HOMEWORK 1, DUE TUESDAY FEBRUARY 9TH

1. Show that a matrix $T \in \mathbb{M}_n(\mathbb{C})$ is normal if and only if there is a polynomial with complex coefficients f of degree at most n such that $T^* = f(T)$.

2. $C_r^*(\mathbb{Z})$ is an abelian, unital C^*-algebra, hence we know it is isomorphic to $C(K)$ for some compact Hausdorff space K. Find a simple description of K and describe the corresponding isomorphism induced by the Gelfand transform from $C_r^*(\mathbb{Z})$ to $C(K)$.

3. Let $\{\delta_k\}_{k\in\mathbb{Z}}$ be the usual orthonormal basis for $\ell^2\mathbb{Z}$. Let $u : \mathbb{Z} \to \mathcal{U}(\ell^2\mathbb{Z})$ be the left regular unitary representation of $\mathbb{Z}$ i.e., $u_j(\Sigma_{k\in\mathbb{Z}}\alpha_k\delta_k) = \Sigma_{k\in\mathbb{Z}}\alpha_k\delta_{k+j}$. Given an operator $x \in C_r^*(\mathbb{Z})$ define the sequence $\{c_k\}_{k\in\mathbb{Z}}$ by $c_k = \langle x\delta_0, \delta_k\rangle$.

Define s_n to be the operator on $\ell^2\mathbb{Z}$ given by $s_n = \Sigma_{k=-n}^{n} c_k u_k$.

Define σ_m to be the operator on $\ell^2\mathbb{Z}$ given by $\sigma_n = \frac{1}{m+1}\Sigma_{n=0}^{m} s_n$.

(a). Show that if $y \in C_r^*(\mathbb{Z})$ such that $\langle y\delta_0, \delta_k\rangle = c_k$, for all $k \in \mathbb{Z}$ then $y = x$.

(b). Show that if $\{c_k\}_{k\in\mathbb{Z}}$ is absolutely summable then $\|s_n - x\| \to 0$.

(c). Show that we have $\|\sigma_n - x\| \to 0$ even if $\{c_k\}_{k\in\mathbb{Z}}$ is not absolutely summable.