# Error Bounds for Minimal Energy Bivariate Polynomial Splines

Manfred von Golitschek [1)], Ming-Jun Lai [2)]
and Larry L. Schumaker [3)]

**Abstract.** We derive error bounds for bivariate spline interpolants which are calculated by minimizing certain natural energy norms.

## §1. Introduction

Suppose we are given values $\{f(v)\}_{v\in\mathcal{V}}$ of an unknown function $f$ at a set $\mathcal{V}$ of scattered points in $\mathbb{R}^2$. To approximate $f$, we choose a linear space $\mathcal{S}$ of polynomial splines of degree $d$ defined on a triangulation $\triangle$ with vertices at the points of $\mathcal{V}$. Let

$$U_f := \{s \in \mathcal{S} :\ \ s(v) = f(v), v \in \mathcal{V}\} \tag{1.1}$$

be the set of all splines in $\mathcal{S}$ that interpolate $f$ at the points of $\mathcal{V}$. We assume that $\mathcal{S}$ is big enough so that $U_f$ is nonempty. Then a commonly used way to create an approximation of $f$ (cf. [6–10]) is to choose a spline $S_f$ such that

$$\mathcal{E}(S_f) = \min_{s\in U_f} \mathcal{E}(s), \tag{1.2}$$

where

$$\mathcal{E}(s) := \sum_{T\in\triangle} \int_T \left[s_{xx}^2 + 2s_{xy}^2 + s_{yy}^2\right]. \tag{1.3}$$

We refer to $S_f$ as the minimal energy interpolating spline.

[1)] Institut für Angewandte Mathematik und Statistik der Universität Würzburg, 97074 Würzburg, Germany, goli@mathematik.uni-wuerzburg.de
[2)] Department of Mathematics, The University of Georgia, Athens, GA 30602, mjlai@math.uga.edu. Supported by the National Science Foundation under grant DMS-9870187,
[3)] Department of Mathematics, Vanderbilt University, Nashville, TN 37240, s@mars.cas.vanderbilt.edu. Supported by the National Science Foundation under grant DMS-9803340 and by the Army Research Office under grant DAAD-19-99-1-0160.

The main result of this paper is Theorem 6.2 which shows that if $\mathcal{S}$ is a space of splines with an appropriate stable local basis (see Def. 2.1) and $\triangle$ is a $\beta$-quasi-uniform triangulation (see Def. 5.1), then for every $f \in W^2_\infty(\Omega)$, the interpolating spline $S_f$ defined in (1.2) satisfies

$$\|f - S_f\|_{L_\infty(\Omega)} \le C|\triangle|^2 |f|_{2,\infty,\Omega}. \tag{1.4}$$

Here $|\triangle|$ is the diameter of the largest triangle in $\triangle$, $\Omega$ is the closed polygonal set consisting of the union of the triangles of $\triangle$, and $W^2_\infty(\Omega)$ is the usual Sobolev space with seminorm

$$|f|_{2,\infty,\Omega} := \sum_{\nu+\mu=2} \|D_x^\nu D_y^\mu f\|_{L_\infty(\Omega)}. \tag{1.5}$$

The proof of this error bound is based on recent results [12] on the $L_\infty$ norms of projections onto bivariate polynomial spline spaces with stable bases. The paper is organized as follows. In Sect. 2 we review basic Bernstein-Bézier tools, discuss stable locally supported basis, and establish a useful approximation theorem for spline spaces with such bases. In Sect. 3 we list a wide variety of spline spaces which possess stable local bases. Sect. 4 is devoted to a Hilbert space formulation of the basic minimization problem, and Sect. 5 contains various estimates on the norm of a key projector. The proof of the error bound (1.4) is presented in Sect. 6, and the sharpness of the result is discussed in Sect. 7, where several numerical examples are presented. Sect. 8 is devoted to an extension involving a kind of bivariate tension spline. We conclude the paper with several remarks in Sect. 9.

## §2. Stable local bases

Throughout the remainder of the paper we shall restrict our attention to spaces of splines $\mathcal{S}$ which are subspaces of the space $\mathcal{S}^0_d(\triangle)$ of continuous splines of degree $d$ on a triangulation $\triangle$. It will be convenient to make use of the well-known Bernstein-Bézier representation of splines (cf. e.g. [2–3,13–16]). Let

$$\mathcal{D}_{d,\triangle} := \{\xi^T_{ijk} : \xi^T_{ijk} = \tfrac{(iu+jv+kw)}{d}, \text{ where } i+j+k=d \\ \text{and } T := \langle u,v,w\rangle \text{ is a triangle in } \triangle\}$$

be the set of **domain points** associated with $d$ and $\triangle$. Then there is a 1-1 correspondence between the space of splines $\mathcal{S}^0_d(\triangle)$ and the set of coefficient vectors $\{c_\xi\}_{\xi\in\mathcal{D}_{d,\triangle}}$. In particular, the restriction $s|_T$ of $s$ has a unique expansion of the form

$$s|_T = \sum_{i+j+k=d} c^T_{ijk} B^T_{ijk},$$

where $B^T_{ijk}$ are the Bernstein basis polynomials of degree $d$ associated with the triangle $T$. The $c_\xi$ are called the **B-coefficients** of $s$.

Given a spline space $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$, recall that a set $\mathcal{M} \subseteq \mathcal{D}_{d,\triangle}$ is called a minimal determining set for $\mathcal{S}$ provided that each spline $s \in \mathcal{S}$ is uniquely determined by its B-coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$. It is well known (cf. [3,14]) that if $\mathcal{M}$ is a minimal determining set for $\mathcal{S}$, then for each $\xi \in \mathcal{M}$, there exists a unique *dual basis spline* $B_\xi \in \mathcal{S}$ with the property

$$\lambda_\eta B_\xi = \delta_{\xi,\eta}, \qquad \text{all } \eta \in \mathcal{M}, \tag{2.1}$$

where $\lambda_\eta$ is the linear functional that picks off the B-coefficient associated with the domain point $\eta$. Note that all vertices of $\triangle$ are in $\mathcal{D}_{d,\triangle}$. By well-known properties of the Bernstein-Bézier representation, it is also clear that if $\mathcal{M}$ contains the set $\mathcal{V}$ of vertices of $\triangle$, then for any given $z_v$, the spline

$$s = \sum_{v\in\mathcal{V}} z_v B_v$$

satisfies $s(v) = z_v$ for all $v \in \mathcal{V}$. This implies that the set $U_f$ in (1.1) is always nonempty.

Throughout this paper we shall assume that $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$ is a spline space with a set of dual basis functions $\{B_\xi\}_{\xi\in\mathcal{M}}$ corresponding to a minimal determining set $\mathcal{M}$ containing the set $\mathcal{V}$ of vertices of $\triangle$. In addition, we shall assume that this basis is a stable local basis in the following sense:

**Definition 2.1.** ([3,15,16]). We say that a basis $\{B_\xi\}_{\xi\in\mathcal{M}}$ for a space $\mathcal{S}$ of splines on a triangulation $\triangle$ is a stable local basis provided there exists an integer $\ell$ and constants $0 < K_1 \le K_2 < \infty$ depending only on $d$ and the smallest angle $\theta_\triangle$ in the triangulation $\triangle$ such that

1) for each $\xi \in \mathcal{M}$,

$$\mathrm{supp}(B_\xi) \subseteq \mathrm{star}^\ell(v_\xi) \text{ for some vertex } v_\xi \text{ of } \triangle,$$

where

$$\mathrm{star}^\ell(v) := \bigcup\{\mathrm{star}^1(w) : \ w \text{ is a vertex of } \mathrm{star}^{\ell-1}(v)\}, \qquad \ell > 1.$$

Here $\mathrm{star}^1(v)$ is the union of all triangles in $\triangle$ that share the vertex $v$.

2) for all $\{c_\xi\}_{\xi\in\mathcal{M}}$,

$$K_1 \max_{\xi\in\mathcal{M}} |c_\xi| \le \| \sum_{\xi\in\mathcal{M}} c_\xi B_\xi \|_{L_\infty(\Omega)} \le K_2 \max_{\xi\in\mathcal{M}} |c_\xi|. \tag{2.2}$$

For a list of some commonly used spaces of bivariate polynomial splines which have stable local bases, see Sect. 3. It is known (cf. eg. [14]) that such spaces have full approximation power. The following theorem shows that when $\mathcal{V} \subseteq \mathcal{M}$, the best order of approximation is even achieved by a *linear interpolation operator*.

**Theorem 2.2.** *Suppose $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$ is a spline space with a stable local basis $\{B_\xi\}_{\xi\in\mathcal{M}}$ corresponding to a minimal determining set $\mathcal{M}$ containing the vertices $\mathcal{V}$ of $\triangle$. Let $0 \le m \le d$. Then there exists a linear operator $Q$ mapping $C(\Omega)$ onto $\mathcal{S}$ such that for every $f \in W_\infty^{m+1}(\Omega)$, the spline $Qf$ interpolates $f$ at the vertices of $\triangle$, and for every triangle $T$ in $\triangle$,*

$$\|D_x^\nu D_y^\mu (f - Qf)\|_{L_\infty(T)} \le C \frac{|T|^{m+1}}{{\rho_T}^{\nu+\mu}} |f|_{m+1,\infty,\Omega}, \tag{2.3}$$

*for all $0 \le \nu + \mu \le m+1$. Here $\rho_T$ is the radius of the incircle inscribed in $T$. The constant $C$ depends only on $d$ and the smallest angle $\theta_\triangle$ in $\triangle$ if $\Omega$ is convex, and also on the Lipschitz constant $L_{\partial\Omega}$ of the boundary of $\Omega$ if $\Omega$ is not convex.*

**Proof:** Given a triangle $T$ in $\triangle$, for all $i+j+k=d$, let $\xi_{ijk}^T$ be the associated domain points, and let $p_{ijk}^T$ be the corresponding Lagrange fundamental polynomials of degree $d$ satisfying

$$p_{ijk}^T(\xi_{\ell,\nu,\mu}^T) = \begin{cases} 1, & \text{if } (i,j,k) = (\ell,\nu,\mu) \\ 0. & \text{otherwise} \end{cases}$$

It is straightforward to check that

$$\|p_{ijk}^T\|_{\infty,T} \le d^d. \tag{2.4}$$

Then, for any function $f \in C(\Omega)$, clearly

$$I_T f := \sum_{i+j+k=d} f(\xi_{ijk}^T)\, p_{ijk}^T \tag{2.5}$$

defines a polynomial of degree $d$ which interpolates $f$ at the domain points of $T$.

Now suppose that $\mathcal{M}$ is a minimal determining set for $\mathcal{S}$ which contains $\mathcal{V}$, and let $\mathcal{B} := \{B_\xi\}_{\xi\in\mathcal{M}}$ be a corresponding stable local basis. For each $\xi$, let $T_\xi$ be the triangle that contains $\xi$. Let $\gamma_\xi$ be the functional such that for any spline $s \in \mathcal{S}_d^0(\triangle)$, $\gamma_\xi s$ is the B-cofficient associated with the domain point $\xi$. For any $f \in C(\Omega)$, let

$$\lambda_\xi f := \gamma_\xi I_{T_\xi} f.$$

Note that $\lambda_\xi$ is a linear functional, and the value of $\lambda_\xi f$ depends on values of $f$ at the domain points in the triangle $T_\xi$. Using (2.4) and Lemma 4.1 in [14], it follows that

$$\begin{aligned} |\lambda_\xi f| &= |\gamma_\xi I_{T_\xi} f| \le K_3 \|I_{T_\xi} f\|_{\infty,T_\xi} \\ &\le K_3 \sum_{i+j+k=d} |f(\xi_{ijk}^{T_\xi})|\, \|p_{ijk}^{T_\xi}\|_{\infty,T_\xi} \le K_3 \binom{d+2}{2} d^d \|f\|_{\infty,T_\xi}, \end{aligned}$$

where $K_3$ is the constant in Lemma 4.1 of [14], and depends only on $d$.

We now define the interpolation operator

$$Qf := \sum_{\xi \in \mathcal{M}} (\lambda_\xi f) \, B_\xi. \tag{2.6}$$

We claim that

$$Qf(v) = f(v), \qquad \text{for all } v \in \mathcal{V} \tag{2.7}$$

$$Qp = p, \qquad \text{for all polynomials of degree } d. \tag{2.8}$$

The interpolation property (2.7) follows immediately from (2.1) and the fact that the minimal determining set $\mathcal{M}$ contains the set of vertices $\mathcal{V}$ of $\triangle$. To establish (2.8), we show even more, namely that $Qs = s$ for all splines $s \in \mathcal{S}$. Fix a triangle $T$ in $\triangle$, and suppose $s \in \mathcal{S}$. Then for each $\xi \in \mathcal{M}$, by the uniqueness of interpolation, $I_{T_\xi} s = s$ on $T_\xi$. But then $\lambda_\xi s$ are just the B-coefficients of $s$ for all $\xi \in \mathcal{M}$. Since $\mathcal{M}$ is a minimal determining set, we conclude that all of the coefficients of $Qf$ agree with those of $s$, and hence $Qs = s$.

Now fix $(x, y)$ in a triangle $T$. By the stability of the basis $\mathcal{B}$,

$$|Qf(x,y)| \le \sum_{\xi \in \mathcal{M}} |\lambda_\xi f| B_\xi(x,y) \le K_2 \max_{\xi \in \mathcal{M}} |\lambda_\xi f| \le K_2 K_3 \binom{d+2}{2} d^d \|f\|_{\infty,\Omega}. \tag{2.9}$$

For any $0 \le m \le d$, this immediately implies

$$|Qf(x,y) - f(x,y)| \le K_4 \sum_{k=0}^{m+1} |T|^k \|f\|_{k,\infty,\Omega},$$

where $|T|$ is the diameter of $T$. Then the well-known Bramble-Hilbert lemma [1] implies

$$|Qf(x,y) - f(x,y)| \le C|T|^{m+1} \|f\|_{m+1,\infty,\Omega},$$

which establishes (2.3) for $\nu = \mu = 0$.

Fix $0 \le \nu + \mu \le m + 1$ and a triangle $T$. Applying the Markov inequality (cf. Lemma 4.2 of [14]) to the polynomial $(Qf)|_T$, we get

$$\|D_x^\nu D_y^\mu Qf\|_{\infty,T} \le \frac{K_5}{{\rho_T}^{\nu+\mu}} \|Qf\|_{\infty,T},$$

where $K_5$ depends only on $d$. Combining (2.9) with the fact that

$$\frac{|T|}{\rho_T} \le \frac{2}{\sin(\theta_\triangle/2)},$$

where $\theta_\triangle$ is the smallest angle in $T$, we get

$$|D_x^\nu D_y^\mu (Qf - f)(x,y)| \le \frac{K_6}{{\rho_T}^{\nu+\mu}} \sum_{k=0}^{m+1} |T|^k \|f\|_{k,\infty,\Omega},$$

where now $K_6$ depends on both $d$ and $\theta_\triangle$. The Bramble-Hilbert lemma then immediately implies (2.3). $\square$

## §3. Spline spaces with stable local bases

In this section we list several spline spaces to which the methods of this paper apply. All of the following have stable local bases $\{B_\xi\}_{\xi\in\mathcal{M}}$ corresponding to a minimal determining set $\mathcal{M}$ containing the set $\mathcal{V}$ of vertices of $\triangle$:

1) The spline spaces $\mathcal{S}_d^0(\triangle)$ for all $d \geq 1$. In this case we can take $\mathcal{M}$ to be the set of domain points $\mathcal{D}_{d,\triangle}$. This leads to basis splines with star$^1$-supports, and thus $\ell = 1$ in Definition 2.1.

2) The spline spaces
$$\mathcal{S}_d^r(\triangle) := \{s \in C^r(\Omega) : \; s|_T \in \mathcal{P}_d \; \text{ for all triangles } T \in \triangle\}$$
with $d \geq 3r+2$. Minimal determining sets leading to stable local bases were constructed in [3] with $\ell = 3$.

3) The superspline spaces
$$\mathcal{S}_d^{r,\rho}(\triangle) := \{s \in \mathcal{S}_d^r(\triangle) : \; s \in C^{\rho_v}(v) \; \text{ for all } v \in \mathcal{V}\},$$
with $d \geq 3r+2$ and $\rho := \{\rho_v\}_{v\in\mathcal{V}}$, where $\mathcal{V}$ is the set of all vertices of $\triangle$ and $\rho_v$ are given integers such that $r \leq \rho_v \leq d$ and $k_v + k_u < d$, where $k_v := \max\{\rho_v, r + \lfloor \frac{r+1}{2} \rfloor\}$. Stable local bases with $\ell = 3$ were constructed for these spaces in [3].

4) The spline spaces $\mathcal{S}_{d(r)}^r(\triangle_{PS})$ with
$$d(r) := \begin{cases} (9m+1)/2, & \text{if } r = 2m \text{ and } m \text{ is odd}, \\ (9m+2)/2, & \text{if } r = 2m \text{ and } m \text{ is even}, \\ (9m+4)/2, & \text{if } r = 2m+1 \text{ and } m \text{ is even}, \\ (9m+5)/2, & \text{if } r = 2m+1 \text{ and } m \text{ is odd}, \end{cases} \tag{3.1}$$
where $\triangle_{PS}$ is the Powell-Sabin refinement of an arbitrary triangulation $\triangle$, see [15]. In this case stable star-supported bases could be constructed, and so $\ell = 1$.

5) The spline spaces $\mathcal{S}_{d(r)}^r(\triangle_{CT})$ with
$$d(r) = \begin{cases} 3r+1, & r \text{ even}, \\ 3r, & r \text{ odd}, \end{cases}$$
where $\triangle_{CT}$ is the Clough-Tocher refinement of an arbitrary triangulation $\triangle$, see [16]. Again $\ell = 1$.

6) Certain other special super-spline spaces with $d \geq 3r+2$ described in [2], where $\ell = \lfloor \frac{r}{2} \rfloor$ and in [14], where $\ell = 3$.

7) The spaces $\mathcal{S}_{3r}^r(\boxtimes)$, where $\boxtimes$ is the triangulation obtained by inserting the diagonals into each quadrangle of an arbitrary quadrangulation, see [13]. Here $\ell = 1$ when $r$ is odd, and $\ell = 2$ when $r$ is even.

## §4. A Hilbert space formulation

In this section we convert the minimal energy interpolation problem (1.2) into a standard approximation problem in Hilbert space. Let

$$X := \{f \in B(\Omega) : \; f|_T \in W^2_\infty(T), \text{ all triangles } T \text{ in } \triangle\},$$

where $B(\Omega)$ is the set of all bounded real-valued functions on $\Omega$. For each triangle $T$ in $\triangle$, let

$$\langle f, g\rangle_{X_T} := \int_T \Big[ f_{xx}\, g_{xx} + 2 f_{xy}\, g_{xy} + f_{yy}\, g_{yy} \Big]. \tag{4.1}$$

Then

$$\langle f, g\rangle_X := \sum_{T\in\triangle} \langle f, g\rangle_{X_T}$$

defines a semi-definite inner-product on $X$. Let $\|f\|_{X_T}$ and $\|f\|_X$ be the associated semi-norms.

Suppose $\mathcal{S} \subseteq \mathcal{S}^0_d(\triangle)$ is a spline space on a triangulation $\triangle$, and that $\mathcal{S}$ has a stable local basis $\{B_\xi\}_{\xi\in\mathcal{M}}$ corresponding to a minimal determining set $\mathcal{M}$ containing the set of vertices $\mathcal{V}$ of $\triangle$. Then it is easy to see that $\langle\cdot,\cdot\rangle_X$ is an inner-product on the linear space

$$\mathcal{W} := \{s \in \mathcal{S} : \; s(v) = 0, v \in \mathcal{V}\}. \tag{4.2}$$

Indeed, if $\langle w, w\rangle_X = 0$ for some $w \in \mathcal{W}$, then $w$ is a piecewise linear function on $\triangle$ which vanishes at all vertices, and thus $w \equiv 0$. Since $\mathcal{W}$ is finite-dimensional, it follows that $\mathcal{W}$ equipped with the inner-product $\langle\cdot,\cdot\rangle_X$ is a Hilbert space.

Given $f$, suppose $s_f$ is any spline in the set $U_f$ defined in (1.1). Then it is easy to see that the solution $S_f$ to the minimal energy problem (1.2) is equal to $s_f - Ps_f$, where $P$ is the linear projector $P : X \to \mathcal{W}$ defined by

$$\mathcal{E}(g - Pg) = \min_{w\in\mathcal{W}} \mathcal{E}(g - w),$$

for all $g \in X$. Since $\mathcal{W}$ is a Hilbert space with respect to $\langle\cdot,\cdot\rangle_X$, $Pg$ is uniquely defined, and is characterized by

$$\langle g - Pg, w\rangle_X = 0, \qquad \text{for all } w \in \mathcal{W}.$$

Moreover, using the Cauchy-Schwarz inequality, it is easy to see that

$$\|Pg\|_X \le \|g\|_X \tag{4.3}$$

for all $g \in X$. We give more refined bounds on the projector $P$ in the following section.

## §5. Bounds on the projector $P$

To get more refined bounds on the projector $P$ defined in the previous section, we need to place some restrictions on the triangulation $\triangle$.

**Definition 5.1.** *Let $\beta < \infty$. A triangulation $\triangle$ is said to be $\beta$-*quasi-uniform *provided that $|\triangle| \le \beta\rho_\triangle$, where $|\triangle|$ is the maximum of the diameters of the triangles in $\triangle$, and $\rho_\triangle$ is the minimum of the radii of the incircles of triangles of $\triangle$.*

It is easy to prove that if $\triangle$ is $\beta$-quasi-uniform, then the smallest angle in $\triangle$ is bounded below by $2/\beta$. We now establish a lemma showing the equivalence of certain norms on the Hilbert space $\mathcal{W}$.

**Lemma 5.2.** *Suppose that $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$ is a spline space defined on a $\beta$-quasi-uniform triangulation $\triangle$. Let $\mathcal{W}$ be the subspace defined in (4.2). Then there exist constants $0 < C_1 \le C_2 < \infty$ depending only on $\beta$ and $d$ such that*

$$C_1 \int_\Omega u^2 \le |\triangle|^4 \|u\|_X^2 \le C_2 \int_\Omega u^2, \tag{5.1}$$

*for all $u \in \mathcal{W}$.*

**Proof:** Let $\mathcal{T}_\beta$ be the set of all triangles $T$ with one vertex at $(0,0)$ and $1/\beta \le \rho_T \le |T| = 1$, where $\rho_T$ is the radius of the incircle inscribed in $T$. Let

$$C_1 := \inf_{T=\langle v_1,v_2,v_3\rangle \in \mathcal{T}_\beta} \{\|p\|_{X_T}^2 : \ p \in \mathcal{P}_d, \ \textstyle\int_T p^2 = 1, \ p(v_i) = 0, \ i = 1,2,3\}.$$

Then there exist sequences $p_k, T_k$ of polynomials and triangles, respectively, such that $p_k \to p \in \mathcal{P}_d$ and $T_k \to T \in \mathcal{T}_\beta$ with $\int_T p^2 = 1$ and $C_1 = \|p\|_{X_T}^2$. We claim that $C_1 > 0$. Indeed, if $\|p\|_{X_T}^2 = 0$, then $p \in \mathcal{P}_1$. But then using the fact that $p$ vanishes at the vertices, it follows that $p \equiv 0$, contradicting $\int_T p^2 = 1$. We have shown that $C_1 > 0$ and that it depends only on $\beta$ and $d$. Now let

$$\tilde{C}_2 := \sup_{T=\langle v_1,v_2,v_3\rangle \in \mathcal{T}_\beta} \{\|p\|_{X_T}^2 : \ p \in \mathcal{P}_d, \ \textstyle\int_T p^2 = 1, \text{ and } p(v_i) = 0, \ i = 1,2,3\} < \infty,$$

Clearly, $\tilde{C}_2$ depends only on $\beta$ and $d$, and using the Markov inequality, it is easy to see that $\tilde{C}_2 < \infty$. Now if $T$ is an arbitrary triangle in $\triangle$, then after translating one vertex to $(0,0)$ and substituting $x = |T|\tilde{x}$ and $y = |T|\tilde{y}$, we see that for any $u \in \mathcal{W}$,

$$C_1 \int_T u^2 \le |T|^4 \|u\|_{X_T}^2 \le \tilde{C}_2 \int_T u^2.$$

Then summing over all triangles in $\triangle$ gives (5.1) with $C_2 := \beta^4 \tilde{C}_2$. $\square$

We now show that under appropriate conditions on $\mathcal{S}$, the $X$-norm on the Hilbert space $\mathcal{W}$ is also equivalent to a certain coefficient norm.

**Corollary 5.3.** *Suppose $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$ is a spline space defined on a $\beta$-quasi-uniform triangulation $\triangle$, and that $\{B_\xi\}_{\xi \in \mathcal{M}}$ is a stable local basis for $\mathcal{S}$ corresponding to a minimal determining set $\mathcal{M}$ containing the set $\mathcal{V}$ of vertices of $\triangle$. Then $\{B_\xi\}_{\xi \in \mathcal{N}}$ is a Riesz basis (with respect to the $X$-norm) for the linear space $\mathcal{W}$ defined in (4.2), where $\mathcal{N} := \mathcal{M} \setminus \mathcal{V}$. In particular, there exist positive constants $C_3, C_4$ depending only on $d, \ell$ and $\beta$ such that*

$$C_3 |\triangle|^{-2} \sum_{\xi \in \mathcal{N}} |c_\xi|^2 \leq \Big\| \sum_{\xi \in \mathcal{N}} c_\xi B_\xi \Big\|_X^2 \leq C_4 |\triangle|^{-2} \sum_{\xi \in \mathcal{N}} |c_\xi|^2, \tag{5.2}$$

*for all $\{c_\xi\}_{\xi \in \mathcal{N}}$.*

**Proof:** By Lemma 6.1 of [12], there exist positive constants $K_7, K_8$ depending only on $d, \ell$ and $\theta_\triangle$, such that

$$K_7 \min_{T \in \triangle} A_T \sum_{\xi \in \mathcal{N}} |c_\xi|^2 \leq \int_\Omega \Big| \sum_{\xi \in \mathcal{N}} c_\xi B_\xi \Big|^2 \leq K_8 \max_{T \in \triangle} A_T \sum_{\xi \in \mathcal{N}} |c_\xi|^2, \tag{5.3}$$

where $A_T$ is the area of $T$. Combining this with (5.1) gives (5.2) with $C_3 := \pi K_7 C_1 / \beta^2$ and $C_4 := K_8 C_2$. $\square$

The next result follows immediately by applying Theorem 2.1 of [12] to the spline space $\mathcal{W}$. Given a triangle $T$, $\mathrm{star}^0(T) = T$, and

$$\mathrm{star}^\ell(T) := \bigcup \{\mathrm{star}^\ell(w) : \ w \text{ is a vertex of } T\}, \qquad \ell \geq 1.$$

**Theorem 5.4.** *Suppose $\mathcal{W}$ is as in Corollary 5.3. Let $g$ be a function in $X$ with support on a triangle $T$ in $\triangle$, and let $\tau$ be another triangle which lies outside of $\mathrm{star}^q(T)$ for some $q \geq 1$. Then*

$$\|Pg\|_{X_\tau} \leq C_5 \sigma^q \|g\|_X, \tag{5.4}$$

*for some constants $0 < \sigma < 1$ and $C_5$ depending only on $d, \ell$ and $\beta$.*

The following result gives a bound on the projector $P$ in terms of the semi-norm defined in (1.5).

**Theorem 5.5.** *Suppose the hypotheses of Corollary 5.3 are satisfied. Then*

$$|Pg|_{2,\infty,\Omega} \leq C_6 |g|_{2,\infty,\Omega} \qquad \text{for all } g \in X, \tag{5.5}$$

*where $C_6$ depends only on $d, \ell$ and $\beta$.*

**Proof:** Let $\tau$ be a fixed triangle in $\triangle$, and let

$$\Omega_0^\tau := \mathrm{star}^1(\tau), \qquad \Omega_q^\tau := \mathrm{star}^{q+1}(\tau) \setminus \mathrm{star}^q(\tau),$$

where

$$\mathrm{star}^\ell(\tau) := \bigcup\{\mathrm{star}^\ell(w):\ w \text{ is a vertex of } \tau\}, \qquad \ell \ge 1.$$

Let $n_0$ be the number of triangles in $\mathrm{star}^1(\tau)$, and let $n_q$ be the number of triangles in $\mathrm{star}^{q+1}(\tau) \setminus \mathrm{star}^q(\tau)$.

Now suppose $s \in \mathcal{P}_d$ with $d \ge 2$. Since $s_{xx}$, $s_{xy}$, and $s_{yy}$ are polynomials of degree at most $d-2$, there exists a constant $K_9 > 0$ depending only on $d$ and the smallest angle $\theta_\triangle$ in $\triangle$ such that

$$\|s\|_{X_\tau} \ge K_9 A_\tau^{1/2} |s|_{2,\infty,\tau}, \tag{5.6}$$

where in general

$$|g|_{2,\infty,\tau} := \sum_{\nu+\mu=2} \|D_x^\nu D_y^\mu g\|_{L_\infty(\tau)}. \tag{5.7}$$

The inequality (5.6) also holds trivially for linear polynomials. Moreover, for all $g \in X$ and $T \in \triangle$,

$$\|g\|_{X_T} \le A_T^{1/2} |g|_{2,\infty,T}. \tag{5.8}$$

We now write $g = \sum_{T\in\triangle} g_T$, where $\mathrm{supp}(g_T) \subseteq T$. Since $P$ is a linear operator, by (5.6)

$$|Pg|_{2,\infty,\tau} \le \sum_{T\in\triangle} |Pg_T|_{2,\infty,\tau} \le \frac{1}{K_9 A_\tau^{1/2}} \sum_{T\in\triangle} \|Pg_T\|_{X_\tau}.$$

Then by (4.3), (5.4), and (5.8),

$$\begin{aligned}
|Pg|_{2,\infty,\tau} &\le \frac{1}{K_9 A_\tau^{1/2}} \sum_{q\ge 0} \sum_{T\in\Omega_q^\tau} \|Pg_T\|_{X_\tau} \\
&\le \frac{1}{K_9 A_\tau^{1/2}} \Big[ \sum_{T\in\Omega_0^\tau} \|g_T\|_X + \sum_{q\ge 1} \sum_{T\in\Omega_q^\tau} C_5 \sigma^q \|g_T\|_X \Big] \\
&\le \frac{\max_{T\in\triangle} A_T^{1/2}}{K_9 A_\tau^{1/2}} \Big[ n_0 + C_5 \sum_{q\ge 1} \sigma^q n_q \Big] |g|_{2,\infty,\Omega}.
\end{aligned}$$

Since $\triangle$ is a $\beta$-quasi-uniform triangulation, $n_q \le (2q+3)^2\beta^2/\pi$ by Lemma 4.2 of [12], and

$$\frac{\max_{T\in\triangle} A_T^{1/2}}{\min_{T\in\triangle} A_T^{1/2}} \le \frac{\beta}{\sqrt{\pi}}.$$

But then (5.5) follows by taking the supremum over all $\tau \in \triangle$ and all $g \in X$. $\square$

## §6. Error bounds for the minimal energy interpolant $S_f$

We begin with a technical lemma.

**Lemma 6.1.** *Let $T = \langle v_1, v_2, v_3 \rangle$ be a triangle. Suppose that $f \in W^2_\infty(T)$ and $f(v_i) = 0$ for $i = 1, 2, 3$. Then for all $v \in T$,*

$$|f(v)| \leq 12|T|^2 |f|_{2,\infty,T}. \tag{6.1}$$

**Proof:** Given $v \in T$, we can write $v = v_1 + t(v_2 - v_1) + u(v_3 - v_1)$ with $(t, u)$ in a standard triangle $S := \{(t, u), t, u \geq 0, t + u \leq 1\}$. Let $g(t, u) = f(v_1 + t(v_2 - v_1) + u(v_3 - v_1))$ for $(t, u) \in S$. By Taylor's expansion, we have

$$0 = f(v_2) = g(1, 0) = g(0, 0) + g_t(0, 0) + \frac{1}{2} g_{tt}(\xi, 0)$$

for some $\xi \in (0, 1)$. It follows that

$$|g_t(0, 0)| \leq \frac{1}{2} |g|_{2,\infty,S}.$$

Similarly, $|g_u(0, 0)| \leq \frac{1}{2} |g|_{2,\infty,S}$. Thus,

$$|f(v)| = |g(t, u)| \leq |g(0, 0)| + |g_t(0, 0)| + |g_u(0, 0)| + 2|g|_{2,\infty,S} \leq 3|g|_{2,\infty,S}.$$

Since $|g|_{2,\infty,S} \leq 4|f|_{2,\infty,T} |T|^2$, we conclude that (6.1) holds. $\square$

We are now in a position to prove the main result of this paper. Note that the following theorem applies to all of the spline spaces listed in Sect. 3.

**Theorem 6.2.** *Suppose $\mathcal{S}$ is a spline space as in Corollary 5.3, and suppose $\triangle$ is a $\beta$-quasi-uniform triangulation. Then there exists a constant $C$ depending only on $d, \ell$ and $\beta$ if $\Omega$ is convex, and also on the Lipschitz constant $L_{\partial\Omega}$ of the boundary of $\Omega$ if $\Omega$ is not convex, such that*

$$\|f - S_f\|_{L_\infty(\Omega)} \leq C|\triangle|^2 |f|_{2,\infty,\Omega}, \tag{6.2}$$

*for all $f \in W^2_\infty(\Omega)$.*

**Proof:** Given a function $f \in W^2_\infty(\Omega)$, let $s_f \in U_f$ be the spline in Theorem 2.2. Then by (2.3),

$$\|f - s_f\|_{L_\infty(\Omega)} \leq K_{10} |\triangle|^2 |f|_{2,\infty,\Omega} \tag{6.3}$$

$$|s_f|_{2,\infty,\Omega} \leq K_{11} |f|_{2,\infty,\Omega}. \tag{6.4}$$

We recall that $Ps_f = s_f - S_f$, and thus by Theorem 5.5,

$$|s_f - S_f|_{2,\infty,\Omega} = |Ps_f|_{2,\infty,\Omega} \leq C_6 |s_f|_{2,\infty,\Omega} \leq C_6 K_{11} |f|_{2,\infty,\Omega}.$$

Since $s_f(v) - S_f(v) = 0$ for all vertices $v$ of $\triangle$, by Lemma 6.1,

$$\|s_f - S_f\|_{L_\infty(\Omega)} \leq 12|\triangle|^2 |s_f - S_f|_{2,\infty,\Omega}$$

and thus

$$\|s_f - S_f\|_{L_\infty(\Omega)} \leq 12 C_6 K_{11} |\triangle|^2 |f|_{2,\infty,\Omega}.$$

But then the error bound (6.2) follows immediately from

$$\|f - S_f\|_{L_\infty(\Omega)} \leq \|f - s_f\|_{L_\infty(\Omega)} + \|s_f - S_f\|_{L_\infty(\Omega)}. \quad \square$$

## §7. Sharpness of the error bound

Theorem 2.2 shows that for smooth functions $f$ there exist interpolating splines $s_f$ which achieve full approximation power of order $d$ for sufficiently smooth functions. On the other hand, in Theorem 6.2 we only prove that the minimal energy interpolating spline has an approximation power of order 2. Based on a variety of numerical tests (see also [9,10]), we conjecture that order 2 approximation is sharp (see Remark 9.5 for a comparison with the univariate case). Here we present numerical examples based on two different spline spaces.

**Example 7.1.** *Let $\mathcal{S}$ be the $C^1$ cubic spline space $\mathcal{S}_3^1(\diamondsuit)$, where $\diamondsuit$ is the triangulation obtained by inserting diagonals into each subsquare of a uniform partition of the unit square $\Omega := [0,1] \times [0,1]$ into $N^2$ subsquares. Consider the following test functions:*

$$
\begin{aligned}
f_1(x,y) &= x^2 + y^2, \\
f_2(x,y) &= \sin(2(x-y)), \\
f_3(x,y) &= \sin(2(x^3+y^3)), \\
f_4(x,y) &= 0.75\exp(-0.25(9x-2)^2 - 0.25(9y-2)^2) \\
&\quad + 0.75\exp(-(9x+1)^2/49 - (9y+1)/10) \\
&\quad + 0.5\exp(-0.25(9x-7)^2 - 0.25(9y-3)^2) \\
&\quad - 0.2\exp(-(9x-4)^2 - (9y-7)^2);
\end{aligned}
$$

**Discussion:** Function $f_4$ is the well-known Franke test function. We approximated these functions for the choices $N = 2, 4, 8, 16, 32, 64$ which corresponds to repeatedly halving the mesh size. The following table give the maximum error $\|f - S_f\|$ computed on $101 \times 101$ equally-spaced points in $\Omega$. The second table presents the corresponding ratios of errors for successive values of $N$. For larger values of $N$, they are very close to 4 which is what we expect for order 2 convergence.

| $N$ | 2 | 4 | 8 | 16 | 32 | 64 |
|---|---|---|---|---|---|---|
| $f_1$ | 0.034150 | 0.01105 | 0.002775 | 0.0006939 | 0.0001618 | 0.00003562 |
| $f_2$ | 0.136660 | 0.02347 | 0.005278 | 0.0012800 | 0.0002934 | 0.00006520 |
| $f_3$ | 0.123500 | 0.04279 | 0.011090 | 0.0026910 | 0.0006262 | 0.00013342 |
| $f_4$ | 0.685100 | 0.08540 | 0.047889 | 0.0034511 | 0.0005555 | 0.00010538 |

| | | | | | |
|---|---|---|---|---|---|
| $f_1$ | 3.0905 | 3.9820 | 3.9991 | 4.2886 | 4.5424 |
| $f_2$ | 5.8228 | 4.4468 | 4.1234 | 4.3626 | 4.5000 |
| $f_3$ | 2.8862 | 3.8584 | 4.1211 | 4.2973 | 4.6934 |
| $f_4$ | 8.0222 | 1.7833 | 13.8812 | 6.2162 | 5.2657 |

**Example 7.2.** *Let $\mathcal{S}$ be the $C^2$ quintic spline space $S_5^2(\triangle_{PS})$ over the Powell-Sabin refinement of the uniform type-I triangulation obtained by dividing the unit square $\Omega$ into $N^2$ equal subsquares and splitting each of them into two triangles by inserting the diagonal pointing in the northeast direction. Let $f_1, f_2, f_3, f_4$ be the functions in Example 7.1.*

**Discussion:** For other experiments using this space, see [5]. As in Example 7.1, we give tables of the maximum error and the ratios of errors for successive values of $N$. For larger values of $N$, they are very close to 4 which is what we expect for order 2 convergence.

| $N$ | 2 | 4 | 8 | 16 | 32 |
|---|---|---|---|---|---|
| $f_1$ | 0.0109019 | 0.0027508 | 0.0006877 | 0.0001541 | 0.0000434 |
| $f_2$ | 0.1338269 | 0.0229111 | 0.0052113 | 0.0012650 | 0.0002804 |
| $f_3$ | 0.1213129 | 0.0420687 | 0.0109061 | 0.0026824 | 0.0005916 |
| $f_4$ | 0.6854575 | 0.0902962 | 0.0484377 | 0.0035372 | 0.0005495 |

| | | | | |
|---|---|---|---|---|
| $f_1$ | 3.963 | 3.999 | 4.461 | 3.550 |
| $f_2$ | 5.841 | 4.396 | 4.119 | 4.510 |
| $f_3$ | 2.883 | 3.857 | 4.065 | 4.533 |
| $f_4$ | 7.591 | 1.864 | 13.693 | 6.436 |

## §8. Tension splines

Let $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$ be a space of splines defined on a triangulation $\triangle$, and for all $T \in \triangle$, let $\lambda_T > 0$ be given real numbers. Then for any $s \in \mathcal{S}$, we define the associated tension energy

$$\mathcal{T}(s) = \mathcal{E}(s) + \sum_{T \in \triangle} \lambda_T \int_T [s_x^2 + s_y^2],$$

where $\mathcal{E}$ is defined in (5.1). Given $f$, let $U_f$ be the set of splines that interpolate $f$ at the vertices of $\triangle$ as defined in (1.1), and let $S_f$ be defined by

$$\mathcal{T}(S_f) = \min_{s \in U_f} \mathcal{T}(s).$$

Then we call $S_f$ a tension spline associated with the weights $\lambda_T$. We now establish the following analog of Theorem 6.2.

**Theorem 8.1.** *Let $\mathcal{S}$ be as in Theorem 6.2, and suppose $\triangle$ is a $\beta$-quasi-uniform triangulation. Then there exists a constant $C$ depending only on $d, \ell$ and $\beta$ if $\Omega$ is* *convex, and also on the Lipschitz constant $L_{\partial\Omega}$ of the boundary of $\Omega$ if $\Omega$ is not convex, such that for all $f \in W^2_\infty(\Omega)$,*

$$\|f - S_f\|_{L_\infty(\Omega)} \le C|\triangle|^2(|f|_{2,\infty,\Omega} + \lambda|f|_{1,\infty,\Omega}), \tag{8.1}$$

*where $\lambda := \max\{\lambda_T\}$.*

**Proof:** The proof is very similar to the proof of Theorem 6.2 after replacing the $X$-inner-product there by

$$\langle f, g\rangle := \sum_{T\in\triangle} \Big[\langle f, g\rangle_{X_T} + \lambda_T \int_T (f_x g_x + f_y g_y)\Big],$$

with $\langle f, g\rangle_{X_T}$ as defined in (4.1). □

## §9. Remarks

**Remark 9.1.** $L_2$ (but not $L_\infty$) error bounds for minimal energy splines where the interpolation points are not necessarily at the vertices of the triangulation can be found in [17]. Results for an arbitrary number of variables, and also with higher order energy functionals can also be found there.

**Remark 9.2.** The minimal energy spline interpolation problem can be generalized in various other ways besides the one discussed in Sect. 8. For example, it is possible to use other energy expressions and/or various forms of Hermite interpolation conditions.

**Remark 9.3.** Here we have given error bounds only in the $L_\infty$ norm, but it is not hard to obtain bounds in terms of general $p$-norms. Like all known results on the approximation power of bivariate splines (cf. [2,3,14]), our results involve constants which depend on the smallest angle in the triangulation. It is a long standing (and open) question of whether this dependence can be removed for splines of smoothness $r \ge 1$.

**Remark 9.4.** For more on the problem of estimating $L_\infty$ norms of spline projectors, see [4,11,12].

**Remark 9.5.** It is well-known that the cubic natural spline minimizes the univariate energy $\int_a^b [s''(x)]^2 dx$ among all smooth functions that interpolate given values at points $a = x_0 < \cdots < x_n = b$. It is also well-known that the full cubic spline space (with no special boundary conditions) has approximation power $\mathcal{O}(h^4)$ where $h$ is the mesh size, but the interpolating natural spline only has approximation order $\mathcal{O}(h^2)$. This loss of accuracy is due to the natural boundary conditions, and indeed the interpolating spline does exhibit $\mathcal{O}(h^4)$ accuracy in a compact subset of $[a, b]$ which stays away from the boundary. Carl de Boor suggested that the analogous situation might also hold for our minimal energy splines, and numerical experiments seem to support this conjecture. In particular, in tests on a variety of smooth functions (including those mentioned in Sect. 7), measuring the error only on a compact subset $\Omega$ of the unit square with a positive distance to the boundary leads to order $\mathcal{O}(h^4)$ accuracy. We are currently looking for a proof that this holds in general.

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