Macro-Elements and Stable Local Bases for Splines on Clough-Tocher Triangulations

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Abstract. Macro-elements of arbitrary smoothness are constructed on Clough-Tocher triangle splits. These elements can be used for solving boundary-value problems or for interpolation of Hermite data, and are shown to be optimal with respect to spline degree. We believe they are also optimal with respect to the number of degrees of freedom. The construction provides local bases for certain superspline spaces defined over Clough-Tocher refinements of arbitrary triangulations. These bases are shown to be stable as a function of the smallest angle in the triangulation, which in turn implies that the associated spline spaces have optimal order approximation power.

§1. Introduction

Let $\triangle$ be triangulation of a polygonal domain Ω in $\mathbb{R}^2$. In this paper we are interested in polynomial spline spaces of the form

$$\mathcal{S}_d^r(\triangle) := \{s \in C^r(\Omega) : \ s|_T \in \mathcal{P}_d \text{ for all } T \in \triangle,\}$$

where $d > r > 0$ are given integers and $\mathcal{P}_d$ is the space of bivariate polynomials of degree d. A basis $\{B_i\}_{i=1}^n$ for a spline space $\mathcal{S}$ is called a stable local basis provided that there exist constants ℓ, K_1, K_2 depending only on the smallest angle in $\triangle$ such that

1) for each $1 \le i \le n$, there is a vertex v_i of $\triangle$ for which $\operatorname{supp}(B_i) \subseteq \operatorname{star}^\ell(v_i)$,
2) for all choices of the coefficient vector $c = (c_1, \ldots, c_n)$,

$$K_1 \|c\|_\infty \ \le \ \|\sum_{i=1}^n c_i B_i\|_\infty \ \le \ K_2 \|c\|_\infty. \tag{1.1}$$

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Here $\text{star}^0(v)$ is defined to be the set of all triangles surrounding a vertex v, and $\text{star}^\ell(v)$ is defined to be the union of the $\text{star}^0(w)$, where w are vertices of $\text{star}^{\ell-1}(v)$.

It is known that if a space of splines $\mathcal{S}$ of degree d contains $\mathcal{P}_d$ and has a stable local basis, then it provides optimal order approximations of smooth functions, see Remark 8.2. These spaces are of particular importance in applications, such as data fitting or the solution of boundary-value problems.

Finding stable local bases for spline spaces $\mathcal{S}_d^r(\triangle)$ is a nontrivial task for $r > 0$, and for general triangulations can only be done when $d \geq 3r+2$, see Remark 8.3. The first constructions were for very special superspline subspaces of $\mathcal{S}_d^r(\triangle)$, and can be found in [2,10]. A construction for arbitrary spline spaces $\mathcal{S}_d^r(\triangle)$ and corresponding superspline subspaces was discovered only very recently, see [4].

To get stable bases for spline spaces with $d < 3r+2$, we have to restrict ourselves to classes of triangulations with a special structure. In this paper we work with Clough-Tocher triangulations $\triangle_{CT}$ which are obtained from an arbitrary triangulation $\triangle$ by splitting each triangle into three subtriangles, see Sect. 2.

The main result of this paper is an explicit construction of stable local bases for the super-spline spaces

$$\begin{aligned} \mathcal{S}_{2m}(\triangle_{CT}) &:= \mathcal{S}_{6m+1}^{2m,3m,5m+1}(\triangle_{CT}), \\ \mathcal{S}_{2m+1}(\triangle_{CT}) &:= \mathcal{S}_{6m+3}^{2m+1,3m+1,5m+2}(\triangle_{CT}), \end{aligned} \tag{1.2}$$

where in general

$$\begin{aligned} \mathcal{S}_d^{r_1,r_2,r_3}(\triangle_{CT}) := \{s \in \mathcal{S}_d^{r_1}(\triangle_{CT}) : \; & s \in C^{r_2}(v) \text{ for all } v \in \mathcal{V}, \\ & s \in C^{r_3}(v) \text{ for all } v \in \mathcal{W}\} \end{aligned} \tag{1.3}$$

for all $m \geq 0$. Here $\mathcal{V}$ is the set of vertices of the original triangulation $\triangle$, and $\mathcal{W}$ is the set of centers which have been inserted to form $\triangle_{CT}$. This is a classical superspline space with variable smoothness at vertices. As a byproduct of the construction, we obtain certain useful macro-elements which can be used in the numerical solution of boundary-value problems, and to solve Hermite interpolation problems. These macro elements are improvements on existing elements obtained in [7–8,12–13], see Sect. 6.

The paper is organized as follows. In Sect. 2 we present several preliminary ideas. The cases $r = 2m$ and $r = 2m+1$ are treated in Sects. 3 and 4, respectively. In Sect. 5 we show how our constructions yield useful new macro-elements for all choices of smoothness r, and in Sect. 6 we compare them with previously available macro-elements. In Sect. 7 we show that our choices of degrees and supersmoothness are optimal in a certain sense. We conclude in Sect. 8 with several remarks.

§2. Preliminaries

We begin by definining what we mean by a Clough-Tocher triangulation.

Definition 2.1. *Given a triangulation $\triangle$ of a set Ω, the* Clough-Tocher refinement *of $\triangle$ is the triangulation obtained by connecting the barycentric center v_T of each triangle T in $\triangle$ to the three vertices of T.*

In the special case where this refinement process is applied to a single triangle $T := \langle v_1, v_2, v_3 \rangle$ with incenter v, we call the resulting Clough-Tocher refinement a Clough-Tocher cell. We denote such a cell by $\triangle_v$. Since the Clough-Tocher split of a triangle T is defined using its barycentric center, it is clear that the smallest angle in $\triangle_{CT}$ is equal to one-half of the smallest angle in T.

To construct stable local bases, we make use of Bernstein-Bézier techniques as in [1,2,4,5,10]. For any given d and triangulation $\triangle$, let

$$\mathcal{D}_{d,\triangle} := \bigcup_{T \in \triangle} \mathcal{D}_{d,T},$$

be the set of domain points, where

$$\mathcal{D}_{d,T} := \{\xi^T_{ijk} := \frac{(iv_1 + jv_2 + kv_3)}{d}, \quad i+j+k = d\},$$

and $T := \langle v_1, v_2, v_3 \rangle$. We recall that if $\mathcal{M}$ is a minimal determining set of domain points for a linear space $\mathcal{S} \subseteq \mathcal{S}^r_d(\triangle)$, then there exists a corresponding set $\{B_\xi\}_{\xi \in \mathcal{M}}$ of dual splines satisfying

$$\lambda_\eta B_\xi = \delta_{\xi,\eta}, \qquad \text{all } \eta \in \mathcal{M}. \tag{2.1}$$

Then the splines $\{B_\xi\}_{\xi \in \mathcal{M}}$ are linearly independent and form a basis for $\mathcal{S}$. The trick is to choose $\mathcal{S}$ and $\mathcal{M}$ carefully to insure that this basis is stable and local.

We close this section by recalling some additional standard notation. Given a triangle $T := \langle v_1, v_2, v_3 \rangle$, the ring of radius n around v_1 is defined by

$$R^T_n(v_1) := \{\xi^T_{ijk} : \ i = d - n\},$$

and the disk of radius n around v_1 is defined by

$$D^T_n(v_1) := \{\xi^T_{ijk} : \ i \ge d - n\}.$$

We have similar definitions at the other vertices of T. If v is a vertex of a triangulation $\triangle$, we define

$$R_n(v) := \bigcup R^T_n(v),$$
$$D_n(v) := \bigcup D^T_n(v),$$

where the union is taken over all triangles attached to v.

§3. The case $r = 2m$

In this section we work with the spaces

$$\mathcal{S}_{2m}(\triangle_{CT}) := \mathcal{S}^{2m,3m,5m+1}_{6m+1}(\triangle_{CT}).$$

To describe a minimal determining set whose corresponding set of dual splines form a stable local basis for $\mathcal{S}_{2m}(\triangle_{CT})$, we first examine the space $\mathcal{S}_{2m}(\triangle_v)$ on a Clough-Tocher cell $\triangle_v$. Suppose the boundary vertices of the cell are v_1, v_2, v_3 in counterclockwise order, and let $T^{[i]} := \langle v, v_i, v_{i+1}\rangle$ for $i = 1, 2, 3$.

Theorem 3.1. *Let $\mathcal{M}$ be the union of the following sets of domain points:*

1) $D^{T^{[i]}}_{3m}(v_i)$ *for* $i = 1, 2, 3$,
2) $\{\xi^{T^{[i]}}_{j,3m,3m-j+1}, \ldots, \xi^{T^{[i]}}_{j,3m-j+1,3m}\}$ *for* $j = 1, \ldots, 2m$ *and* $i = 1, 2, 3$,
3) $D^{T^{[1]}}_{2m-2}(v)$.

Then $\mathcal{M}$ is a minimal determining set for $\mathcal{S}_{2m}(\triangle_v)$, and the corresponding dual basis $\{B_\xi\}_{\xi\in\mathcal{M}}$ is a stable basis for $\mathcal{S}_{2m}(\triangle_v)$. Moreover,

$$\dim \mathcal{S}_{2m}(\triangle_v) = \frac{43m^2 + 31m + 6}{2}. \tag{3.1}$$

Proof: We first show that $\mathcal{M}$ is a minimal determining set. Suppose s is a spline in $\mathcal{S}_{2m}(\triangle_v)$ whose B-coefficients corresponding to points in $\mathcal{M}$ are set to prescribed values. We now show that all of its remaining B-coefficients associated with domain points on $\triangle_v$ are uniquely and stably determined. Clearly, the coefficients corresponding to domain points in the disks $D_{3m}(v_i)$ can be uniquely and stably computed from those corresponding to domain points in item 1) by the classical smoothness conditions.

We now show how to compute the coefficients corresponding to the remaining domain points in the disk $D_{5m+1}(v)$. By the super-smoothness at v, these coefficients can be regarded as the coefficients of a polynomial p of degree $5m+1$ on the triangle $\widetilde{T} := \{u_1, u_2, u_3\}$, where $u_i := \xi^{T^{[i]}}_{m,5m+1,0}$ for $i = 1, 2, 3$. The coefficients which have already been set or computed determine the derivatives $D^\alpha_x D^\beta_y p(u_i)$ for $0 \le \alpha + \beta \le 2m$ and $i = 1, 2, 3$. For each edge e_i of $\widetilde{T}$, the prescribed information also determines the cross derivatives of p of order κ at $\kappa + m$ equally spaced points in the interior of e_i for $0 \le \kappa \le m$. We also note that setting the data in 3) uniquely determines the derivatives $D^\alpha_x D^\beta_y p(v)$ for $0 \le \alpha + \beta \le 2m - 2$.

Suppose we now represent p as a single polynomial in B-form on the triangle $\widetilde{T}$. The above derivative information uniquely determines the corresponding B-coefficients $\{\tilde{c}_{ijk}\}_{i+j+k=5m+1}$ in the disks $D_{2m}(u_i)$ and in the set $E_1 := \{\tilde{c}_{ijk} \notin D_{2m}(u_1) \cup D_{2m}(u_2) :\ 0 \le i \le m\}$ and the two analogous sets E_2 and E_3 along the other two edges. At this point we can uniquely compute the coefficients $\tilde{c}_{ijk}$ of p for $i, j, k \ge m+1$ from the derivatives $D^\alpha_x D^\beta_y p(v)$ for $0 \le \alpha + \beta \le 2m-2$. The stability of this computation is governed by the smallest angle of $\widetilde{T}$, which is bounded below by the smallest angle of $\triangle$. We can now convert the coefficients of p to coefficients of s in the disk $D_{5m+1}(v)$ by subdivision using the stable de Casteljau algorithm.

Finally, to compute the dimension of $\mathcal{S}_{2m}(\triangle_v)$, we observe that

$$\#\mathcal{M} = 3\left[\binom{3m+2}{2} + \binom{2m+1}{2}\right] + \binom{2m}{2},$$

which reduces to the number in (3.1). □

We now illustrate Theorem 3.1 with three examples.

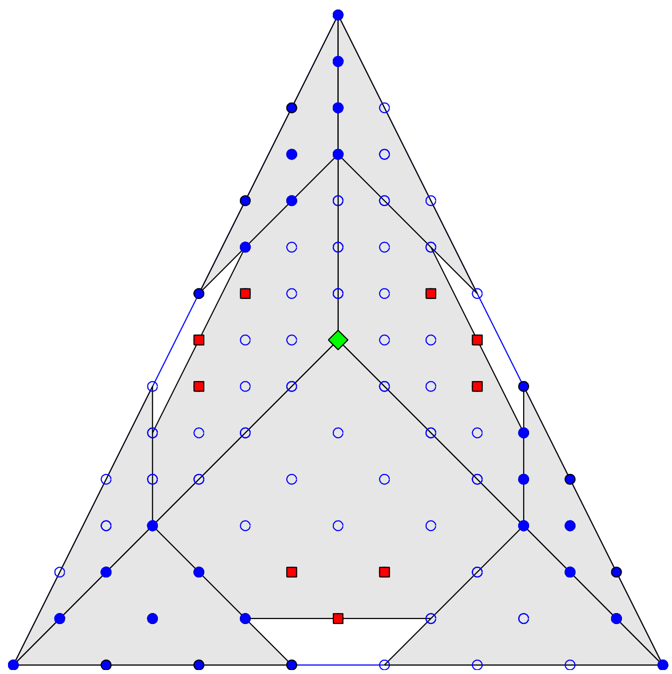

Fig. 1. The macro-element $\mathcal{S}_7^{2,3,6}(\triangle_{CT})$.

Example 3.2. *The stable local MDS of Theorem 3.1 for* $\mathcal{S}_7^{2,3,6}(\triangle_{CT})$ *is shown in Fig. 1. It contains 40 domain points. There are 10 points in each of the disks* $D_3(v_i)$ *(marked with dark circles), three points associated with each edge (marked with dark squares), and one at the incenter (marked with a dark diamond).*

Discussion: We have shaded the disks $D_3(v_i)$ and $D_6(v)$. The unmarked coefficients in the disks $D_3(v_i)$ are computed from the usual smoothness conditions. The remaining coefficients in $D_6(v)$ are then computed by the method described in the proof of Theorem 3.1. □

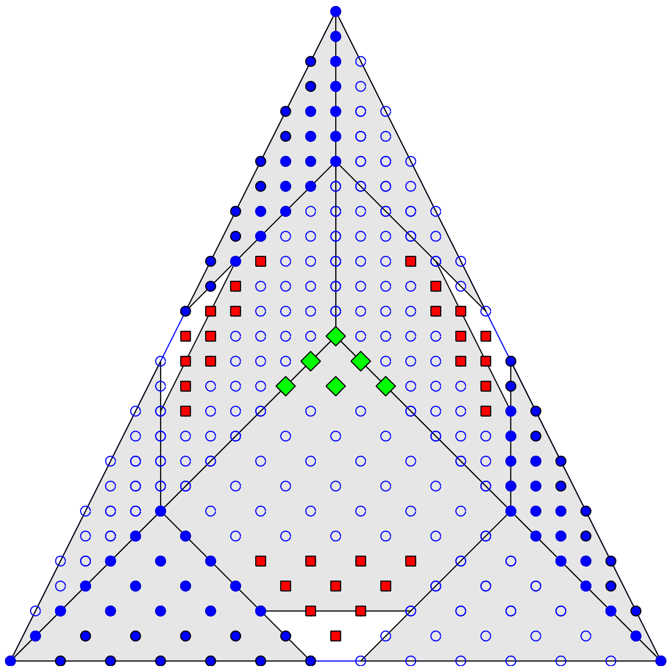

Fig. 2. The macro-element $\mathcal{S}_{13}^{4,6,11}(\triangle_{CT})$.

Example 3.3. *The stable local MDS of Theorem 3.1 for* $\mathcal{S}_{13}^{4,6,11}(\triangle_{CT})$ *is shown in Fig. 2. It contains 120 domain points. There are 28 points in each of the disks* $D_6(v_i)$ *(marked with dark circles), 10 points associated with each edge (marked with dark squares), and 6 at the incenter (marked with a dark diamond).*

Discussion: We have shaded the disks $D_6(v_i)$ and $D_{11}(v)$. The unmarked coefficients in the disks $D_6(v_i)$ are computed from the usual smoothness conditions. The remaining coefficients in $D_{11}(v)$ are then computed by the method described in the proof of Theorem 3.1. □

Example 3.4. *The stable local MDS of Theorem 3.1 for* $\mathcal{S}_{19}^{6,9,16}(\triangle_{CT})$ *is shown in Fig. 3. It contains 243 domain points. There are 55 points in each of the disks* $D_9(v_i)$ *(marked with dark circles), 21 points associated with each edge (marked with dark squares), and 15 at the incenter (marked with a dark diamond).*

Discussion: We have shaded the disks $D_9(v_i)$ and $D_{16}(v)$. The unmarked coefficients in the disks $D_9(v_i)$ are computed from the usual smoothness conditions. The remaining coefficients in $D_{16}(v)$ are then computed by the method described in the proof of Theorem 3.1. □

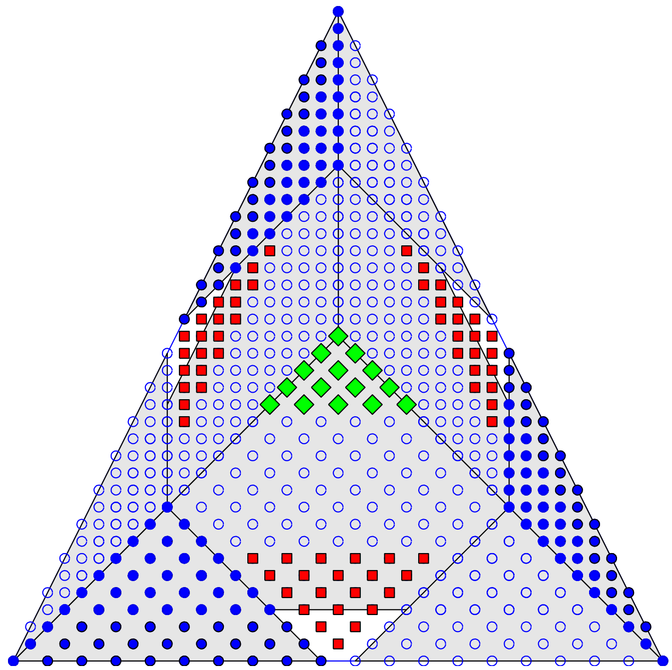

Fig. 3. The macro-element $\mathcal{S}_{19}^{6,9,16}(\triangle_{CT})$.

We can now use the construction of Theorem 3.1 to create a stable local basis for $\mathcal{S}_{2m}(\triangle)$.

Theorem 3.5. *Let $\mathcal{M}$ be the following set of domain points:*

1) *for each vertex v of $\triangle$, choose a triangle T of $\triangle_{CT}$ attached to v and include $D_{3m}^{T}(v)$,*
2) *for each edge $e = \langle v_1, v_2 \rangle$ of $\triangle$, let $T = \langle v, v_1, v_2 \rangle$ be a triangle of $\triangle_{CT}$ containing the edge e. Then include the points $\{\xi^T_{j,3m,3m-j+1}, \ldots, \xi^T_{j,3m-j+1,3m}\}$ for $j = 1, \ldots, 2m$ and $i = 1, 2, 3$,*
3) *for each triangle $T := \langle v_1, v_2, v_3 \rangle$ of $\triangle$ include $D^{T^{[1]}}_{2m-2}(w)$ where w is the barycenter of T and $T^{[1]} := \langle v, v_1, v_2 \rangle$.*

Then $\mathcal{M}$ is a minimal determining set for $\mathcal{S}_{2m}(\triangle)$, and the corresponding dual basis $\{B_\xi\}_{\xi \in \mathcal{M}}$ forms a stable star-supported basis for $\mathcal{S}_{2m}(\triangle)$. Moreover,

$$\dim \mathcal{S}_{2m}(\triangle) = \binom{3m+2}{2} V + \binom{2m+1}{2} E + \binom{2m}{2} N, \tag{3.2}$$

where V, E, N are the number of vertices, edges, and triangles in $\triangle$.

Proof: Following the arguments used in the proof of Theorem 3.1, it is easy to verify that $\mathcal{M}$ is a minimal determining set, and that the construction of a dual basis can be carried out in a stable way. It is also easy to see that each dual basis spline has support on $star(v)$ for some vertex v. To get the dimension, we simply count the number of points in $\mathcal{M}$. □

§4. The case $r = 2m+1$

In this section we work with the superspline spaces

$$\mathcal{S}_{2m+1}(\triangle) := \mathcal{S}_{6m+3}^{2m+1,3m+1,5m+2}(\triangle_{CT}).$$

Theorem 4.1. *Let $\mathcal{M}$ be the union of the following sets of domain points:*

1) $D_{3m+1}^{T^{[i]}}(v_i)$ *for* $i = 1, 2, 3$,
2) $\{\xi_{j,3m+1,3m-j+2}^{T^{[i]}}, \ldots, \xi_{j,3m-j+2,3m+1}^{T^{[i]}}\}$ *for* $j = 1, \ldots, 2m+1$ *and* $i = 1, 2, 3$,
3) $D_{2m-1}^{T^{[1]}}(v)$.

Then $\mathcal{M}$ is a minimal determining set for $\mathcal{S}_{2m+1}(\triangle_v)$, and the corresponding dual basis $\{B_\xi\}_{\xi\in\mathcal{M}}$ is a stable basis for $\mathcal{S}_{2m+1}(\triangle_v)$. Moreover,

$$\dim \mathcal{S}_{2m+1}(\triangle_v) = \frac{43m^2 + 65m + 24}{2}. \tag{4.1}$$

Proof: The proof is very similar to the proof of Theorem 3.1. Suppose s is a spline in $\mathcal{S}_{2m+1}(\triangle_v)$ whose B-coefficients corresponding to points in $\mathcal{M}$ are set to prescribed values. We now show that all of its remaining B-coefficients associated with domain points on $\triangle_v$ are uniquely and stably determined. First the data in 1) is used to uniquely compute all coefficients corresponding to domain points in the disks $D_{3m+1}(v_i)$.

To compute the remaining coefficients of s, we consider it to be a polynomial p of degree $5m+2$ on the triangle $\widetilde{T} := \langle u_1, u_2, u_3\rangle$ with $u_i := \xi_{m+1,5m+2,0}^{T^{[i]}}$ for $i = 1, 2, 3$. By what we have already computed, we have uniquely determined the derivatives $D_x^\alpha D_y^\beta p(u_i)$ for $0 \le \alpha+\beta \le 2m$ and $i = 1, 2, 3$. For each edge e_i of $\widetilde{T}$, the prescribed information also determines the cross derivatives of p of order κ at $\kappa + m + 1$ equally spaced points in the interior of e_i for $0 \le \kappa \le m$. We also note that setting the data in 3) uniquely determines the derivatives $D_x^\alpha D_y^\beta p(v)$ for $0 \le \alpha + \beta \le 2m-1$. This information uniquely and stably determines all of the coefficients of p. We can then stably convert them to coefficients of s by subdivision.

The dimension of $\mathcal{S}_{2m+1}(\triangle_v)$ is given by

$$\#\mathcal{M} = 3\left[\binom{3m+3}{2} + \binom{2m+2}{2}\right] + \binom{2m+1}{2},$$

which reduces to the number in (4.1). □

We now illustrate Theorem 4.1 with three examples.

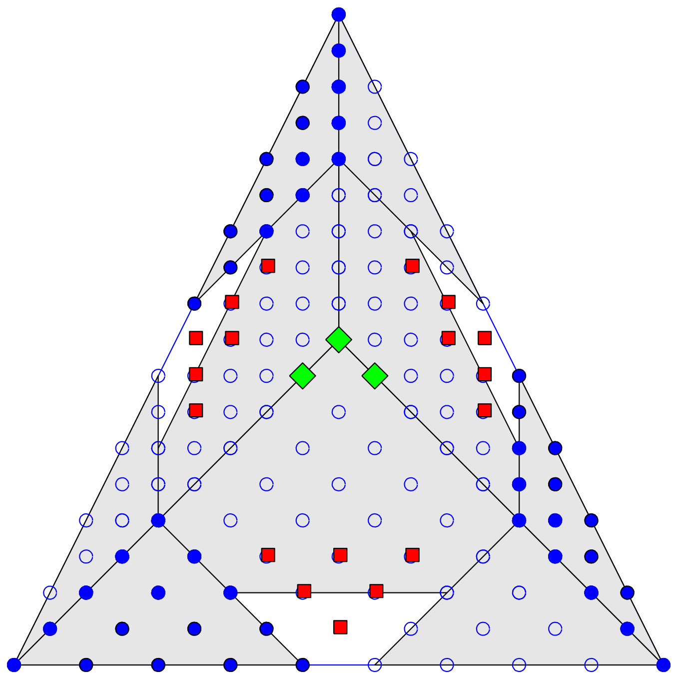

Fig. 4. The macro-element $\mathcal{S}_9^{3,4,7}(\triangle_{CT})$.

Example 4.2. *The stable local MDS of Theorem 4.1 for $\mathcal{S}_9^{3,4,7}(\triangle_{CT})$ is shown in Fig. 4. It contains 66 domain points. There are 15 points in each of the disks $D_4(v_i)$ (marked with dark circles), 6 points associated with each edge (marked with dark squares), and 3 at the incenter (marked with a dark diamond).*

Discussion: We have shaded the disks $D_4(v_i)$ and $D_7(v)$. The unmarked coefficients in the disks $D_4(v_i)$ are computed from the usual smoothness conditions. The remaining coefficients in $D_7(v)$ are then computed by the method described in the proof of Theorem 4.1. □

Example 4.3. *The stable local MDS of Theorem 3.1 for $\mathcal{S}_{15}^{5,7,12}(\triangle_{CT})$ is shown in Fig. 5. It contains 163 domain points. There are 36 points in each of the disks $D_7(v_i)$ (marked with dark circles), 15 points associated with each edge (marked with dark squares), and 10 at the incenter (marked with a dark diamond).*

Discussion: We have shaded the disks $D_7(v_i)$ and $D_{12}(v)$. The unmarked coefficients in the disks $D_7(v_i)$ are computed from the usual smoothness conditions. The remaining coefficients in $D_{12}(v)$ are then computed by the method described in the proof of Theorem 4.1. □

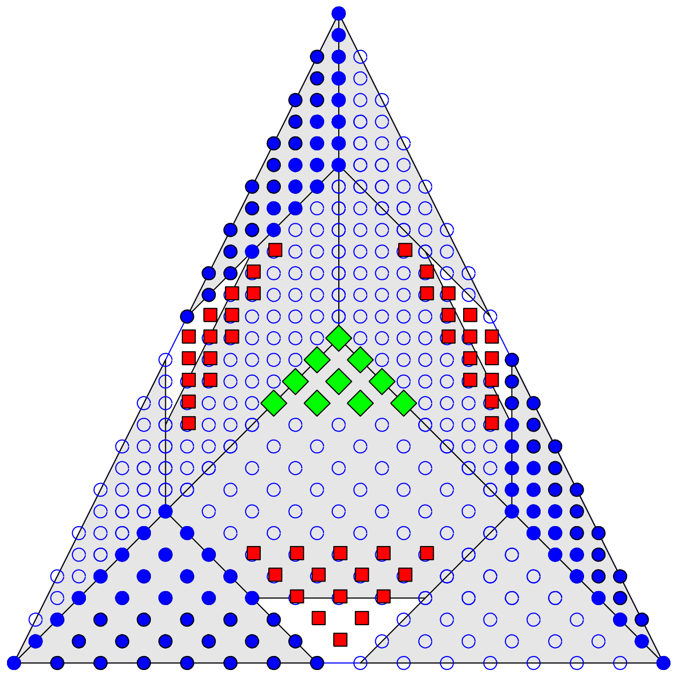

Fig. 5. The macro-element $\mathcal{S}_{15}^{5,7,12}(\triangle_{CT})$.

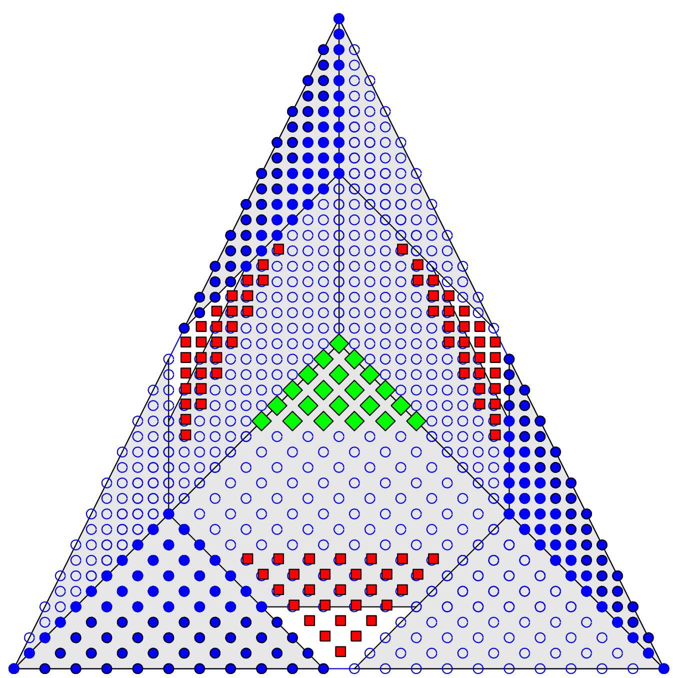

Fig. 6. The macro-element $\mathcal{S}_{21}^{7,10,17}(\triangle_{CT})$.

Example 4.4. *The stable local MDS of Theorem 3.1 for $\mathcal{S}_{21}^{7,10,17}(\triangle_{CT})$ is shown in Fig. 6. It contains 303 domain points. There are 66 points in each of the disks $D_{10}(v_i)$ (marked with dark circles), 28 points associated with each edge (marked with dark squares), and 21 at the incenter (marked with a dark diamond).*

Discussion: We have shaded the disks $D_{10}(v_i)$ and $D_{17}(v)$. The unmarked coefficients in the disks $D_{10}(v_i)$ are computed from the usual smoothness conditions. The remaining coefficients in $D_{17}(v)$ are then computed by the method described in the proof of Theorem 4.1. $\square$

We can now use the construction of Theorem 4.1 to create a stable local basis for $\mathcal{S}_{2m+1}(\triangle)$.

Theorem 4.5. *Let $\mathcal{M}$ be the following set of domain points:*

1) *for each vertex v of $\triangle$, choose a triangle T of $\triangle_{CT}$ attached to v and include $D^T_{3m+1}(v)$,*
2) *for each edge $e = \langle v_1, v_2\rangle$ of $\triangle$, let $T = \langle v, v_1, v_2\rangle$ be a triangle of $\triangle_{CT}$ containing the edge e. Then include the points $\{\xi^T_{j,3m+1,3m-j+2}, \ldots, \xi^T_{j,3m-j+2,3m+1}\}$ for $j = 1, \ldots, 2m+1$ and $i = 1, 2, 3$,*
3) *for each triangle $T := \langle v_1, v_2, v_3\rangle$ of $\triangle$ include $D^{T^{[1]}}_{2m-1}(w)$ where w is the barycenter of T and $T^{[1]} := \langle v, v_1, v_2\rangle$.*

Then $\mathcal{M}$ is a minimal determining set for $\mathcal{S}_{2m+1}(\triangle)$, and the corresponding dual basis $\{B_\xi\}_{\xi\in\mathcal{M}}$ forms a stable star-supported basis for $\mathcal{S}_{2m+1}(\triangle)$. Moreover,

$$\dim \mathcal{S}_{2m+1}(\triangle) = \binom{3m+3}{2} V + \binom{2m+2}{2} E + \binom{2m+1}{2} N. \tag{4.2}$$

§5. Macro-elements

The constructions of minimal determining sets for superspline spaces $\mathcal{S}_m(\triangle_v)$ on the Clough-Tocher split $\triangle_v$ of a single triangle T given in Theorems 3.1 and 4.1 can be regarded as defining macro-elements. In the finite-element literature, such macro-elements are typically defined in terms of nodal parameters, *i.e.*, derivatives. Here we have described them in terms of minimal determining sets of B-coefficients, but it is easy to translate to derivatives.

We give three examples. Given a triangle $T = \langle v_1, v_2, v_3\rangle$, let $\triangle_v$ be the corresponding Clough-Tocher cell with center v and boundary vertices $\{v_1, w_1, v_2, w_2, v_3, w_3\}$ in counterclockwise order. We denote the perpendicular cross-derivative across the edge $\langle v_i, v_{i+1}\rangle$ by D_i. For any $1 \le j$, let

$$p^i_{j\ell} := \frac{(j-\ell+1)v_i + \ell v_{i+1}}{j+1}, \qquad \text{for } \ell = 1, \ldots, j \text{ and } i = 1, 2, 3.$$

Example 5.1. *Any element in the superspline space* $\mathcal{S}_3^1(\triangle_{CT})$ *is uniquely defined by the following twelve data:*

1) $D_x^\alpha D_y^\beta s(v_i)$ *for* $0 \le \alpha+\beta \le 2$ *and* $i = 1,2,3$,
2) $D_i s(p_{11}^i)$ *for* $i = 1,2,3$.

Discussion: It is well-known from the Bernstein-Bézier theory that specifying the data in item 1) for an $s \in \mathcal{S}_3^1(\triangle_{CT})$ is equivalent to setting the B-coefficients of s corresponding to the domain points in the disks $D_2(v_i)$, $i = 1,2,3$. Having set these, then setting the derivatives in item 2) here is equivalent to setting the coefficients in item 2) of Theorem 4.1. □

Example 5.2. *Any element in the superspline space* $\mathcal{S}_7^{2,3,6}(\triangle_v)$ *of Theorem 3.1 is uniquely defined by the following set of 40 data (cf. Example 3.2 and Fig. 1):*

1) *the derivatives* $D_x^\alpha D_y^\beta s(v_i)$ *for* $0 \le \alpha+\beta \le 3$ *and* $i = 1,2,3$,
2) *the derivatives* $D_i^j s(p_{j1}^i), \ldots, D_i^j s(p_{jj}^i)$, *for* $j = 1,2$ *and* $i = 1,2,3$,
3) *the value* $s(w)$ *at the barycenter* w *of* $\triangle_v$.

Example 5.3. *Any element in the space* $\mathcal{S}_9^{3,4,7}(\triangle_v)$ *of Theorem 4.5 is uniquely defined by the following set of 66 data (cf. Example 4.2 and Fig. 4):*

1) *the derivatives* $D_x^\alpha D_y^\beta s(v_i)$ *for* $0 \le \alpha+\beta \le 4$ *and* $i = 1,2,3$,
2) *the derivatives* $D_i^j s(p_{i1}), \ldots, D_i^j s(p_{ij})$ *for* $j = 1,2,3$ *and* $i = 1,2,3$,
3) *the derivatives* $D_x^\alpha D_y^\beta s(v)$ *for* $0 \le \alpha+\beta \le 1$.

§6. Comparison with earlier Clough-Tocher macro-elements

Macro-elements based on Clough-Tocher splits have been proposed in several earlier papers. For $r = 2$, see [13]. For general r, see [7,8,12]. The following formulae (which can easily verified using the above Bernstein-Bézier techniques) can be found in [8,14]:

$$\begin{aligned} \dim \mathcal{S}_{6m+1}^{2m,3m,3m}(\triangle_{CT}) &= (51m^2+27m+6)/2, \\ \dim \mathcal{S}_{6m+3}^{2m+1,3m+1,3m+1}(\triangle_{CT}) &= (51m^2+69m+24)/2. \end{aligned} \tag{6.1}$$

The macro-elements constructed Sects. 3–4 have the following advantage over these macro-elements:

- they use a smaller number of degrees of freedom.

Table 1 shows a comparison of the macro-elements in (6.1) with our new macro elements for $1 \le r \le 10$. The column d gives the degrees, and the columns n and $\tilde{n}$ give the number of degrees of freedom, where n is the number otained from (3.1) and (4.1) while $\tilde{n}$ is from (6.1).

r	d	n	$\tilde{n}$
2	7	40	42
3	9	66	72
4	13	120	132
5	15	163	183
6	19	243	273
7	21	303	345
8	25	409	465
9	27	486	558
10	31	618	708

Tab. 1. Comparison of macro elements.

§7. Optimality of the macro-elements

In this section we explore to what extent the spaces chosen in (1.2) are optimal with respect to the degrees of the splines and the number of degrees of freedom of the corresponding macro-elements.

Fix the smoothness r. By Theorem 10.1 of [11], a necessary condition for constructing a macro-element on a Clough-Tocher cell is that we use splines with super-smoothness

$$\rho_i \geq \left\lceil \frac{3r-1}{2} \right\rceil = \begin{cases} 3m, & \text{if } r = 2m, \\ 3m+1, & \text{if } r = 2m+1 \end{cases} \tag{7.1}$$

at each vertex v_i of T.

This means that in order to construct a macro-element on the Clough-Tocher cell, we have to include the disks D_{ρ_i} in the minimal determining set. In order to insure that this data can be specified independently, we have to be sure that the disks do not overlap, and it follows that we need

$$d \geq 2\left\lceil \frac{3r-1}{2} \right\rceil + 1 = \begin{cases} 6m+1, & \text{if } r = 2m, \\ 6m+3, & \text{if } r = 2m+1. \end{cases} \tag{7.2}$$

We now examine what values can be chosen for the super-smoothness ρ at the center v of the cell $\triangle_v$. Let

$$\mathcal{S}_d^{r,\rho}(\triangle_v) := \{s \in \mathcal{S}_d^r(\triangle_v) : \ s \in C^\rho(v)\}.$$

By Lemma 3.2 of [5],

$$\dim \mathcal{S}_d^{r,\rho}(\triangle_v) = \binom{\rho+2}{2} + 6\left[\binom{d-r+1}{2} - \binom{\rho-r+1}{2}\right] + \sigma,$$

where

$$\sigma := \sum_{j=\rho-r+1}^{d-r+1} (r+j+1-je)_+$$

and e is the number of edges attached to the center vertex with different slopes. For stability of dimension, we need $\sigma = 0$. Since $e = 3$ for the Clough-Tocher cell, stability of dimension is guaranteed as soon as we enforce super-smoothness

$$\rho \geq \left\lceil \frac{3r-1}{2} \right\rceil = \begin{cases} 3m, & \text{if } r = 2m, \\ 3m+1, & \text{if } r = 2m+1 \end{cases} \tag{7.3}$$

at the barycentric center v of T.

Clearly, it is advantageous to choose larger values of ρ if possible, since this reduces the dimension of the corresponding super-spline space. However, we cannot choose ρ too large, since it can lead to incompatible information on certain of the rings $R_j(v_i)$. Suppose we choose ρ_i and d as given in equations (7.1)–(7.2), and that we enforce C^ρ continuity at v. Now consider the domain points on the ring $R_{\rho_i+1}(v_1)$ which lie inside the disk $D_\rho(v)$. It is easy to see that this is a set of $2(\rho_i+1-d+\rho)+1$ points. Now setting derivatives up to order C^r across the edges of $\triangle_v$ implies that $r-d+\rho+1$ of these points are determined at each end of this arc. This leaves $2(\rho_i-r)+1$ points in the center of this arc which are undetermined. But these points must satisfy $\rho_i+1-d+\rho$ continuity conditions across edge $\langle v_1, v\rangle$, which leads to an incompatibility unless $2(\rho_i-r)+1 \geq \rho_i+1-d+\rho$, or equivalently

$$d + \rho_i - 2r \geq \rho.$$

Then from (7.2) and (7.1) we conclude that we need

$$\rho \leq \begin{cases} 5m+1, & r = 2m, \\ 5m+2, & r = 2m+1. \end{cases}$$

which is exactly what we have used in (1.2).

§8. Remarks

Remark 8.1. Clough-Tocher splits were introduced in [3].

Remark 8.2. It was shown in Sect. 10 of [10] that if a space of splines $\mathcal{S}$ of degree d contains $\mathcal{P}_d$ and has a stable local basis, then it provides optimal order approximations of smooth functions. In particular, for every $0 \leq k \leq d$, there exists a quasi-interpolation operator Q_k such that for every function $f \in W_p^{k+1}(\Omega)$,

$$\|D_x^\alpha D_y^\beta (f - Q_k f)\|_p \leq K |\triangle|^{k+1-\alpha-\beta} |f|_{k+1,p} \tag{8.1}$$

for $0 \leq \alpha+\beta \leq k$, where $|\triangle|$ is the mesh size of $\triangle$ (ie., the diameter of the largest triangle), and $|f|_{k+1,p}$ is the usual Sobolev semi-norm. If Ω is convex, then the constant K depends only on d, p, k, and on the smallest angle $\theta_\triangle$ in $\triangle$. If Ω is nonconvex, it also depends on the Lipschitz constant $L_{\partial\Omega}$ associated with the boundary of Ω. In view of our construction of stable local bases for the spaces $\mathcal{S}_{2m}(\triangle_{CT})$ and $\mathcal{S}_{2m+1}(\triangle_{CT})$ for general $m \geq 0$, we can conclude that all of these spaces have full approximation power.

Remark 8.3. For $d < 3r+2$, it is known [6] that the spaces $\mathcal{S}_d^r(\triangle)$ do not possess optimal order approximation order for arbitrary triangulations. This means that neither they (nor any subspace $\mathcal{S}$ containing $\mathcal{P}_d$) has a stable local basis.

Remark 8.4. Macro-elements and stable local bases can be constructed for several other refinement methods. In [11] we do this for the well-known Powell-Sabin split.

Remark 8.5. We would like to thank Peter Alfeld for writing a beautiful JAVA program which computes dimensions of spline spaces (in exact arithmetic) and which can also be used for verifying minimal determining sets. We have made extensive use of this software in preparing this paper and checking our results. It can be accessed at `http://www.math.utah/~alfeld/MDS/index.html`.

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