compute the coefficients of s corresponding to the domain points whose distance to the boundary of T are greater than or equal to $r+n+1$, i.e., coefficients of the form c^T_{ijkl} with $i, j, k, l \geq r+n+1$. Lemma 2.1 shows how to compute these coefficients from the data corresponding to $\mathcal{N}^4_T$.

We have shown that $\mathcal{N}$ is a nodal determining set. We claim it is m-stable in the sense of (2.2) with $m = 4r$. This follows from the fact that all coefficients are computed directly from derivatives using well-known formulae from the Bernstein–Bézier theory, or by solving linear systems of equations whose matrices are nonsingular and whose determinants depend only on the smallest angles in $\triangle$. The highest derivative involved is of order $4r$.

To show that $\mathcal{N}$ is minimal, we have to show that its cardinality is equal to the dimension of $\mathcal{S}_r(\triangle)$ as given in (3.2). It is clear that for every $v \in \mathcal{V}$,

$$\#\mathcal{N}_v = \#\mathcal{M}_v = \binom{4r+3}{3} = \frac{32r^3 + 48r^2 + 22r + 3}{3}.$$

and for every $e \in \mathcal{E}$,

$$\#\mathcal{N}_e = \#\mathcal{M}_e = \frac{(r+1)(2r+1)(4r)}{3} = \frac{8r^3 + 12r^2 + 4r}{3}.$$

This gives the first two terms in (3.2). To get the term involving n_F, we note that there is a one-to-one correspondence between the functionals in the sets $\mathcal{N}^1_F \cup \mathcal{N}^2_F$

and the points in $\mathcal{M}_F$, and so the two sets have the same cardinality. To see this directly, note that the cardinality of $A_{F,\ell}$ is $\binom{2r+2\ell-3\lfloor \ell/2\rfloor}{2}$. Thus,

$$\#\big(\mathcal{N}_F^1 \cup \mathcal{N}_F^2\big) = \sum_{\ell=0}^{r} \binom{2r+2\ell-3\lfloor \ell/2\rfloor}{2} + 3\sum_{\ell=2}^{r}\sum_{m=1}^{\lfloor \ell/2\rfloor}(2r+m),$$

which reduces to $(25r^3+21r^2-4r)/6 = \#\mathcal{M}_F$. Finally, we deal with the term in (3.2) involving N. We have

$$\#\big[\mathcal{N}_T^1 \cup \mathcal{N}_T^2 \cup \mathcal{N}_T^3 \cup \mathcal{N}_T^4\big] = \sum_{\ell=r+1}^{r+n} \Big[12\ell + \sum_{m=1}^{2r-2\ell+\lfloor \ell/2\rfloor} 12(2\ell+m) + 4\binom{2r+2\ell-3\lfloor \ell/2\rfloor}{2}\Big] + \binom{4r-4n}{3},$$

which reduces to $10r^3-6r^2 = \#\mathcal{M}_T$. $\square$

Theorem 5.1 asserts that if we assign values to all of the derivatives of s listed there, then s is uniquely determined. We emphasize that some of this data applies to the polynomial pieces of s rather than s itself. For example (cf. $\mathcal{N}_T^3$), for every interior face F of T, every $r+1 \le \ell \le r+n$, and every point $t \in A_{F,\ell}$, we have to assign values to both $D_F^\ell s|_T(t)$ and $D_F^\ell s|_{\widetilde{T}}(t)$, where T and $\widetilde{T}$ are the tetrahedra sharing the face F. We are allowed to assign different values to these derivatives since s is not not required to be C^ℓ across the face F. The situation is similar for the data associated with $\mathcal{N}_T^1$ and $\mathcal{N}_T^2$ since they also involve derivatives of order greater than r.

§6. Hermite Interpolation

Theorem 5.1 shows that for any function $f \in C^{4r}(\Omega)$, there is a unique spline $s \in \mathcal{S}_r(\triangle)$ solving the Hermite interpolation problem

$$\lambda s = \lambda f, \qquad \text{for all } \lambda \in \mathcal{N}.$$

This defines a linear projector $\mathcal{I}$ mapping $C^{4r}(\Omega)$ onto the superspline space $\mathcal{S}_r(\triangle)$. Since $\mathcal{S}_r(\triangle)$ contains $\mathcal{P}_{8r+1}$, $\mathcal{I}$ reproduces all polynomials of degree $d := 8r+1$. Since the NMDS of Theorem 5.1 is local and stable, we can establish the following error bound, see [2–4,15,17,18] for similar results for other macro-elements.

Theorem 6.1. *For every $f \in C^{m+1}(\Omega)$ with $4r-1 \le m \le 8r+1$,*

$$\|D^\alpha(f-\mathcal{I}f)\|_\Omega \le K|\triangle|^{m+1-|\alpha|}|f|_{m+1,\Omega}, \tag{6.1}$$

for all $0 \le |\alpha| \le m$. Here K depends only on r and the smallest angles in $\triangle$.

Proof: Fix $T \in \triangle$, and let $f \in C^{m+1}(\Omega)$. Fix α with $|\alpha| \le m$. By Lemma 4.3.8 of [9], there exists a polynomial $p \in \mathcal{P}_m$ such that

$$|f-p|_{|\beta|,T} \le K_1|T|^{m+1-|\beta|}|f|_{m+1,T}, \tag{6.2}$$

for all $0 \le |\beta| \le m$. Since $\mathcal{I}\, p = p$,

$$\|D^\alpha(f - \mathcal{I} f)\|_T \le \|D^\alpha(f-p)\|_T + \|D^\alpha \mathcal{I}(f-p)\|_T.$$

In view of (6.2) with $\beta = \alpha$, it suffices to estimate the second quantity. Applying the Markov inequality [15,20] to each of the polynomials $\mathcal{I}(f-p)|_T$, we have

$$\|D^\alpha \mathcal{I}(f-p)\|_T \le K_2 |T|^{-|\alpha|} \|\mathcal{I}(f-p)\|_T,$$

where K_2 is a constant depending only on r and the smallest angles in $\triangle$. Let $\{c_\xi\}$ be the B-coefficients of the polynomial $\mathcal{I}(f-p)|_T$ relative to the tetrahedron T. Then combining (2.2) with the fact that the Bernstein basis polynomials form a partition of unity, it is easy to see that

$$\|\mathcal{I}(f-p)\|_T \le K_3 \max_{\xi \in \mathcal{D}_{d,T}} |c_\xi| \le K_4 \sum_{i=0}^{4r} |T|^i |f-p|_{i,T}.$$

Combining this with (6.2), we have

$$\|\mathcal{I}(f-p)\|_T \le K_5 |T|^{m+1} |f|_{m+1,T},$$

which gives

$$\|D^\alpha(f - \mathcal{I} f)\|_T \le K_6 |T|^{m+1-|\alpha|} |f|_{m+1,T}.$$

Finally, we take the maximum over all tetrahedra T in $\triangle$ to get (6.1). □

§7. Remarks

Remark 1. The idea of a macro-element space is that the nodal data can be used to compute a Hermite interpolating spline s as in Sect. 6, where the coefficients of the polynomial $s|_T$ can be computed locally one tetrahedron at a time using only the data associated with points in T. As observed already in [23], this implies that to construct a trivariate C^r macro-element, we need to enforce at least C^{2r} supersmoothness around the edges (since otherwise polynomials constructed locally will not join together with C^r smoothness). This in turn implies that we need C^{4r} supersmoothness at the vertices, which implies that to create a macro-element without splitting the tetrahedra, we need to use polynomials of degree at least $8r+1$.

Remark 2. Lemma 2.3 can also be established using a Bezout-type argument similar to the one used to prove Lemma 2.1.

Remark 3. It is possible to construct trivariate macro-element spaces using lower degree polynomials provided we are willing to split the tetrahedra into subtetrahedra. For $r = 1$ this idea has been used in [1,21,22] to create macro-element spaces using splines of degree five, three, and two, respectively. For some recently constructed C^2 macro-element spaces based on split tetrahedra, see [2,3,4].

Remark 4. It is also possible to construct macro-elements based on splits of octahedra. For the C^1 case, see [12], and for the C^2 case, see [13].

Remark 5. Using the MDS $\mathcal{M}$ of Theorem 3.1, it is easy to construct a stable local basis for $\mathcal{S}_r(\triangle)$. For each $\xi \in \mathcal{M}$, let $\psi_\xi \in \mathcal{S}_r(\triangle)$ be the spline whose B-coefficients satisfy $c_\eta = \delta_{\eta,\xi}$ for all $\xi, \eta \in \mathcal{M}$. It is clear that the splines in the set $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ are linearly independent, and since $\#\mathcal{M} = \dim \mathcal{S}_r(\triangle)$, it follows that they form a basis for $\mathcal{S}_r(\triangle)$. They are called the dual basis splines associated with $\mathcal{M}$. It is easy to see that they have the following small supports:

1) if $\xi \in \mathcal{M}_v$ for some vertex v of $\triangle$, then the support of ψ_ξ lies in star(v),
2) if $\xi \in \mathcal{M}_e$ for some edge e of $\triangle$, then the support of ψ_ξ is contained in the union of the tetrahedra containing e,
3) if $\xi \in \mathcal{M}_F$ for some face F of $\triangle$, then the support of ψ_ξ is contained in the union of the tetrahedra containing F,
4) if $\xi \in \mathcal{M}_T$ for some tetrahedron T of $\triangle$, then the support of ψ_ξ is contained in T.

Remark 6. We can construct a different stable local basis for $\mathcal{S}_r(\triangle)$ using the NMDS $\mathcal{N}$ of Theorem 5.1. For each $\lambda \in \mathcal{N}$, let $\phi_\lambda \in \mathcal{S}_r(\triangle)$ be such that $\gamma\phi_\lambda = \delta_{\gamma,\lambda}$ for all $\gamma, \lambda \in \mathcal{N}$. It is clear that the splines in the set $\{\phi_\xi\}_{\xi\in\mathcal{N}}$ are linearly independent, and since $\#\mathcal{N} = \dim \mathcal{S}_r(\triangle)$, it follows that they form a basis for $\mathcal{S}_r(\triangle)$. They are called the dual basis splines associated with $\mathcal{N}$. Each of these splines has support on a set that is at most the union of all tetrahedra surrounding a vertex.

Remark 7. Assuming that Conjecture 2.2 holds, it is possible to describe a different nodal minimal determining set for $\mathcal{S}_r(\triangle)$ which replaces some of the higher order edge derivatives by face derivatives. Given a face F with edges $\mathcal{E}_F$ and vertices $\mathcal{V}_F$, let

$$\tilde{A}_{F,\ell} := \{\xi_{ijk}^{F,8r+1-\ell} \,:\, i,j,k \geq 2r-\ell+1\} \setminus \bigcup_{v\in\mathcal{V}_F} D_{4r-\ell}(v),$$

for $0 \leq \ell \leq r$, and

$$\tilde{A}_{F,\ell} := \{\xi_{ijk}^{F,8r+1-\ell} \,:\, i,j,k \geq \ell\} \setminus \bigcup_{v\in\mathcal{V}_F} D_{4r-\ell}(v),$$

for $r+1 \leq \ell \leq r+n$. Then we define an alternative nodal minimal determining set $\tilde{\mathcal{N}}$ by replacing the sets $\mathcal{N}_F^1 \cup \mathcal{N}_F^2$ by

$$\tilde{\mathcal{N}}_F := \bigcup_{\ell=0}^{r} \{\varepsilon_\xi D_F^\ell\}_{\xi\in\tilde{A}_{F,\ell}},$$

and replacing the sets $\mathcal{N}_T^1 \cup \mathcal{N}_T^2 \cup \mathcal{N}_T^3$ by

$$\tilde{\mathcal{N}}_T^1 := \bigcup_{F\in\mathcal{F}_T} \bigcup_{\ell=r+1}^{r+n} \{\varepsilon_\xi D_F^\ell\}_{\xi\in\tilde{A}_{F,\ell}},$$

where $\mathcal{F}_T$ is the set of faces of T. A simple count shows that $\tilde{\mathcal{N}}$ has the same cardinality as $\mathcal{N}$. To show that it is a nodal minimal determining set, we can follow the proof of Theorem 5.1. The proof changes slightly in dealing with layers of a tetrahedron. Given that we have already determined coefficients associated with all balls $D_{4r}(v)$ and tubes $E_{2r}(e)$, we determine the remaining coefficients associated with domain points on a typical face F_ℓ of the ℓ-th layer by solving an interpolation problem whose matrix is $\left[B_\xi^{F,8r+1-\ell}(\eta)\right]_{\xi,\eta\in\tilde{A}_{F,\ell}}$. To ensure this is a nonsingular system, we appeal to the conjecture.

Remark 8. It was also conjectured by the second author that the trivariate version of Conjecture 2.2 holds, i.e., that given a tetrahedron T, matrices of the form $[B_\xi^{T,d}(\eta)]_{\xi,\eta\in\Gamma}$ are nonsingular for any choice of $\Gamma \subseteq \mathcal{D}_{d,T}$. Assuming this conjecture, it is possible to give another interesting nodal minimal determining set for $\mathcal{S}_r(\triangle)$. Now we define $\tilde{\mathcal{N}}$ by replacing $\mathcal{N}_T^1 \cup \mathcal{N}_T^2 \cup \mathcal{N}_T^3 \cup \mathcal{N}_T^4$ in Theorem 5.1 by $\tilde{\mathcal{N}}_T := \{\varepsilon_\xi\}_{\mathcal{M}_T}$, where $\mathcal{M}_T$ is as in Theorem 3.1, i.e.,

$$\mathcal{M}_T := \{\xi_{ijkl}^{T,8r+1} \ : \ i,j,k,l \geq r+1\} \setminus \bigcup_{v\in\mathcal{V}_T} D_{4r}(v),$$

where $\mathcal{V}_T$ is the set of vertices of T. Using $\tilde{\mathcal{N}}$, we now determine the B-coefficients corresponding to $\mathcal{M}_T$ by interpolating at the domain points of $\mathcal{M}_T$.

Remark 9. The proof of Theorem 4.2 showing that $\mathcal{S}_r(\triangle)$ has full approximation power is based on the construction of a quasi-interpolation operator. The general idea has been used in spline theory for some time. It was used explicitly in the trivariate setting in [17], and can also be used to show that any trivariate spline space with a stable local minimal determining set has full approximation power, see [15]. The idea of the proof of Theorem 6.1 has also been used in several recent papers on trivariate macro-elements, see [2–4,17,18].

Remark 10. The problem of finding the dimension of C^r trivariate spline spaces with $r > 0$ seems to be quite difficult. There are dimension formulae in the literature for the macro-element spaces mentioned in Remarks 3 and 4. There are also results for certain spaces on other special partitions, see e.g. [17,18]. However, for general tetrahedral partitions, formulae are not known for the spline spaces $\mathcal{S}_d^r(\triangle) := \{s \in C^r(\Omega) \,:\, s|_T \in \mathcal{P}_d, \text{all } T \in \triangle\}$ or their superspline subspaces, even for large values of d compared to r. As observed in [6], we cannot expect to get dimension formulae for these spaces without first solving the dimension problem for bivariate splines on cells (clusters of triangles surrounding a single vertex) for *all* values of d and r. For generic partitions, the case $r = 1$ and $d \geq 8$ was treated in [6]. Surprisingly, without giving dimension formulae, it is possible to show that the spaces $\mathcal{S}_d^r(\triangle)$ have local bases for $d \geq 8r+1$, see [5,7,8]. Numerical methods for constructing stable local bases of multivariate spline spaces can be found in [11].

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