Recipe for the Cauchy-Euler Equation

The Cauchy-Euler equation looks like this:

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y \;=\; g(x).$$

The first step is to write the homogeneous proble (i.e., replace $g(x)$ with 0), and substitute $y = x^m$. This leads to the polynomial equation

$$\begin{aligned} 0 = \quad & a_n\, m(m-1)(m-2)\cdots(m-n+2)(m-n+1) \\ + \quad & a_{n-1} m(m-1)(m-2)\cdots(m-n+2) \\ + \quad & \cdots \\ + \quad & a_2 m(m-1) \\ + \quad & a_1 m \\ + \quad & a_0 \end{aligned}$$

which you should now solve, obtaining some roots $m_1, m_2, \ldots, m_n$. If those roots are distinct, and if the original problem was homogeneous (i.e., if we actually had $g(x) = 0$ in the given problem), then the solution is simply

$$y \;=\; c_1 x^{m_1} + c_2 x^{m_2} + \cdots + c_n x^{m_n}.$$

But if there are repeated roots or if the problem was nonhomogeneous, the solution is more complicated. In that case we can proceed as follows: Multiply out the polynomial equation; it will then look something like this:

$$b_n m^n + b_{n-1} m^{n-1} + \cdots + b_2 m^2 + b_1 m + b_0 = 0$$

where the b_j's are numbers that you'll have to find. (The first and last coefficients will agree with the previous polynomial — that is, you will find $b_n = a_n$ and $b_0 = a_0$ — but in general you may have $b_j \neq a_j$ for all $j = 1, 2, 3, \ldots, n-1$, so you'll have to compute those b_j's.)

Now write down this *constant-coefficient* differential equation:

$$b_n \frac{d^n y}{dt^n} + b_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + b_2 \frac{d^2 y}{dt^2} + b_1 \frac{dy}{dt} + b_0 y = g(e^t).$$

Note that this equation **differs** from the original one in these respects:

- We have to work to find the b_j's, which differ from the a_j's.
- The x's have been replaced by t's.
- It's now a constant-coefficient equation — i.e., the derivatives are no longer preceded by polynomials.
- The right side of the equation has changed: we've replaced $g(x)$ with $g(e^t)$.

We will use D as an abbreviation for $\dfrac{d}{dt}$. (Do *not* confuse that with $\dfrac{d}{dx}$.) Then the transformed problem is

$$b_n D^n y + b_{n-1} D^{n-1} y + \cdots + b_2 D^2 y + b_1 D y + b_0 y = g(e^t).$$

Solve this constant-coefficient differential equation, by methods that we've solved earlier. The solution y is a function of t, with n arbitrary constants.

Finally, substitute $\underline{t = \ln x}$ or $\underline{x = e^t}$, and the resulting function of x is the solution of the original problem.

An Example: Repeated Roots

We will solve Example 2 from page 195 of the textbook. The problem is

$$4x^2 \frac{d^2y}{dx^2} + 8x \frac{dy}{dx} + y = 0.$$

That yields the polynomial equation $4m(m-1) + 8m + 1 = 0$ which simplifies to

$$4m^2 + 4m + 1 = 0; \qquad \text{that is,} \qquad (2m+1)^2 = 0.$$

We have repeated roots, so the answer is not just $y = c_1 x^{-1/2} + c_2 x^{-1/2}$. Start from the equation

$$4m^2 + 4m + 1 = 0$$

which gives us the constant-coefficient equation

$$4 \frac{d^2y}{dt^2} + 4 \frac{dy}{dt} + y = 0.$$

I've only written that down to make it more evident what we're doing; writing that down isn't really essential for the computation. The computation proceeds from a polynomial equation that we already worked out a few steps ago:

$$(2m+1)^2 = 0$$

$$m = -1/2, \qquad m = -1/2$$

which gives us (by the usual recipe)

$$y = c_1 e^{-t/2} + c_2 t e^{-t/2}.$$

Finally, substitute $t = \ln x$ and $x = e^t$; we get

$$\boxed{y = c_1 x^{-1/2} + c_2 (\ln x) x^{-1/2}} \qquad \text{or more concisely} \qquad \boxed{y = \frac{c_1 + c_2 \ln x}{\sqrt{x}}}.$$

Check: If $y = \frac{c_1 + c_2 \ln x}{\sqrt{x}}$ then $y\sqrt{x} = c_1 + c_2 \ln x$. Differentiate both sides to get $y' x^{1/2} + \frac{1}{2} y x^{-1/2} = c_2 x^{-1}$. Multiply both sides by x to get $y' x^{3/2} + \frac{1}{2} y x^{1/2} = c_2$. Differentiate both sides of that to get $y''x^{3/2} + 2y'x^{1/2} + \frac{1}{4}yx^{-1/2} = 0$. Multiply both sides by $4x^{1/2}$ to get the original problem.

Another Example: A Nonhomgeneous Problem

We'll do Example 6 from page 198 of the textbook:

$$x^2y'' - xy' + y = \ln x.$$

To avoid any possible confusion, let's first rewrite the problem with a differentiation notation that explicitly displays the independent variable:

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \ln x.$$

The associated polynomial equation (for the homogeneous problem) is

$$m(m-1) - m + 1 = 0$$

$$m^2 - 2m + 1 = 0$$

$$(m-1)(m-1) = 0$$

$$m = 1 \qquad \text{or} \qquad m = 1$$

The transformed problem is

$$\begin{aligned}\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t \\ (D-1)^2(D-0)^0 y \;=\; 0e^{1t} + te^{0t}.\end{aligned}$$

That has general solution of the form

$$y \;=\; [a_0 + a_1 t]\, e^{1t} + [p_0 + p_1 t]\, e^{0t}$$

where a_0 and a_1 are arbitrary, but we must find p_0 and p_1. Compute:

$$\begin{array}{lcccc} y_p & = & p_0 & + & p_1 t \\ y_p' & = & p_1 & & \\ y_p'' & = & 0 & & \end{array}$$

$$y_p'' - 2y_p' + y_p \;=\; (-2p_1 + p_0) \;+\; p_1 t$$

and that last expression must equal t. So we need $p_1 = 1$ and $-2p_1 + p_0 = 0$, hence $p_0 = 2$. Thus we obtain

$$y \;=\; (a_0 + a_1 t)e^t + 2 + t.$$

Finally, substitute $t = \ln x$ and $x = e^t$; we get $\boxed{y = (a_0 + a_1 \ln x) + 2 + \ln x}$.

Why It Works (a partial explanation)

It works because of the relationship between the original variables x and y, and the auxiliary variable $t = \ln x$ that we've introduced. Note that $x = e^t$. Don't worry about whether t is a function of x or x is a function of t. Think of them as *linked quantities*: when we vary one, then the other varies too — not quite at the same rate, but at rates determined by the chain rule. Note that

$$\frac{dx}{dt} = e^t = x \qquad \text{and} \qquad \frac{dt}{dx} = \frac{1}{x} = e^{-t}.$$

Let's denote $D = \frac{d}{dt}$; that's the operator we use to solve constant-coefficient differential equations. Then what is $\frac{d}{dx}$? (It's not D.) And what is D in terms of x?

Consider any function u, which depends on x or on t. (Don't worry about whether u is a "function of" x or a "function of" t; just think of u as another quantity that varies when we vary x and t.) The chain rule tells us

$$\frac{du}{dx} \;=\; \frac{du}{dt} \cdot \frac{dt}{dx} \;=\; (Du) \cdot \frac{1}{x}.$$

Or, in other words,

$$x\frac{du}{dx} \;=\; Du.$$

Consequently, using the product rule for derivatives, we can find

$$D^2u = x\frac{d}{dx}\left(x\frac{du}{dx}\right) = x \cdot \left[1 \cdot \frac{du}{dx} + x \cdot \frac{d^2u}{dx^2}\right] = x\frac{du}{dx} + x^2\,\frac{d^2u}{dx^2}$$

and so on. In general, $D^k u$ will look like the kinds of terms that appear in the left side of a Cauchy-Euler equation — they will be a sum of constants times terms of the form $x^k \frac{d^k u}{dx^k}$. But the constants get changed by this transformation. A full description of how the constants get changed would take longer; we'll skip that part.

Still Harder Problems

Example 5 on page 197 requires techniques that we haven't discussed yet.