Bounds on the Dimensions of
Trivariate Spline Spaces

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Abstract. We derive upper and lower bounds on the dimensions of trivariate spline spaces defined on tetrahedral partitions. The results hold for general partitions, and for all degrees of smoothness r and polynomial degrees d.

§1. Introduction

Suppose $\triangle$ is a finite collection of nondegenerate tetrahedra such that if any two tetrahedra in $\triangle$ intersect, then their intersection is exactly one vertex, one edge, or one face. Then we call $\triangle$ a **tetrahedral partition of the set** Ω, where Ω is the union of all tetrahedra in $\triangle$. This definition of a tetrahedral partition allows Ω to have holes and cavities as well as pinch points where just two tetrahedra meet in a single point.

Given $0 \le r \le d$, let $\mathcal{P}_d$ be the $\binom{d+3}{3}$-dimensional space of trivariate polynomials of degree d. In this paper we study the space

$$\mathcal{S}_d^r(\triangle) = \{s \in C^r(\Omega) : s|_T \in \mathcal{P}_d, \text{ for all } T \in \triangle\}$$

of **trivariate splines of smoothness** r **and degree** d associated with $\triangle$. Here the condition $s \in C^r(\Omega)$ means that s belongs to $C^r(v)$ for every point $v \in \Omega$, i.e., if v is contained in the tetrahedra $T_1, \dots, T_m$, then all of the polynomials $s|_{T_1}, \dots, s|_{T_m}$ have common derivatives up to order r at v.

Trivariate splines are important tools in approximation theory and numerical analysis, and have attracted considerable interest in the past few years, see [3] and references therein.

Clearly $\mathcal{S}_d^r(\triangle)$ is a finite dimensional linear space. However, finding explicit formulae for its dimension for general r, d, and $\triangle$, is an extremely difficult problem due to the fact that the dimension depends not only on the way in which the

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tetrahedra are connected to each other, but also on the precise location of the vertices. In general, an arbitrarily small change of the location of the vertices can change the dimension of $\mathcal{S}_d^r(\triangle)$.

The dimension of $\mathcal{S}_d^r(\triangle)$ can be trivially determined when $r = 0$ or when $r = d$, see Remark 1. Our aim in this paper is to establish upper and lower bounds on the dimension of $\mathcal{S}_d^r(\triangle)$ for all choices of $0 < r < d$, and for arbitrary tetrahedral partitions. Our bounds will depend on d and r, and also on the nature of $\triangle$, but not on the precise locations of the vertices.

The paper is organized as follows. In Section 2 we describe how completely general tetrahedral partitions can be built inductively using 28 specific types of assembly steps. In Section 3 we show that lower and upper bounds on the dimension of $\mathcal{S}_d^r(\triangle)$ for arbitrary $0 < r < d$ and arbitrary $\triangle$ can be given in terms of lower and upper bounds ℓ_i, u_i on how much the dimension of a spline space changes when we add just one tetrahedron using assembly step i to an existing tetrahedral partition. Section 4 is devoted to a review of basic Bernstein–Bézier theory, while in the following section we present several formulae for counting certain sets of domain points. In Section 6 we derive explicit formulae for suitable ℓ_i and u_i in terms of r and d. In Section 7 we show that our lower and upper bounds take a strikingly simple form in the special case where the partition $\triangle$ is shellable. Several numerical examples illustrating our bounds are given in Section 8. We conclude the paper with remarks and references.

§2. Tetrahedral Partitions

Every tetrahedral partition $\triangle$ consisting of N tetrahedra can be assembled in N steps, starting with a single tetrahedron, and adding one additional tetrahedron in each subsequent step. Generally, there will be many different ways to assemble a given partition, and we will see later that the order in which the tetrahedra are added will affect our bounds on the dimension of the associated spline spaces.

For our purposes, we need to identify and classify the different ways in which a tetrahedron can be added to a partially completed partition in order to build a partition $\triangle$. There are twenty-eight types of assembly steps, which we number as in Tab. 1. For example, simply adding one tetrahedron that doesn't touch any existing tetrahedron at all is classified as type 0. Adding a tetrahedron that touches the existing partition at one vertex only is classified as type 1. A step of type 27 corresponds to adding a tetrahedron that touches the existing partition along exactly one face and also at the opposite vertex. The last column in the table is designed to help visualize these steps. In particular, assuming the tetrahedron being added is $T := \langle v_1, v_2, v_3, v_4 \rangle$, this column lists the vertices, edges, and faces of T where T touches the existing partition. Here e_{ij} denotes the edge $\langle v_i, v_j \rangle$, and f_{ijk} denotes the face $\langle v_i, v_j, v_k \rangle$.

These twenty-eight types of assembly steps allow the step-wise construction of completely general tetrahedral partitions. However, most partitions likely to be used in practice, including those with holes and cavities, can be built with only a few of these steps, see Remark 4.

Type	Nature of Touching Facets	List of Touching Facets
0	Isolated tetrahedron	
1	one vertex	v_1
2	two vertices	v_1, v_2,
3	three vertices	v_1, v_2, v_3
4	four vertices	v_1, v_2, v_3, v_4
5	one edge	e_{12}
6	two opposite edges	e_{12}, e_{34}
7	two edges sharing a vertex	e_{12}, e_{23}
8	three edges forming a path	e_{12}, e_{23}, e_{34}
9	three edges sharing a vertex	e_{12}, e_{13}, e_{14}
10	three edges on one face	e_{12}, e_{23}, e_{31}
11	four edges forming a loop	e_{12}, e_{23}, e_{34}, e_{41}
12	four edges not forming a loop	e_{12}, e_{23}, e_{31}, e_{14}
13	five edges	e_{12}, e_{13}, e_{14}, e_{23}, e_{24}
14	six edges	e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}
15	one edge and one vertex	e_{12}, v_3
16	one edge and two vertices	e_{12}, v_3, v_4
17	two edges and one vertex	e_{12}, e_{23}, v_4
18	three edges and one vertex	e_{12}, e_{23}, e_{31}, v_4
19	one face and one edge	f_{123}, e_{14}
20	one face and two edges	f_{123}, e_{14}, e_{24}
21	one face and three edges	f_{123}, e_{14}, e_{24}, e_{34}
22	two faces and one edge	f_{123}, f_{124}, e_{34}
23	one face	f_{123}
24	two faces	f_{123}, f_{124}
25	three faces	f_{123}, f_{124}, f_{134}
26	four faces	f_{123}, f_{124}, f_{134}, f_{234}
27	one face and the opposite vertex	f_{123}, v_4

Tab. 1. Assembly steps for tetrahedral partitions.

§3. The Key Inequalities

It is clear that when $\triangle$ consists of a single tetrahedron, the dimension of $\mathcal{S}_d^r(\triangle)$ is equal to the dimension of $\mathcal{P}_d$ which is $\binom{d+3}{3}$. To get bounds on the dimension of $\mathcal{S}_d^r(\triangle)$ on an arbitrary tetrahedral partition $\triangle$, we proceed by induction. Suppose $\triangle$ can be obtained from a tetrahedral partition $\tilde{\triangle}$ by adding one tetrahedron T. To get bounds on $\mathcal{S}_d^r(\triangle)$, we need to account for how the dimension changes as we add T to $\tilde{\triangle}$. Clearly, this will depend on how T joins $\tilde{\triangle}$, i.e., to which of the 28 classes identified in the previous section it belongs. For each $0 \leq i \leq 27$, let ℓ_i and u_i be such that

$$\ell_i \leq \dim \mathcal{S}_d^r(\triangle) - \dim \mathcal{S}_d^r(\tilde{\triangle}) \leq u_i,$$

whenever T is of type i. We now establish the key inequalities.

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Theorem 3.1. *Fix $0 < r < d$, and suppose $\triangle$ is a tetrahedral partition that can be built inductively using τ_i assembly steps involving tetrahedra of type i for $i = 0, \ldots, 27$. Then*

$$L(r,d) \leq \dim \mathcal{S}_d^r(\triangle) \leq U(r,d), \tag{3.1}$$

where

$$L(r,d) := \max\left[\binom{d+3}{3}, \sum_{i=0}^{27} \tau_i \ell_i\right], \qquad U(r,d) := \sum_{i=0}^{27} \tau_i u_i.$$

Proof: Let $T_1, \ldots, T_N$ be a numbering of the tetrahedra in the order that they are added in building $\triangle$. The spline space $\mathcal{S}_d^r(\triangle)$ restricted to the single tetrahedron T_1 has dimension $\ell_0 = u_0 = \binom{d+3}{3}$. Now each time we add a tetrahedron of type i, the dimension changes by an amount δ_i satisfying $\ell_i \leq \delta_i \leq u_i$. Summing gives the formulae for $L(r,d)$ and $U(r,d)$, where for $L(r,d)$ we have taken the maximum with $\binom{d+3}{3}$ since $\mathcal{P}_d \subseteq \mathcal{S}_d^r(\triangle)$ for all $0 \leq r \leq d$. $\square$

Formulae for the ℓ_i and u_i will be derived in Sect. 6 below. We refer to the τ_i as the assembly parameters of $\triangle$. In general they are not uniquely determined. A given tetrahedral partition can usually be built in many different ways, leading to different bounds. We give an example to illustrate this in Sect. 8, which also contains several other numerical examples illustrating Theorem 3.1.

§4. Bernstein–Bézier Methods

Our analysis of the dimension of $\mathcal{S}_d^r(\triangle)$ is based on well-known Bernstein–Bézier techniques, see [3] and references therein. For convenience, we review some of the basic ideas and notation. Suppose $T := \langle v_1, v_2, v_3, v_4 \rangle$ is a tetrahedron with vertices v_1, v_2, v_3, v_4. The key fact is that every polynomial $p \in \mathcal{P}_d$ has a unique Bernstein–Bézier representation of the form

$$p := \sum_{i+j+k+l=d} c_{ijkl} B_{ijkl},$$

where B_{ijkl} are the Bernstein basis polynomials of degree d associated with T. It is standard practice to associate each coefficient c_{ijkl} with a corresponding domain point

$$\xi_{ijkl} := \frac{iv_1 + jv_2 + kv_3 + lv_4}{d},$$

for $i + j + k + l = d$. Given $r \geq 0$, we often work with the following subsets of domain points:

- ball of radius r around v_1: $B_r(v_1) := \{\xi_{ijkl} : i \geq d - r\}$,
- shell of radius r around v_1: $R_r(v_1) := \{\xi_{ijkl} : i = d - r\}$,
- tube of radius r around $e := \langle v_i, v_j \rangle$: $t_r(e) := \{\xi_{ijkl} : i + j \geq d - r\}$,
- r-slab containing the face $F := \langle v_1, v_2, v_3 \rangle$: $G_r(F) := \{\xi_{ijkl} : l \leq r\}$.

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Analogous sets can be defined in the same way for the other vertices, edges, and faces.

One of the reasons that the Bernstein–Bézier representation is so useful is that it provides a simple way to describe when two polynomials defined on tetrahedra $T := \langle v_1, v_2, v_3, v_4 \rangle$ and $\widetilde{T} := \langle v_5, v_2, v_3, v_4 \rangle$ sharing the face $F := \langle v_2, v_3, v_4 \rangle$ join with C^r smoothness. The condition is that

$$\tilde{c}_{mijk} = \sum_{\nu+\mu+\kappa+\delta=m} c_{\nu,i+\mu,j+\kappa,k+\delta} B_{\nu\mu\kappa\delta}^m(v_5), \qquad \text{all } i+j+k = d-m, \quad (4.1)$$

for $m = 0, \ldots, r$, where $\{c_{ijkl}\}_{i+j+k+l=d}$ and $\{\tilde{c}_{ijkl}\}_{i+j+k+l=d}$ are the B-coefficients of p and $\tilde{p}$ relative to T and $\widetilde{T}$, respectively.

We refer to the domain point $\tilde{\xi}_{mijk}$ corresponding to the coefficient $\tilde{c}_{mijk}$ on the left in (4.1) as the **tip of the smoothness condition**. It is clear from (4.1) that the conditions for two polynomials to join with C^r smoothness across a common face involve only coefficients corresponding to domain points in the r–slabs containing F in both T and $\widetilde{T}$.

§5. Formulae for Intersections of Balls, Tubes, and Slabs

In this section we provide formulae for the sizes of various subsets of domain points. They are used in the following section to derive formulae for the expressions ℓ_i and u_i appearing in Theorem 3.1. Given $0 \le r < d$, let

- $n_r :=$ the number of points in $B_r(v)$, where v is a vertex of T,
- $m_{2b}(r, d) :=$ the number of points in $B_r(v_1) \cap B_r(v_2)$, where v_1, v_2 are vertices of T,
- $m_{3b}(r, d) :=$ the number of points in $B_r(v_1) \cap B_r(v_2) \cap t_r(v_3)$, where v_1, v_2, v_3 are vertices of T,
- $m_{4b}(r, d) :=$ the number of points in $B_r(v_1) \cap B_r(v_2) \cap B_r(v_3) \cap B_r(v_4)$, where v_1, v_2, v_3, v_4 are the vertices of T,
- $m_{1t}(r, d) :=$ the number of points in $t_r(e)$, where e is an edge of T,
- $m_{2t}(r, d) :=$ the number of points in $t_r(e_1) \cap t_r(e_2)$, where e_1, e_2 are edges of T sharing a vertex v,
- $m_{to}(r, d) :=$ the number of points in $t_r(e_1) \cap t_r(e_2)$, where e_1, e_2 are edges of T not sharing any vertex,
- $m_{3t}(r, d) :=$ the number of points in $t_r(e_1) \cap t_r(e_2) \cap t_r(e_3)$, where e_1, e_2, e_3 are edges of T sharing a vertex v of T,
- $m_{3tf}(r, d) :=$ the number of points in $t_r(e_1) \cap t_r(e_2) \cap t_r(e_3)$, where e_1, e_2, e_3 are edges of T lying in one face,
- $m_{3tp}(r, d) :=$ the number of points in $t_r(e_1) \cap t_r(e_2) \cap t_r(e_3)$, where e_1, e_2, e_3 are edges of T forming a path, i.e., not all lying on one face and not all sharing a single vertex,

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- $m_{4tl}(r, d) :=$ the number of points in $t_r(e_1) \cap t_r(e_2) \cap t_r(e_3) \cap t_r(e_4)$, where $e_1, \ldots, e_4$ are edges of T forming a loop, i.e., such that no three of the edges share a single vertex,
- $m_{tb}(r, d) :=$ the number of points in $t_r(e) \cap B_r(v)$, where e is an edge of T and v is a vertex not contained in e,
- $m_{t2b}(r, d) :=$ the number of points in $t_r(e) \cap B_r(v) \cap B_r(w)$, where e is an edge of T and v, w are vertices not contained in e,
- $m_{2tb}(r, d) :=$ the number of points in $t_r(e_1) \cap t_r(e_2) \cap B_r(v)$, where e_1, e_2 are edges of T and v is a vertex not contained in either edge,
- $m_{ts}(r, d) :=$ the number of points in $t_r(e) \cap G_r(F)$, where e is an edge of T not lying in the face F.
- $m_{2t2b}(r, d) :=$ the number of points in $\big(B_r(v_1) \cap B_r(v_2)\big) \setminus t_r(e)$, where e is the edge with vertices v_3 and v_4,
- $m_s(r, d) :=$ the number of points in the slab $G_r(F)$ associated with a face F of T.

We now derive formulae for all of these expressions.

Lemma 5.1. *For all $0 \leq r < d$,*

$$n_r := \binom{r + 3}{3},$$

$$m_{2b}(r, d) := \binom{2r - d + 3}{3},$$

$$m_{3b}(r, d) := \binom{3r - 2d + 3}{3},$$

$$m_{4b}(r, d) := \binom{4r - 3d + 3}{3},$$

$$m_{1t}(r, d) := \binom{r + 3}{3} + \sum_{\nu=r+1}^{d} \binom{r + 2}{2},$$

$$m_{2t}(r, d) := \binom{r + 3}{3} + \sum_{\nu=r+1}^{d} \binom{2r + 2 - \nu}{2},$$

$$m_{3t}(r, d) := \binom{r + 3}{3} + \sum_{\nu=r+1}^{d} \binom{3r + 2 - 2\nu}{2},$$

$$m_{to}(r, d) := \begin{cases} \sum_{\nu=d-r}^{r} (\nu + 1)(d + 1 - \nu), & \text{if } d \leq 2r, \\ 0, & \text{otherwise,} \end{cases}$$

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$$m_{3tf}(r, d) := \sum_{\nu=0}^{r} \left[\binom{3r - 2\nu - d + 2}{2} - 3\binom{2r - \nu - d + 1}{2} \right],$$

$$m_{3tp}(r, d) := \sum_{\nu=d-r}^{r} \sum_{\mu=0}^{\nu} \left(d + 1 - \max(\nu, d - r + \mu) \right),$$

$$m_{4tl}(r, d) := \sum_{\nu=d-r}^{r} \sum_{\mu=0}^{\nu} \left(\min(d, r + \mu) + 1 - \max(\nu, d - r + \mu) \right),$$

$$m_{tb}(r, d) := \sum_{\nu=0}^{2r-d} (\nu + 1)(r - \nu + 1),$$

$$m_{t2b}(r, d) := \sum_{\nu=0}^{3r-2d} (\nu + 1)(2r - d - \nu + 1),$$

$$m_{2tb}(r, d) := \sum_{\nu=0}^{2r-d} \left[(\nu + 1)^2 - \binom{r - d + \nu + 1}{2} \right],$$

$$m_{ts}(r, d) := \sum_{\nu=0}^{r} \binom{\min(r, d - \nu) + 2}{2},$$

$$m_{2t2b}(r, d) := \sum_{i=d-r}^{r} \sum_{j=\max(d-r, r+1-i)}^{d-i} (d - i - j + 1),$$

$$m_s(r, d) := \binom{d + 3}{3} - \binom{d + 2 - r}{3}.$$

Proof: The formula for $n_r, m_{2b}, m_{3b}, m_{4b}$ and m_s can be obtained by inspection. We now consider the other cases individually.

m_{1t}: To get the formula for m_{1t}, note that the tube contains the ball of radius r around one of the vertices v of e. In addition, for each $\nu = r + 1, \ldots, d$, it contains $\binom{r+2}{2}$ points on the shell $R_\nu(v)$. Summing over the shells gives m_{1t}.

m_{2t}: To get the formula for m_{2t}, note that the ball of radius r around v is in both tubes, which accounts for the term $\binom{r+3}{3}$. To count the remaining points in the intersection of the two tubes, we consider the shells $R_\nu(v)$ for $\nu = r + 1, \ldots, d$. It is easy to check that on the shell $R_\nu(v)$, there are $\binom{2r+2-\nu}{2}$ points in the intersection. Summing over the shells gives the second term in the formula for m_{2t}.

m_{3t}: For the case of three intersecting tubes sharing the vertex v, we note that the number of points in the intersection lying on the shell $R_\nu(v)$ is $\binom{\nu+2-3(\nu-r)}{2}$. Summing over the shells gives the formula for m_{3t}.

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m_{to}: Without loss of generality we may assume the first tube is along the edge $e_1 := \langle v_1, v_2 \rangle$ while the second is along the edge $e_2 := \langle v_3, v_4 \rangle$. Then ξ_{ijkl} lies in both tubes if and only if $i + j \geq d - r$ and $k + l \geq d - r$ which implies that $i + j \leq d - (d - r) = r$. Thus, for a point ξ_{ijkl} to be in the intersection we need $d - r \leq i + j \leq r$. It follows that the intersection of the two tubes is empty if $d > 2r$. Letting $\nu = i + j$, there are $\nu + 1$ ways to choose i and j, and for each such choice, there are $d + 1 - \nu$ ways to choose k and l with $k + l = d - \nu$. The count follows.

m_{3tf}: Suppose the face is $F := \langle v_1, v_2, v_3 \rangle$. The expression in brackets is just the number of points in the intersection of the shell $R_{d-\nu}(v_4)$ and the tubes around the three edges of F. The result follows by summing over the shells.

m_{3tp}: For each $0 \leq i, j \leq 4$, let t_{ij} be the tube along edge $e_{ij} := \langle v_i, v_j \rangle$. We suppose the edges forming the path are e_{12}, e_{23}, e_{34}, and consider the three tubes t_{12}, t_{23}, and t_{34}. We need to count the number of 4-tuples (i, j, k, l) of nonnegative integers that satisfy

$$i + j + k + l = d, \quad i + j \geq d - r, \quad j + k \geq d - r, \quad \text{and} \quad k + l \geq d - r.$$

These inequalities imply that $i + j \leq r$ and $k + l \leq r$. It is convenient to use the substitution $\mu = i$, $\nu = i + j$, and $\kappa = i + j + k$. Then, according to the above inequalities, admissible values of ν range from $d - r$ to r, and admissible values of μ range from 0 to ν. Since $j + k = \kappa - \mu \geq d - r$, we need $\kappa \geq d - r + \mu$. Given μ and ν, there are thus $d + 1 - \max\{\nu, d - r + \mu\}$ choices of κ ranging from $\max\{\nu, d - r + \mu\}$ to d. Counting the total number of possibilities gives the formula for m_{3tp}.

m_{4tl}: The proof is similar to the proof for m_{3tp} except that now we also have to satisfy $\kappa - \mu = j + k \leq r$.

m_{tb}: For each $d - r \leq \nu \leq r$, points in the intersection lie on $\nu - d + r + 1$ lines parallel to e, each of which contains $d + 1 - \nu$ points. Thus, the total number is $\sum_{\nu=d-r}^{r} (\nu - d + r + 1)(d + 1 - \nu)$. A change of summation index gives the stated formula.

m_{t2b}: For each $2d - 2r \leq \nu \leq r$, points in the intersection lie on $\nu - 2d + 2r + 1$ lines parallel to e, each of which contains $d + 1 - \nu$ points. Thus, the total number is $\sum_{\nu=2d-2r}^{r} (\nu - 2d + 2r + 1)(d + 1 - \nu)$. A change of summation index gives the stated formula.

m_{2tb}: Suppose the two edges are e_1 and e_2, and that the vertex not on the edges is v_4. For each $d - r \leq \ell \leq r$, the number of points in $t_r(e_1) \cap t_r(e_2) \cap R_\ell(v_4)$ is $(r - d + \ell + 1)^2 - \binom{2r - 2d + \ell + 1}{2}$. Summing over ℓ and changing the index gives the stated formula.

m_{ts}: Let v be the vertex opposite the slab. For each $0 \leq \nu \leq r$, the number of points in the intersection and lying on the shell $R_{d-\nu}(v)$ is $\binom{\min(r, d-\nu)+2}{2}$, and the formula follows.

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m_{2t2b}: A domain point ξ_{ijkl} lies in the set $B_r(v_1) \cap B_r(v_2)$ if and only if $i \geq d - r$ and $j \geq d - r$. It lies outside the tube $t_r(e)$ if and only if $k + l < d - r$. We compute the number of 4-tuples (i, j, k, l) satisfying all three inequalities. The first two inequalities imply that $i, j \leq r$. The third inequality implies that $j \geq r + 1 - i$. Given i and j there are $d + 1 - i - j$ possible choices of k and l. Summing gives the formula. $\square$

§6. The Change in Dimension When Adding One Tetrahedron

Suppose $\triangle$ is a tetrahedral partition that can be obtained from a tetrahedral partition $\tilde{\triangle}$ by adding one tetrahedron T. In this section we give upper and lower bounds u_i and ℓ_i on the change in dimension

$$\delta_i := \dim \mathcal{S}_d^r(\triangle) - \dim \mathcal{S}_d^r(\tilde{\triangle})$$

when T is of type i, $0 \leq i \leq 27$, according to the classification scheme of Sect. 3. Note that δ_i can be negative in some cases.

Suppose $s \in \mathcal{S}_d^r(\triangle)$ and that we already know the coefficients of s corresponding to domain points in $\tilde{\triangle}$. If T touches $\tilde{\triangle}$ at a vertex v, then the C^r smoothness at v imposes n_r conditions on the coefficients of s associated with domain points in $B_r(v)$. Similarly, if T touches $\tilde{\triangle}$ along an edge e, then the C^r smoothness along e imposes m_{1t} conditions on the coefficients in the tube $t_r(e)$. If T touches $\tilde{\triangle}$ along a face F, then the C^r smoothness across F imposes m_s conditions on the coefficients in the slab $G_r(F)$. Recall that $n_d = \binom{d+3}{3}$ is the number of coefficients of $s|_T$.

To get a *lower bound* ℓ_i for δ_i, we can take $n_d - n_{\mathrm{cond}}$, where n_{cond} is the number of conditions on $s|_T$ imposed by the fact that T joins with $\tilde{\triangle}$ with C^r smoothness. However, some of these conditions may be redundant in the sense that one or more of them are automatically satisfied whenever the others are satisfied. Removing as many redundant conditions as possible leads to a better lower bound. There are two situations where we know that conditions on a coefficient c_ξ are redundant:

1) $\xi \in B_r(v)$, where v is contained in more than one of the edges or faces where T touches $\tilde{\triangle}$. There is one smoothness condition associated with each such edge or face, but we need only count one of them as the rest are redundant.

2) $\xi \in t_r(e)$, where e is contained in more than one of the faces where T touches $\tilde{\triangle}$. There is one smoothness condition associated with each such face, but we need only count one of them as the rest are redundant.

Note that ℓ_i computed in this way can be negative as well as nonnegative.

The change in dimension δ_i is equal to the number of coefficients of $s|_T$ that can be set to arbitrary values while maintaining all smoothness conditions. Such coefficients cannot correspond to domain points which are tips of any smoothness condition. We call such points **constrained points**. Thus, as an *upper bound* we can take $u_i = n_d - n_{\mathrm{tips}}$, where n_{tips} is the number of points in T that are constrained. While counting unconstrained points seems like a straightforward process, it is

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complicated in some cases due to the fact that balls, tubes, and slabs can intersect in various ways depending on r, d and the class to which T belongs. Note that u_i computed in this way is nonnegative for all $0 \leq i \leq 27$.

In terms of the notation of the previous section, the number of domain points in a ball of radius r around a vertex is n_r. For ease of notation we write $m_{1t} := m_{1t}(r, d)$, with a similar short-hand notation for all the other quantities defined in the previous section. For $d > 2r$, we define $\tilde{m}_{1t} := m_{1t}(r, d-r-1)$, $\tilde{m}_{2t} := m_{2t}(r, d-r-1)$, $\tilde{m}_{3t} := m_{3t}(r, d-r-1)$. Let $\hat{m}_{1t} := m_{1t}(2r-d, r)$, $\hat{m}_{2t} := m_{2t}(2r-d, r)$, and $\hat{m}_{3tf} := m_{3tf}(2r-d, r)$ for $d \leq 2r$, and define them to be zero when $d > 2r$. Given an edge $e := \langle u, v \rangle$ of a tetrahedron, we define a 1-partial tube around e to be $t_r(e) \setminus B_r(u)$. Similarly, we define a 2-partial tube to be $t_r(e) \setminus (B_r(u) \cup B_r(v))$.

Lemma 6.1. *The expressions given in Tab. 2 provide upper and lower bounds for δ_i for all $0 \leq i \leq 27$.*

Proof: The cases 0, 1, 5, 23, and 27 are obvious.

2: (T touches $\tilde{\triangle}$ at two vertices v_1, v_2). There are n_r smoothness conditions associated with each of the balls $B_r(v_i)$, $i = 1, 2$, and the lower bound follows. The number of constrained points in T is $2n_r - m_{2b}$, since points in the intersection of two balls should be counted only once. This gives the upper bound.

3: (T touches $\tilde{\triangle}$ at three vertices v_1, v_2, v_3). There are n_r smoothness conditions associated with each of the balls $B_r(v_i)$, $i = 1, 2, 3$, and the lower bound follows. The number of points in the union of the three balls is $3n_r - 3m_{2b} + m_{3b}$, and the upper bound follows.

4: (T touches $\tilde{\triangle}$ at four vertices v_1, v_2, v_3, v_4). There are n_r smoothness conditions associated with each of the balls $B_r(v_i)$, $i = 1, \ldots, 4$, and the lower bound follows. The number of points in the union of the four balls is $4n_r - 6m_{2b} + 4m_{3b} + m_{4b}$, and the upper bound follows.

6: (T touches $\tilde{\triangle}$ along two edges e_1, e_2 that do not share a vertex). There are m_{1t} smoothness conditions associated with each of these edges, and the lower bound follows. The number of points in $t_r(e_1) \cup t_r(e_2)$ is $2m_{1t} - m_{to}$, where the term m_{to} takes account of the fact that the tubes can intersect when $d \leq 2r$. This gives the upper bound.

7: (T touches $\tilde{\triangle}$ along two edges e_1 and e_2 that share a vertex v). There are m_{1t} smoothness conditions associated with each of these edges. However, since both tubes contain $B_r(v)$, n_r of these conditions are redundant, and the lower bound follows. The number of points in $t_r(e_1) \cup t_r(e_2)$ is $2m_{1t} - m_{2t}$, and the upper bound follows.

8: (T touches $\tilde{\triangle}$ along three edges e_1, e_2, e_3 forming a path). There are m_{1t} smoothness conditions associated with each of these edges. However, since the first two tubes $t_r(e)$ both contain $B_r(v_1)$, n_r of the conditions with tips in $B_r(v_1)$ are redundant. Similarly, n_r of the conditions with tips in $B_r(v_2)$ are redundant, and the

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i	ℓ_i	u_i
0	n_d	n_d
1	$n_d - n_r$	$n_d - n_r$
2	$n_d - 2n_r$	$n_d - 2n_r + m_{2b}$
3	$n_d - 3n_r$	$n_d - 3n_r + 3m_{2b} - m_{3b}$
4	$n_d - 4n_r$	$n_d - 4n_r + 6m_{2b} - 4m_{3b} + m_{4b}$
5	$n_d - m_{1t}$	$n_d - m_{1t}$
6	$n_d - 2m_{1t}$	$n_d - 2m_{1t} + m_{to}$
7	$n_d - 2m_{1t} + n_r$	$n_d - 2m_{1t} + m_{2t}$
8	$n_d - 3m_{1t} + 2n_r$	$n_d - 3m_{1t} + 2m_{2t} + m_{to} - m_{3tp}$
9	$n_d - 3m_{1t} + 2n_r$	$n_d - 3m_{1t} + 3m_{2t} - m_{3t}$
10	$n_d - 3m_{1t} + 3n_r - m_{3b}$	$n_d - 3m_{1t} + 3m_{2t} - m_{3tf}$
11	$n_d - 4m_{1t} + 4n_r - m_{4b}$	$n_d - 4m_{1t} + 4m_{2t} + 2m_{to} - 4m_{3tp} + m_{4tl}$
12	$n_d - 4m_{1t} + 4n_r - m_{3b}$	$\begin{cases} 0, & \text{if } d \leq 2r \\ n_d - 4m_{1t} + 5m_{2t} - m_{3t} - m_{3tf}, & \text{if } d > 2r \end{cases}$
13	$n_d - 5m_{1t} + 6n_r - 2m_{3b},$	$\begin{cases} 0, & \text{if } d \leq 2r \\ n_d - 5m_{1t} + 8m_{2t} - 2m_{3tf} - 2m_{3t}, & \text{if } d > 2r \end{cases}$
14	$n_d - 6m_{1t} + 8n_r - 4m_{3b} + m_{4b}$	$\begin{cases} 0, & \text{if } d \leq 2r \\ n_d - 6m_{1t} + 12m_{2t} - 4m_{3t} - 4m_{3tf}, & \text{if } d > 2r \end{cases}$
15	$n_d - m_{1t} - n_r$	$n_d - m_{1t} - n_r + m_{tb}$
16	$n_d - m_{1t} - 2n_r$	$n_d - m_{1t} - 2n_r + 2m_{tb} + m_{2b} - m_{t2b}$
17	$n_d - 2m_{1t}$	$n_d - 2m_{1t} - n_r + m_{2t} + 2m_{tb} - m_{2tb}$
18	$n_d - 3m_{1t} + 2n_r - m_{3b}$	$n_d - n_r - 3(m_{1t} - \hat{m}_{1t} - m_{2t} + \hat{m}_{2t})$ $-(m_{3tf} - \hat{m}_{3tf})$
19	$n_d - m_s - m_{1t} + n_r$	$n_d - m_s - m_{1t} + m_{ts}$
20	$n_d - m_s - 2m_{1t} + 3n_r - m_{3b}$	$\begin{cases} 0, & \text{if } d \leq 2r \\ n_d - m_s - 2\tilde{m}_{1t} + \tilde{m}_{2t}, & \text{if } d > 2r \end{cases}$
21	$n_d - m_s - 3m_{1t} + 5n_r - 3m_{3b} + m_{4b}$	$\begin{cases} 0, & \text{if } d \leq 2r \\ n_d - m_s - 3\tilde{m}_{1t} + 3\tilde{m}_{2t} - \tilde{m}_{3t}, & \text{if } d > 2r \end{cases}$
22	$n_d - 2m_s + 2n_r - m_{2b} + m_{2t2b}$	$\begin{cases} 0, & \text{if } d \leq 3r \\ n_{d-2r-2} - m_{1t} + 2m_{ts}, & \text{if } d > 3r \end{cases}$
23	$n_d - m_s$	n_{d-r-1}
24	$n_d - 2m_s + m_{1t}$	n_{d-2r-2}
25	$n_d - 3m_s + 3m_{1t} - m_{3t}$	n_{d-3r-3}
26	$\begin{cases} -4(m_s - 3m_{1t} + 3m_{2t} - m_{3tf}), & \text{if } d \leq 2r \\ n_d - 4m_s + 6m_{1t} - 4m_{3t}, & \text{if } d > 2r \end{cases}$	n_{d-4r-4}
27	$n_d - m_s - n_r$	$(n_d - m_s - n_r)_+$

Tab. 2. Formulae for ℓ_i and u_i.

lower bound follows. The upper bound follows from the fact that the number of points in $t_r(e_1) \cup t_r(e_2) \cup t_r(e_3)$ is $3m_{1t} - 2m_{2t} - m_{to} + m_{3tp}$, as can be seen by a careful examination of a Venn diagram.

9: (T touches $\tilde{\triangle}$ along three edges e_1, e_2, e_3 sharing a vertex v). There are m_{1t} smoothness conditions associated with each of these edges. However, since all three tubes contain $B_r(v)$, $2n_r$ of these conditions are redundant, and the lower bound follows. The upper bound follows from the fact that the number of points in the intersection of all three tubes is $3m_{1t} - 3m_{2t} + m_{3t}$.

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10: (T touches $\tilde{\triangle}$ along three edges e_1, e_2, e_3 on one face). The three 2-partial tubes contain $3(m_{1t} - 2n_r + m_{2b})$ points. The three balls contain $3n_r - 3m_{2b} + m_{3b}$ points, cf. the argument for the upper bound in case 3. Thus, there are at most $3m_{1t} - 3n_r + m_{3b}$, nonredundant smoothness conditions, and the lower bound follows. The upper bound follows from the fact that the number of points in $t_r(e_1) \cup t_r(e_2) \cup t_r(e_3)$ is $3m_{1t} - 3m_{2t} + m_{3tf}$.

11: (T touches $\tilde{\triangle}$ along four edges $e_1, \ldots, e_4$ forming a loop). Given a domain point ξ in T, let $b(\xi)$ and $t(\xi)$ be the number of balls $B_r(v_i)$ and tubes $t_r(e_i)$, $1 \leq i \leq 4$, that contain ξ. Suppose ξ is not in the intersection of all four balls. Then the coefficient associated with the point ξ is subject to $t(\xi)$ conditions, but $b(\xi)$ of them are redundant, and thus the change in dimension corresponding to this point is bounded below by $1 - t(\xi) + b(\xi)$. Points that are in all four balls are also in all four tubes. The associated coefficients are subject to four conditions, but three of them are redundant, and thus there is no change in dimension. It follows that a lower bound for the change in dimension when we add T to $\tilde{\triangle}$ is given by $\sum_{\xi \in T}(1 - t(\xi) + b(\xi)) - m_{4b} = n_d - 4m_{1t} + 4n_r - m_{4b}$. To get the upper bound, we need to count the number of points in $\Gamma := \cup_{i=1}^4 t_r(e_i)$. Looking at a Venn diagram, we see that there are four pairs of neighboring tubes intersecting in m_{2t} points and two pairs of opposite tubes that intersect in m_{to} points. There are four paths of three tubes, and m_{3tp} points in the intersection of the three tubes in such a path. The intersection of all four tubes contains m_{4tl} points. It follows that the number of points in Γ is $4m_{1t} - 4m_{2t} - 2m_{to} + 4m_{3tf} - m_{4tl}$, and the upper bound follows.

12: (T touches $\tilde{\triangle}$ along four edges $e_1, \ldots, e_4$ not forming a loop). Three of these edges lie on one face F. Let v_1, v_2, v_3 be the vertices on this face, and suppose v_3 is the vertex where three of the touching edges meet. There are $m_{1t} - 2n_r + m_{2b}$ points in each of the three 2-partial tubes along the edges of F. In addition, there are $m_{1t} - n_r$ points in the 1-partial tube along the fourth edge. There are $3n_r - 3m_{2b} + m_{3b}$ points in the three balls around the vertices of F, cf. the argument for the upper bound in case 3. Thus, the total number of nonredundant smoothness conditions is at most $4m_{1t} - 4n_r + m_{3b}$, and the lower bound follows. To get the upper bound, we count the number of points in $\Gamma := \cup_{i=1}^4 t_r(e_i)$. It is easy to see that when $d \leq 2r$, Γ contains all points in T, and so $u_{12} = 0$. Suppose now $d > 2r$. In this case $m_{3tp} = m_{to} = 0$. Looking at a Venn diagram, we see there are five pairs of neighboring tubes intersecting in m_{2t} points, and one pair of opposite tubes intersecting in m_{to} points. There is one set of three tubes intersecting in m_{3tf} points, one set of three tubes meeting at v_3 intersecting in m_{3t} points, and 2 pairs of three tubes forming paths such that each pair intersects in m_{3tp} points. The intersection of all four tubes is empty. It follows that the number of points in Γ is $4m_{1t} - 5m_{2t} + m_{3t} + m_{3tf}$. This gives the upper bound.

13: (T touches $\tilde{\triangle}$ along five edges $e_1, \ldots, e_5$). Let v_1, v_2 be the vertices where three of these edges meet, and e_{34} the single non-touching edge. Then two of the touching edges meet at v_3, and two meet at v_4. Each of the five 2-partial tubes contains

$m_{1t} - 2n_r + m_{2b}$ points. As discussed in case 3, $B_r(v_1) \cup B_r(v_2) \cup B_r(v_3)$ contains $3n_r - 3m_{2b} + m_{3b}$ points. There are $n_r - 2m_{2b} + m_{3b}$ points in $B_r(v_4)$ but not in the balls $B_r(v_1)$ or $B_r(v_2)$. We conclude that the total number of nonredundant smoothness conditions is at most $5m_{1t} - 6n_r + 2m_{3b}$, and the lower bound follows. To get the upper bound, we count the number of points in $\Gamma := \cup_{i=1}^{5} t_r(e_i)$. It is easy to see that Γ contains all the points of T when $d \leq 2r$, and so $u_{13} = 0$. Suppose now $d > 2r$. In this case $m_{3tp} = m_{to} = 0$. Looking at a Venn diagram, we see that there are eight pairs of neighboring tubes intersecting in m_{2t} points, and two pairs of opposite tubes intersecting in $m_{to} = 0$ points. There are 2 sets of three tubes intersecting in m_{3tf} points and 2 sets of three tubes intersecting in m_{3t} points. The remaining 6 intersecting sets of triples of three tubes each contain $m_{3tp} = 0$ points. There are 6 sets where four tubes intersect, but the intersections are all empty since opposite tubes don't overlap. It follows that the number of constrained points is $5m_{1t} - 8m_{2t} + 2m_{3tf} + 2m_{3t}$. This gives the upper bound.

14: (T touches $\tilde{\triangle}$ along six edges). There are $m_{1t} - 2n_r + m_{2b}$ points in each of the six 2-partial tubes around the edges. In addition, there are $4n_r - 6m_{2b} + 4m_{3b} - m_{4b}$ points in the four balls, cf. the discussion of the upper bound in case 4. Thus, the total number of nonredundant smoothness conditions is at most $6m_{1t} - 8n_r + 4m_{3b} - m_{4b}$, and the lower bound follows. To get the upper bound, we count the number of points in the union Γ of the tubes around the six edges. It is easy to see that Γ contains all the points of T when $d \leq 2r$, and so $u_{14} = 0$. Suppose now $d > 2r$. In this case $m_{3tp} = m_{to} = 0$. There are 15 pairs of intersecting tubes. Twelve of these intersect in m_{2t} points, and the other three intersect in m_{to} points. There are 20 sets of intersections of three tubes. Four of these contain m_{3tf} points, four of them contain m_{3t} points, and the remaining 12 contain $m_{3tp} = 0$ points. There are 6 sets where four tubes intersect, but the intersections are all empty since opposite tubes don't overlap. It follows that the number of points in Γ is $6m_{1t} - 12m_{2t} + 4m_{3t} + 4m_{3tf}$. This gives the upper bound.

15: (T touches $\tilde{\triangle}$ on one edge e and at a vertex v not on e). Clearly the number of nonredundant smoothness conditions is at most $m_{1t} + n_r$, and the lower bound follows. The upper bound follows from the fact that there are $m_{1t} + n_r - m_{tb}$ points in $t_r(e) \cup B_r(v)$.

16: (T touches $\tilde{\triangle}$ on one edge e and at two vertices v_1, v_2 not on e). The number of smoothness conditions with tips in T is $m_{1t} + 2n_r$, and the lower bound follows. Examining a Venn diagram, we see that there are $m_{1t} + 2n_r - 2m_{tb} - m_{2b} + m_{t2b}$ points in $t_r(e) \cup B_r(v_1) \cup B_r(v_2)$, and the upper bound follows.

17: (T touches $\tilde{\triangle}$ on two edges e_1, e_2 and at a vertex v not on those edges). Clearly the number of nonredundant smoothness conditions is at most $2m_{1t} - n_r + n_r = 2m_{1t}$, and the lower bound follows. The upper bound follows from the fact that there are $2m_{1t} + n_r - 2m_{tb} - m_{2t} + m_{2tb}$ points in $t_r(e_1) \cup t_r(e_2) \cup B_r(v)$.

18: (T touches $\tilde{\triangle}$ on three edges e_1, e_2, e_3 and at a vertex v not on those edges). The number of nonredundant smoothness conditions associated with the three tubes is

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at most $3m_{1t} - 3n_r + m_{3b}$, cf. the discussion of case 10. Adding the n_r conditions associated with $B_r(v)$ leads to the lower bound. To get the upper bound we first notice that are $3m_{1t} - 3m_{2t} + m_{3tf}$ points in $\Lambda := t_r(e_1) \cup t_r(e_2) \cup t_r(e_3)$, cf. the upper bound in case 10. There are n_r points in the ball $B_r(v)$. However, if $d \leq 2r$, the ball $B_r(v)$ intersects Λ creating three smaller tubes of radius $2r - d$. The number of points in these tubes is $3\hat{m}_{1t} - 3\hat{m}_{2t} + \hat{m}_{3tf}$, and the upper bound follows.

19: (T touches $\tilde{\triangle}$ on one face F and one edge e not in the face). The number of nonredundant smoothness conditions is at most $m_s + m_{1t} - n_r$, and the lower bound follows. To get the upper bound, we observe that the number of points in $G_r(F) \cup t_r(e)$ is $m_s + m_{1t} - m_{ts}$.

20: (T touches $\tilde{\triangle}$ on one face F and two edges e_1, e_2 not in the face). Let e_3 be the edge lying on the same face as e_1 and e_2. The number of nonredundant smoothness conditions in the three tubes surrounding the edges e_1, e_2, e_3 is $3m_{1t} - 3n_r + m_{3b}$, cf. the lower bound for case 10. The number of nonredundant smoothness conditions in the slab $G_r(F)$ but not in the tube $t_r(e_3)$ is $m_s - m_{1t}$. Thus, the total number of nonredundant smoothness conditions is at most $m_s + 2m_{1t} - 3n_r + m_{3b}$, and the lower bound follows. Let Γ be the union of the slab $G_r(F)$ with the tubes around the two edges e_1, e_2. It is easy to see that Γ contains all points of T when $d \leq 2r$, and so $u_{20} = 0$ in this case. Now suppose $d > 2r$. In this case there are m_s constrained points in the slab $G_r(F)$. The set of domain points resulting from cutting off the slab can be considered as corresponding to a polynomial of degree $d - r - 1$. Referring to case 7, the number of points in the two tubes sharing the vertex v_4 is $2\tilde{m}_{1t} - \tilde{m}_{2t}$. The upper bound follows.

21: (T touches $\tilde{\triangle}$ on one face F and three edges e_1, e_2, e_3 not in the face). To count nonredundant smoothness conditions, we note that T touches on all six edges. By case 14, the number of nonredundant smoothness conditions associated with these edges is at most $6m_{1t} - 8n_r + 4m_{3b} - m_{4b}$. The number of nonredundant smoothness conditions associated with points in the slab $G_r(F)$ but not in the tubes around the three edges of F is $m_s - (3m_{1t} - 3n_r + m_{3b})$. Thus the total number of nonredundant smoothness conditions is at most $m_s + 3m_{1t} - 5n_r + 3m_{3b} - m_{4b}$, and the lower bound follows. Let $\Gamma := G_r(F) \cup t_r(e_1) \cup t_r(e_2) \cup t_r(e_3)$. It is easy to see that Γ contains all points of T when $d \leq 2r$, and so $u_{21} = 0$ in this case. Suppose now $d > 2r$. Then the number of points in the slab $G_r(F)$ is m_s, while the number of points in the rest of Γ is $3\tilde{m}_{1t} - 3\tilde{m}_{2t} + \tilde{m}_{3t}$, cf. case 9. This leads immediately to the upper bound.

22: (T touches $\tilde{\triangle}$ on two faces F_1, F_2 and one edge e not in either face). If $d > 2r$, then the number of nonredundant conditions with tips in the two slabs is $2m_s - m_{1t}$, while the number of nonredundant conditions with tips in the tube $t_r(e)$ is at most $m_{1t} - 2n_r + m_{2b}$. The lower bound follows since $m_{2t2b} = 0$ in this case. Now suppose $d \leq 2r$ and that $e := \langle v_3, v_4 \rangle$ is the edge between the two faces. In this case we have counted each point ξ in $B_r(v_1) \cap B_r(v_2) \setminus t_r(e)$ twice, but the two smoothness conditions with tip at ξ are are consistent because they are associated with the

tube $t_r(e)$. Thus in this case we have to adjust the lower bound by adding m_{2t2b}. It is easy to check that the upper bound is zero whenever $d \leq 3r$. Suppose now $d > 3r$. Let $\widetilde{T}$ be the tetrahedron obtained by stripping off two slabs of size r from T. Clearly, $\widetilde{T}$ contains n_{d-2r-2} points. The number of points lying in the tube of radius r around the edge $e \cap \widetilde{T}$ is $m_{1t} - 2m_{ts}$, and the upper bound follows.

24: (T touches $\widetilde{\triangle}$ on two faces). The number of nonredundant smoothness conditions in the slabs associated with the two faces is $2m_s - m_{1t}$, and the lower bound follows. Since all points in these two slabs are constrained, it is clear that the number of unconstrained points is equal to the number of points in the tetrahedron that is obtained by slicing off the two slabs, namely n_{d-2r-2}.

25: (T touches $\widetilde{\triangle}$ on three faces F_1, F_2, F_3). Let v be the vertex opposite the nontouching face, and let e_1, e_2, e_3 be the three edges meeting at v. The number of points contained in T but not in the tubes around these edges is $u_9 = n_d - 3m_{1t} + 3m_{2t} - m_{3t}$. The number of points in one of the three slabs $G_r(F_i)$ but not in the tubes is $m_s - 2m_{1t} + m_{2t}$. Multiplying the latter by three and subtracting gives the lower bound. The upper bound follows as in the previous case by removing three slabs.

26: (T touches $\widetilde{\triangle}$ on four faces). Consider first the case where $d \leq 2r$. In this case the number of constrained points is n_d, and so the upper bound is 0. To get the lower bound, we subtract the number of smoothness conditions with tips outside the tubes around edges of T from the number of unconstrained points (which is 0). For each face, the number of smoothness conditions outside the three tubes around the edges of that face is $m_s - (3m_{1t} - 3m_{2t} + m_{3tf})$, cf. case 9, and the lower bound follows. Now suppose $d > 2r$. In this case the number of points contained in T but not in the six tubes is $u_{14} = n_d - 6m_{1t} + 12m_{2t} - 4m_{3t} - 4m_{3tf}$. The number of points in each of the four slabs but not in the tubes is $m_s - 3m_{1t} + 3m_{2t} - m_{3tf}$. Multiplying the latter by four and subtracting gives the lower bound. To get the upper bound we strip off four slabs.

§7. The Case Where $\triangle$ Is Shellable

A tetrahedral partition $\triangle$ is called shellable provided it consists of a single tetrahedron, or can be obtained from a shellable tetrahedral partition $\widetilde{\triangle}$ by adding one tetrahedron T such that T intersects $\widetilde{\triangle}$ precisely along one, two, or three triangular faces. In this section we show that if $\triangle$ is shellable, then there are simple formulae for the bounds on $\dim \mathcal{S}_d^r(\triangle)$ in terms of the combinatorics of $\triangle$. Let F_I, E_I, V_I be the number of interior faces, edges, and vertices of $\triangle$. Given $0 \leq r < d$, let $\beta := m_{1t} - n_d$ and $\gamma := n_d - m_{3t}$. Similarly, let $\tilde{\beta} := n_{d-2r-2} - 2n_{d-r-1}$ and $\tilde{\gamma} := 3n_{d-r-1} - 3n_{d-2r-2} + n_{d-3r-3}$.

Theorem 7.1. *Suppose $\triangle$ is a shellable tetrahedral partition. Then for every $0 \leq r < d$,*

$$L(r, d) \leq \dim \mathcal{S}_d^r(\triangle) \leq U(r, d),$$

where

$$L(r, d) = n_d + (n_{d-r-1}F_I + \beta E_I + \gamma V_I)_+,$$
$$U(r, d) = n_d + n_{d-r-1}F_I + \tilde{\beta}E_I + \tilde{\gamma}V_I.$$

(7.1)

Proof: If $\triangle$ is shellable, then it can be assembled using only steps of types 0, 23, 24, and 25. Clearly, $\tau_0 = 1$. We get a new interior vertex each time we carry out step 25, and thus $V_I = \tau_{25}$. We get new interior edges each time we carry out steps of types 24 and 25, and so $E_I = \tau_{24} + 3\tau_{25}$. Similarly, $F_I = \tau_{23} + 2\tau_{24} + 3\tau_{25}$. Solving these equations for the assembly parameters, we find that $\tau_{23} = F_I - 2E_I + 3V_I$, $\tau_{24} = E_I - 3V_I$, and $\tau_{25} = V_I$. Referring to Tab. 2 and using the fact that $n_d - m_s = n_{d-r-1}$, the formulae (7.1) follows immediately from Theorem 3.1. $\square$

§8. Numerical Examples

We have extensively tested and evaluated our bounds using a java program which is capable of computing the exact dimension, see Remark 6. In this section we describe the results of a few of our tests. Our first example shows that (at least for a range of r and d), our lower bounds cannot be improved.

Example 8.1. *For each $0 \le i \le 27$, let $\triangle_i$ be the smallest tetrahedral partition such that to build $\triangle_i$ we use an assembly step of type i as the last step.*

Discussion: The $\triangle_i$ are actually quite small. For example, for $i = 1, \ldots, 4$, $\triangle_i$ consists of just $i + 1$ tetrahedra. The cases $i = 4$ and $i = 5$ require just two and three tetrahedra, respectively. For each $r = 1, \ldots, 6$ and $d = r + 1, \ldots, 4r + 1$, we computed the change in dimension due to the last step. In all cases it turns out that $\ell_i = \delta_i$. As a check on our upper bounds, we also directly counted the number of unconstrained points. $\square$

Our next example shows that in general the bounds in Theorem 3.1 depend on the order in which the tetrahedral partition is assembled.

Example 8.2. *Let $\triangle$ be the partition described in Tab. 3.*

Discussion: This partition is obtained by splitting a given tetrahedron into 15 subtetrahedra. It is the trivariate analog of the well-known Morgan-Scott split of a triangle, see [5]. The table lists the coordinates (x_i, y_i, z_i) of each of the 8 vertices of $\triangle$, and the vertex numbers $(v_0^i, v_1^i, v_2^i, v_3^i)$ of each of the 15 tetrahedra in $\triangle$. Tab. 4 shows the results obtained for selected values of r and d. The column labeled *symdim* gives the exact dimension of the splines spaces for the specific choice of vertices given in the table, which corresponds to a high level of symmetry. The column labeled *gendim* gives the corresponding results for a generic partition obtained by perturbing each of the vertices slightly. We give two sets of upper and lower bounds. The columns labeled $L_1(r, d)$ and $U_1(r, d)$ correspond to an assembly scheme for which $\tau_0 = \tau_5 = \tau_7 = \tau_{10} = \tau_{26} = 1$, $\tau_{24} = 6$, and $\tau_{25} = 4$.

The columns labeled $L_2(r, d)$ and $U_2(r, d)$ correspond to an assembly scheme for which $\tau_0 = 1$, $\tau_{23} = 4$, $\tau_{24} = 6$, and $\tau_{25} = 4$. The table shows that the bounds

i	v_0^i	v_1^i	v_2^i	v_3^i	x_i	y_i	z_i
0	4	5	6	7	14	42	42
1	3	4	5	6	14	0	42
2	2	4	5	7	-14	21	21
3	1	4	6	7	0	21	0
4	0	5	6	7	2	18	24
5	2	3	4	5	2	24	24
6	1	3	4	6	6	21	27
7	1	2	4	7	4	21	30
8	0	3	5	6			
9	0	2	5	7			
10	0	1	6	7			
11	0	1	2	7			
12	0	1	3	6			
13	0	2	3	5			
14	1	2	3	4			

Tab. 3. The tetrahedral partition for Example 8.2.

depend on the assembly order. For this example, the second set of bounds are better. □

Our next example gives results for a partition with a hole.

Example 8.3. *Let $\triangle$ be the tetrahedral partition described in Tab. 5.*

Discussion: $\triangle$ is a partition of a square-shaped torus into 24 tetrahedra. In Tab. 6 we give results for selected values of r and d, both for the symmetric case corresponding to the listed vertices, and for the generic case corresponding to slightly perturbed vertices. In both cases we give bounds corresponding to assembly parameters $\tau_0 = 1$, $\tau_{19} = 4$, $\tau_{23} = 12$, $\tau_{24} = \tau_{25} = 3$, and $\tau_{27} = 1$. □

Our final example involves a partition with a cavity.

Example 8.4. *Let $\triangle$ be the partition obtained by removing the tetrahedron numbered 0 in Tab. 3 from the partition described there.*

Discussion: Removing the tetrahedron creates a cavity. This partition can be built using the assembly parameters $\tau_0 = \tau_5 = \tau_7 = \tau_{10} = 1$, $\tau_{24} = 6$, and $\tau_{25} = 3$. In Tab. 7 we give numerical results for selected values of r and d, where as before we consider both the generic and symmetric case. For $r = 1$ we note that the lower bounds give the correct dimensions for all $d \geq 4$. □

r	d	$L_1(r,d)$	$L_2(r,d)$	gendim	symdim	$U_2(r,d)$	$U_1(r,d)$
1	2	10	10	10	11	14	15
1	3	20	20	20	25	36	39
1	4	43	43	43	54	81	88
1	5	104	104	104	113	160	172
1	6	214	214	214	222	288	306
1	7	388	388	388	396	480	504
1	8	641	641	641	649	751	781
1	9	988	988	988	996	1116	1152
2	3	20	20	20	20	24	26
2	4	35	35	35	35	51	57
2	5	56	56	56	56	96	109
2	6	84	84	84	87	170	193
2	7	120	120	132	144	284	320
2	8	203	207	243	252	449	500
2	9	380	384	420	428	680	748
3	4	35	35	35	35	39	42
3	5	56	56	56	56	72	81
3	6	84	84	84	84	124	143
3	7	120	120	120	120	200	233
3	8	165	165	165	167	311	363
3	9	220	220	220	234	468	543
3	10	286	286	332	348	682	784
3	11	364	364	524	534	964	1096
3	12	581	593	799	807	1329	1494
3	13	936	948	1172	1180	1792	1992
4	5	56	56	56	56	60	64
4	6	84	84	84	84	100	112
4	7	120	120	120	120	160	185
4	8	165	165	165	165	245	288
4	9	220	220	220	220	360	427
4	10	286	286	286	287	516	613
4	11	364	364	364	371	724	857
4	12	455	455	455	486	995	1169
4	13	560	560	640	667	1340	1560
4	14	680	680	926	944	1770	2040

Tab. 4. Results for Example 8.2.

§9. Remarks

Remark 1. Given a tetrahedral partition $\triangle$, let $\mathcal{D}_{d,\triangle}$ be the set of all domain points associated with tetrahedra of $\triangle$, where common domain points are included just once. Then the dimension of the trivariate spline space $\mathcal{S}_d^0(\triangle)$ is equal to the

i	v_0^i	v_1^i	v_2^i	v_3^i	x_i	y_i	z_i
0	0	3	7	15	-1	1	0
1	0	4	7	15	-1	-1	0
2	0	4	12	15	1	-1	0
3	0	8	12	15	1	1	0
4	0	8	11	15	-2	2	0
5	0	3	11	15	-2	-2	0
6	0	1	4	12	2	-2	0
7	1	4	5	12	2	2	0
8	1	5	12	13	-1	1	1
9	1	9	12	13	-1	-1	1
10	1	8	9	12	1	-1	1
11	0	1	8	12	1	1	1
12	1	2	5	13	-2	2	1
13	2	5	6	13	-2	-2	1
14	2	6	13	14	2	-2	1
15	2	10	13	14	2	2	1
16	2	9	10	13			
17	1	2	9	13			
18	2	3	6	14			
19	3	6	7	14			
20	3	7	14	15			
21	3	11	14	15			
22	3	10	11	14			
23	2	3	10	14			

Tab. 5. The tetrahedral partition for Example 8.3.

the number of points in the set $\mathcal{D}_{d,\triangle}$. This is $n_V + (d-1)n_E + \binom{d-1}{2}n_F + \binom{d-1}{3}n_T$, where n_V, n_E, n_F, and n_T are the numbers of vertices, edges, faces, and tetrahedra in $\triangle$. At the other extreme, the dimension of $\mathcal{S}_d^d(\triangle)$ is equal to $m\binom{d+3}{3}$, where m is the number of connected components of $\triangle$.

Remark 2. By definition, any shellable tetrahedral partition is homeomorphic to a ball in $\mathbb{R}^3$. The simplest example of a nonshellable partition is the partition consisting of two tetrahedra touching at one vertex. The fact that a tetrahedral partition is homeomorphic to a ball does not imply that it is shellable. For an example consisting of 41 tetrahedra and 14 vertices, see Rudin [6].

Remark 3. For the special case of shellable tetrahedral decompositions, the upper bounds given here agree with those given in [1]. Our lower bounds agree for $r = 1$, and provide a significant improvement when $r > 1$. That paper contains two typographical errors: the summation indices in (10) and (15) there should be κ instead of i, and the upper limits of the summation should be $k + 1$ and k instead of $\kappa + 1$ and κ, respectively.

r	d	$L(r,d)$	gendim	symdim	$U(r,d)$
1	2	10	10	12	22
1	3	48	48	48	68
1	4	144	144	144	176
1	5	320	320	320	364
1	6	600	600	600	656
1	7	1008	1008	1008	1076
1	8	1568	1568	1568	1648
1	9	2304	2304	2304	2396
2	3	20	20	22	32
2	4	35	35	46	83
2	5	93	96	108	176
2	6	237	240	256	353
2	7	477	480	500	629
2	8	837	840	864	1025
2	9	1341	1344	1372	1565
3	4	35	35	37	47
3	5	56	56	64	104
3	6	84	84	112	204
3	7	151	176	208	360
3	8	351	380	420	623
3	9	663	696	744	1004
3	10	1111	1148	1204	1524
3	11	1719	1760	1824	2204
3	12	2511	2556	2628	3068
3	13	3511	3560	3640	4140
4	5	56	56	58	68
4	6	84	84	92	132
4	7	120	120	140	240
4	8	165	165	220	405
4	9	220	288	356	640
4	10	483	560	644	1006
4	11	875	960	1060	1509
4	12	1419	1512	1628	2170
4	13	2139	2240	2372	3010
4	14	3059	3168	3316	4050

Tab. 6. Results for Example 8.3.

Remark 4. A shellable partition can be built using only assembly steps of types 0, 23, 24, and 25. However, a tetrahedral partition of a set Ω with holes and/or cavities cannot be shellable. In [3], a tetrahedral partition $\triangle$ is said to have a regular cavity provided that it can be obtained from a larger tetrahedral partition by removing a shellable subpartition $\mathcal{T}$, all of whose vertices are interior vertices

r	d	$L(r,d)$	gendim	symdim	$U(r,d)$
1	2	10	10	11	15
1	3	20	20	25	39
1	4	46	46	57	88
1	5	112	112	121	172
1	6	228	228	236	306
1	7	408	408	416	504
1	8	666	666	674	780
1	9	1016	1016	1024	1148
2	3	20	20	20	26
2	4	35	35	35	57
2	5	56	56	57	109
2	6	84	84	91	193
2	7	120	144	157	320
2	8	242	270	279	500
2	9	436	464	472	748
3	4	35	35	35	42
3	5	56	56	56	81
3	6	84	84	84	143
3	7	120	120	120	233
3	8	165	165	168	363
3	9	220	224	245	543
3	10	286	368	381	784
3	11	428	584	593	1096
3	12	718	886	894	1494
3	13	1108	1288	1296	1992
4	5	56	56	56	64
4	6	84	84	84	112
4	7	120	120	120	185
4	8	165	165	165	288
4	9	220	220	220	427
4	10	286	286	289	613
4	11	364	364	380	857
4	12	455	478	518	1169
4	13	560	712	736	1560
4	14	680	1036	1053	2040

Tab. 7. Results for Example 8.4.

of $\triangle$. A tetrahedral partition $\triangle$ of a set Ω is said to have a regular hole provided that it can be obtained from a larger tetrahedral partition by removing a shellable subpartition $\mathcal{T}$ with the property that if F is a triangular face of a tetrahedron in $\mathcal{T}$ and F lies on the boundary of Ω, then $F \in t_1 \cup t_2$, where

1) t_1 and t_2 are shellable triangulations,

2) the triangles of t_1 do not touch those in t_2.

In [3], a tetrahedral partition $\triangle$ is said to be regular provided that one of the following holds:

1) $\triangle$ is shellable, or

2) $\triangle$ can be obtained from a regular tetrahedral partition by creating a regular hole or a regular cavity.

This definition is general enough to include all of the tetrahedral partitions typically used in practice. We conjecture that regular partitions can be built using only assembly steps of types 0, 23, 24, 25, and 27.

Remark 5. It was shown in [1] that there exists a tetrahedral decomposition such an arbitrarily small change in the location of the vertices changes the dimension of the corresponding spline space $\mathcal{S}_d^r(\triangle)$ by a term which is linear in d. It follows that any upper and lower bounds on the dimension of $\mathcal{S}_d^r(\triangle)$ that work for arbitrary partitions and do not take account of the precise locations of the vertices must differ by a term that is at least $\mathcal{O}(d)$. We claim that for $d > 2r$, the upper and lower bounds of Theorem 3.1 satisfy $U(r,d) - L(r,d) = \mathcal{O}(d)$, and thus can be considered to be asymptotically optimal. To see this, we note that the upper bound is $U(r,d) = n_d - n_{\text{tips}}$, where n_{tips} is the number of points that are tips of smoothness conditions. By the discussion in Sect. 6, $L(r,d) \geq n_d - n_{\text{cond}}$, where n_{cond} is the number of conditions imposed by smoothness. Thus $U(r,d) - L(r,d) \geq n_{\text{cond}} - n_{\text{tips}}$. When $d > 2r$, a point can be the tip of more than one smoothness condition only if it lies in a tube $t_r(e)$ around an edge of $\triangle$, and it follows that $n_{\text{cond}} - n_{\text{tips}} = \mathcal{O}(d)$. This proves the claim.

Remark 6. The java software that we used for testing our bounds is available at http://www.math.utah.edu/~pa/3DMDS/. It can be used as an applet directly from the web page, or can be downloaded and used offline.

Remark 7. Explicit formulae for the dimension of $\mathcal{S}_d^1(\triangle)$ for $d \geq 8$ were established in [2] for generic shellable tetrahedral partitions. Comparing with Theorem 7.1, it is easy to see that for such partitions the dimension of $\mathcal{S}_d^1(\triangle)$ is equal to the lower bound in (7.1).

Remark 8. Recently Lau [4] gave a lower bound for the dimension of $\mathcal{S}_d^r(\triangle)$ for tetrahedral partitions of simply connected domains. His lower bound differs from our lower bound in Theorem 7.1 in two ways. First, it is missing the term involving V_I, and thus is generally much too low. For example, for the partition of Example 8.2 and for the range of r and d shown in Tab. 4, it gives negative numbers for almost all cases. On the other hand, his bound contains an additional nonnegative term which takes account of the geometry of faces surrounding interior edges. We have not attempted to take account of geometry here.

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