# The Dehn Function and a Regular Set of Normal Forms for R. Thompson's Group $F$

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**Abstract**

We show that the group $F$ discovered by Richard Thompson in 1965 has a subexponential upper bound for its Dehn function. This disproves a conjecture by S. Gersten. We also prove that $F$ has a regular terminating confluent presentation.

In this paper we study the group, denoted by $F$, which is defined by the following presentation:

$$< x_0, x_1, x_2, \ldots \mid x_j^{x_i} = x_{j+1},\ i < j > . \tag{1}$$

(By definition, $a^b = b^{-1}ab$.) This group was discovered by Richard Thompson in 1965. It was rediscovered in 1977-1979 by Dydak, Freyd and Heller in their work on homotopy idempotents. Since 1965 the group $F$ has arisen in many group theoretic and topological investigations. It was proved, in particular, that:

- $F$ has a presentation with 2 generators $x_0, x_1$ and two relations.
- $F$ has a solvable word problem.
- $F$ does not satisfy any non-trivial identities but does not contain free subgroups of rank ¿ 1 and is not elementary amenable.
- $F$ is the group of orientation preserving piecewise linear homeomorphisms from the closed unit interval to itself that are differentiable except at finitely many dyadic rational numbers and such that all slopes are powers of 2.
- $F$ is isomorphic to the group of all piecewise integral projective homeomorphisms of the unit interval.
- $F$ is the diagram group of the presentation $< x \mid x^2 = x >$ of the trivial group.

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For the history of the group $F$ and for the proofs of these and other results about the group $F$ see [MT, CFP, BS, BG, Br, Kil, GbS].

Let us recall the definition of the isoperimetric function of a group presentation. We do not assume that the number of generators or the number of defining relations is finite. Let

$$H = < a_1, a_2, \dots \mid R_1, R_2, \dots > \tag{2}$$

be a presentation of the group $H$. Denote by $\mathcal{F}$ the free group over $a_1, a_2, \dots$ and let $N$ be the normal closure of the defining relations $R_1, R_2, \dots$. Thus $H = \mathcal{F}/N$. For any word $w \in N$ we consider the smallest number $k = k(w)$ such that $w$ is equal in the free group $\mathcal{F}$ to a product of the form

$$U_1^{-1} R_{i_1}^{\pm 1} U_1 U_2^{-1} R_{i_2}^{\pm 1} U_2 \dots U_k^{-1} R_{i_k}^{\pm 1} U_k$$

where $U_i$ are elements of $\mathcal{F}$. In other words $k(w)$ is the minimal number of applications of relations $R_i$ needed to reduce $w$ to the empty word. The function $p(n)$ is called an *isoperimetric function* for the presentation (2) if $p(n) \geq k(w)$ for every word $w \in N$ of length $\leq n$. Note that if the set of generators is infinite then the presentation (2) may not have an everywhere defined isoperimetric function because in this case the set of words of any given positive length is infinite. If the set of generators in (2) is finite, then isoperimetric functions exist. The smallest isoperimetric function for a group presentation (2) is called the *Dehn function* of this presentation.

Let us define a partial order $\preceq$ on functions from the set of natural numbers $\mathbf{N}$ into itself. By definition $p \preceq q$ means that there are integer constants $C_1, C_2, C_3$ such that $p(n) \leq C_1 q(C_2 n) + C_3 n$ for all $n$. This order induces an equivalence relation $\simeq$ on functions from $\mathbf{N} \to \mathbf{N}$: $p \simeq q$ if and only if $p \preceq q$ and $q \preceq p$.

The following result is well-known (see [MO, Ger1, BMS]): if $H$ is a finitely presented group and $p, q$ are Dehn functions for two finite presentations of $H$, then $p \simeq q$. Thus for a finitely presented group $H$ there exists a unique (up to the equivalence relation $\simeq$) Dehn function $p(n)$. Notice that infinite presentations of the same group may have different (non-equivalent) Dehn functions.

Gersten conjectured in [Ger2] that the Dehn function of $F$ is exponential and presented an argument to justify this conjecture.

The main goal of the paper is to prove the following result.

**Theorem 1** *The Dehn function $\Phi(n)$ of the R. Thompson group $F$ is strictly subexponential, namely $\Phi(n) \preceq 2^{(\log_2 n)^2}$.*

As far as the lower bounds for $\Phi(n)$ are concerned, we know only that $n^2 \preceq \Phi(n)$. Indeed $F$ contains the free Abelian group with 2 generators [CFP] so it is not hyperbolic [Gro], and thus the Dehn function must exceed $Cn^2$ for some constant $C$ (see [Gro], [1]). It is unknown whether or not $F$ has a polynomial or even quadratic Dehn function.

The plan of the proof of Theorem 1 is the following. First we prove that the presentation (1) has a quadratic isoperimetric function. This means that every word $w$ of length $\leq n$ which is equal to 1 in $F$ can be reduced to the empty word in a quadratic (as a function of $n$) number of applications of relations from (1). Then we shall prove that each application of a relation (1) in this reduction process is equivalent to a sequence of at most $O(2^{(\log_2 n)^2})$ applications of relations from a finite presentation of $F$ .

In order to prove that the presentation (1) has a quadratic isoperimetric function we need a set of normal forms for elements of $F$. One set of normal forms in generators $x_0, x_1, x_2, \ldots$ is well known (see [BG], [CFP]). Each word over the alphabet $\{x_0, x_1, x_2, \ldots\}$ is equal in $F$ to a unique word of the form

$$x_{i_1} x_{i_2} \ldots x_{i_k} x_{j_l}^{-1} \ldots x_{j_2}^{-1} x_{j_1}^{-1}, \tag{3}$$

where $k, l \geq 0$, $i_1 \leq \ldots i_k \neq j_l \geq \ldots j_1$, and if both $x_i$ and $x_i^{-1}$ occur in this product then either $x_{i+1}$ or $x_{i+1}^{-1}$ also occurs. It is possible to use these normal forms in the proof of Theorem 1. Nevertheless we introduce a new set of normal forms. One of the reasons is that these normal forms make the proof of our Theorem 1 easier. Another, more important, reason is that these normal forms allowed us to construct a regular terminating and confluent presentation for $F$ and to find a regular set of normal forms for elements of $F$ in generators $x_0, x_1$. Thus we prove the following result.

**Theorem 2.** *The group $F$ admits a regular terminating confluent presentation and a regular set of normal forms in generators $x_0, x_1$.*

Notice that it is not known whether or not $F$ admits a finite terminating confluent presentation (this question was raised by D. Cohen in [Coh]), but regular confluent presentations are almost as good as the finite ones.

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# 1 Rewriting systems

In this section we first recall some general and well known facts about (string) rewriting systems (for more information see [DJ] or [Sq]). Then we construct a suitable rewriting system for $F$ which leads to some new normal forms for elements of $F$ in generators $x_0, x_1, x_2, \ldots$ .

Let $\Sigma$ be an alphabet. The sequences of elements of $\Sigma$ (including the empty sequence) form a monoid under concatenation. This monoid is called the *free monoid over* $\Sigma$; it is denoted by $\Sigma^*$. The elements of this monoid are called *words*; the symbol 1 denotes the *empty word* (that is the empty sequence). By a *rewriting system over* $\Sigma$ we mean a subset $R$ of $\Sigma^* \times \Sigma^*$. The elements of $R$ are called the *rewriting rules*. We shall write $U \to V$ instead of $(U, V)$. A rewriting system $R$ is called *regular* if $\Sigma$ is finite and the left parts of the rules from $R$ form a regular language (for the definition of regular languages see, for example, [Eil]). We can consider a rewriting system as a presentation of a monoid. If the rewriting system is regular, we call the presentation *regular* also.

By definition, we write $X \to Y$ for words $X, Y$ if there are words $Z_1, Z_2$ and a rule $(U \to V) \in R$ such that $X = Z_1 U Z_2,\ Y = Z_1 V Z_2$. We denote the reflexive transitive closure of the relation $\to$ by $\Rightarrow$. The reflexive symmetric and transitive closure of $\to$ will be denoted by $\Leftrightarrow$. If two words are in this relation, we call them *R-equivalent*. It is clear that $\Leftrightarrow$ is a congruence and $\mathcal{F}/\Leftrightarrow$ is the monoid presented by $R$.

The rewriting system $R$ is *terminating*, if there are no infinite sequences of words of the form $X_1 \to X_2 \to \ldots$. We say that $R$ is *confluent* if for any words $X, Y, Z$ such that $X \Rightarrow Y$ and $X \Rightarrow Z$ there exists a word $W$ such that $Y \Rightarrow W, Z \Rightarrow W$.

Suppose that $R$ satisfies the following two conditions:

1. If $(XY, P), (YZ, Q) \in R$, where $Y$ is nonempty, then there exists a word $W$ such that $PZ \Rightarrow W, XQ \Rightarrow W$;

2. if $(XYZ, P), (Y, Q) \in R$, then there exists a word $W$ such that $P \Rightarrow W, XQZ \Rightarrow W$.

In this case we say that the system $R$ is *locally confluent*. Obviously each confluent system satisfies these conditions. The following fact is well known (see, for example, [DJ]).

**Lemma 1** (Newman, [New]) *Suppose that $R$ is a terminating rewriting system. If $R$ is locally confluent, then it is confluent.*

The word $X$ is *reduced* for $R$ (or, simply, reduced) if there is no word $Y$ such that $X \to Y$. We say that the word $Z$ is a *reduced form* of the word $X$, if $X \Rightarrow Z$ and $Z$ is reduced. It is easy to see that in the case when $R$ is terminating and confluent each element has exactly one reduced form. Moreover, in this case two words are $R$-equivalent if and only if they have the same reduced forms (see [DJ] for the details). This shows that if we have a terminating confluent rewriting system $R$ which defines the quotient monoid $M = \Sigma^*/\Leftrightarrow$, then we have a nice solution to the word problem in $M$. If in addition the rewriting system $R$ is regular then the set of reduced forms is a regular language because it consists of those and only those words which do not contain left sides of the rules from $R$ (see [Eil]).

Now we present a rewriting system for $F$. Let $X^{\pm 1}$ be the infinite alphabet $\{x_0, x_0^{-1}, x_1, x_1^{-1}, \ldots\}$. Consider the following rewriting system $R$ over this alphabet:

1. $x_i x_i^{-1} \to 1$ for all $i \geq 0$;

2. $x_i^{-1} x_i \to 1$ for all $i \geq 0$;

3. $x_j x_i \to x_i x_{j+1}$ for all $i < j$;

4. $x_j^{-1} x_i \to x_i x_{j+1}^{-1}$ for all $i < j$;

5. $x_{j+1} x_i^{-1} \to x_i^{-1} x_j$ for all $i < j$;

6. $x_{j+1}^{-1} x_i^{-1} \to x_i^{-1} x_j^{-1}$ for all $i < j$.

**Lemma 2** *The system $R$ defined above is terminating and confluent.*

**Proof.** Let us check that the system $R$ is terminating. Obviously, if $Y \to Z$, then either $Y, Z$ have the same length or $Z$ is shorter. With every word $W = x_{j_1}^{\beta_1} x_{j_2}^{\beta_2} \ldots x_{j_n}^{\beta_n}$ where $\beta_i = \pm 1$ we associate the following vector $\hat{W} = (j_1, j_2, \ldots, j_n, \beta_1, \ldots, \beta_n)$. These vectors correspond bijectively with $W$. Each application of the rules $1 - 6$ either makes this vector shorter or makes it smaller in the lexicographic order. Thus if $W \to W_1$ then $\hat{W}$ is strictly bigger than $\hat{W}_1$ in the ShortLex order. (Recall that under this order the vectors are compared first by length and then lexicographically if the lengths are equal.) It is well known [DJ] that the ShortLex order satisfies the descending chain condition. This implies that there cannot be an infinite sequence $W \to W_1 \to W_2 \to \dots$ because otherwise we would have an infinite ShortLex descending sequence of vectors $\hat{W} > \hat{W}_1 > \hat{W}_2 > \dots$. Thus our rewriting system $R$ is terminating.

Now we check that $R$ is locally confluent.

It is easy to check that $R$ satisfies the second condition of the definition of the property to be locally confluent. Indeed if we have rules $(XYZ, P), (Y, Q) \in R$, then obviously $X = Z = 1, P = Q$.

Now let us check the first condition. Take $(XY, P), (YZ, Q)$ from $R$, where $Y$ is nonempty. We see that $X, Y, Z$ are letters. The word $XYZ$ may have one of the following 18 forms (i¡j¡k):

$$x_i x_i^{-1} x_i,\ x_i^{-1} x_i x_i^{-1};$$

$$x_j^{-1} x_j x_i,\ x_j x_j^{-1} x_i,\ x_{j+1}^{-1} x_{j+1} x_i^{-1},\ x_{j+1} x_{j+1}^{-1} x_i^{-1},$$

$$x_j x_i x_i^{-1},\ x_j^{-1} x_i x_i^{-1},\ x_{j+1} x_i^{-1} x_i,\ x_{j+1}^{-1} x_i^{-1} x_i;$$

$$x_k x_j x_i,\ x_{k+1} x_{j+1} x_i^{-1},\ x_{k+1} x_j^{-1} x_i,\ x_{k+2} x_{j+1}^{-1} x_i^{-1},$$

$$x_k^{-1} x_j x_i,\ x_{k+1}^{-1} x_{j+1} x_i^{-1},\ x_{k+1}^{-1} x_j^{-1} x_i,\ x_{k+2}^{-1} x_{j+1}^{-1} x_i^{-1}.$$

It is trivial to check each of these cases, so we will check only one of them as an example. Let us consider the case when $XYZ = x_{k+2} x_{j+1}^{-1} x_i^{-1}$. Here $P = x_{j+1}^{-1} x_{k+1}$, $Q = x_i^{-1} x_j^{-1}$. We have to find a word $W$ such that $PZ \Rightarrow W$ and $XQ \Rightarrow W$. To find this word, we will only reduce the words $PZ$ and $XQ$, and it will be obvious that they can be reduced to the same word $W$. We have:

$$PZ = x_{j+1}^{-1} x_{k+1} x_i^{-1} \to x_{j+1}^{-1} x_i^{-1} x_k \to x_i^{-1} x_j^{-1} x_k = W;$$

$$XQ = x_{k+2} x_i^{-1} x_j^{-1} \to x_i^{-1} x_{k+1} x_j^{-1} \to x_i^{-1} x_j^{-1} x_k = W.$$

All other cases can be checked analogously (most of them are even easier).

The quotient monoid $(X^{\pm 1})^* / \Leftrightarrow$ is a group because of the rules 1, 2. If we write each rule 3 – 6 in the form $U = V$ instead of $U \to V$ we obtain a presentation which is obviously equivalent to the presentation (1) for $F$.

Now we can describe the new set of normal forms for $F$.

**Lemma 3** *Every word $w$ over the alphabet $X^{\pm 1}$, where $X = \{x_0, x_1, \dots\}$ is equal in the group $F$ to a unique word $\bar{w}$ of the following form:*

$$x_{i_1}^{\delta_1} x_{i_2}^{\delta_2} \dots x_{i_m}^{\delta_m}, \tag{4}$$

*where $m \geq 0$, $i_1, \dots, i_m \geq 0$, $\delta_1, \dots, \delta_m = \pm 1$ such that for any $j$, $1 \leq j < m$ one of the following three conditions is true: (a) $i_j < i_{j+1}$ or (b) $i_j = i_{j+1}$ & $\delta_j = \delta_{j+1}$ or (c) $i_j = i_{j+1} + 1$ & $\delta_{j+1} = -1$.*

**Proof.** Two words are equal in $F$ if and only if they have the same reduced form over $R$ which is defined above by the rules 1 – 6. A word is reduced if and only if it does not contain any left side of any rule. But this condition means that the word has a form (4), where for each $1 \le j < m$ one of the conditions (a)–(c) holds. So each word $w$ over $X^{\pm 1}$ is equal in $F$ to exactly one word of the form (4).

This word $\bar{w}$ will be called the *normal form* of $w$ (in $F$). We shall use these normal forms in the proof of Theorem 1.

**Remark.** Unlike the well known set of normal forms (3), the set of normal forms defined in Lemma 3 is 2-testable [Eil]: in order to check whether a given word over $X^{\pm 1}$ is a normal form it is enough to check all subwords of length 2. Another advantage of our set of normal forms is that it is a set of reduced forms for a simple rewriting system. One can also get normal forms from (3) as reduced forms for some rewriting system but this rewriting system won't be as simple.

# 2 Proof of Theorem 2

Relations (1) show that for every $i > 0$ the element $x_i$ is equal in $F$ to $x_0^{-i+1} x_1 x_0^{i-1}$. Let us replace each $x_i$ in the rules 1 – 6 from the previous section by $x_0^{-i+1} x_1 x_0^{i-1}$. It is easy to check that after some cancellations and removing rules which trivially follow from the other rules we get the following rewriting system over the alphabet $\{x_0, x_1, x_0^{-1}, x_1^{-1}\}$:

1. $x_0 x_0^{-1} \to 1$
2. $x_0^{-1} x_0 \to 1$
3. $x_1^{-1} x_1 \to 1$
4. $x_1 x_1^{-1} \to 1$
5. $x_1 x_0^i x_1 \to x_0^i x_1 x_0^{-i-1} x_1 x_0^{i+1}$
6. $x_1^{-1} x_0^i x_1 \to x_0^i x_1 x_0^{-i-1} x_1^{-1} x_0^{i+1}$
7. $x_1 x_0^{i+1} x_1^{-1} \to x_0^{i+1} x_1^{-1} x_0^{-i} x_1 x_0^i$
8. $x_1^{-1} x_0^{i+1} x_1^{-1} \to x_0^{i+1} x_1^{-1} x_0^{-i} x_1^{-1} x_0^i$

Here $i$ is an arbitrary positive integer.

By construction this rewriting system defines the group $F$. It is clear that the left sides of these rules form a regular language. Thus we have a regular presentation of $F$. In order to check that this presentation is locally confluent it is enough to check 20 words of the form $XYZ$ from the first condition from the definition of locally confluent systems. (Note that it is not necessary to check the second condition from this definition since no left sides of the rules contain each other.) We will illustrate this checking by two examples. All other cases are quite analogous and so we leave the checking of them to the reader as an exercise.

a) Let us consider an overlap of the left sides of rules 6 and 4. Namely, take $XY \equiv x_1x_0^ix_1$, $YZ \equiv x_1x_1^{-1}$, that is, $Y \equiv x_1$, $X \equiv x_1x_0^i$, $Z \equiv x_1^{-1}$. Here $P \equiv x_0^ix_1x_0^{-i-1}x_1x_0^{i+1}$, $Q \equiv 1$. Now $PZ \equiv x_0^ix_1x_0^{-i-1}(x_1x_0^{i+1}x_1^{-1}) \to x_0^ix_1x_0^{-i-1}x_0^{i+1}x_1^{-1}x_0^{-i}x_1x_0^i \Longrightarrow x_1x_0^i$ (we applied rule 7 and several free cancellations $1-4$); $XQ \equiv x_1x_0^i$. It remains to take $x_1x_0^i$ as $W$.

b) Now we consider an overlap of rules 7 and 8:

$$XY \equiv x_1x_0^{i+1}x_1^{-1},$$

$$YZ \equiv x_1^{-1}x_0^{i+1}x_1^{-1},$$

that is,

$$Y \equiv x_1^{-1}, X \equiv x_1x_0^{i+1}, Z \equiv x_0^{i+1}x_1^{-1}, P \equiv x_0^{i+1}x_1^{-1}x_0^{-i}x_1x_0^i, Q \equiv x_0^{i+1}x_1^{-1}x_0^{-i}x_1^{-1}x_0^i.$$

Now we have

$$\begin{aligned}
&PZ \equiv x_0^{i+1}x_1^{-1}x_0^{-i}(x_1x_0^{2i+1}x_1^{-1}) \to x_0^{i+1}x_1^{-1}x_0^{-i}x_0^{2i+1}x_1^{-1}x_0^{-2i}x_1x_0^{2i} \Longrightarrow\\
&x_0^{i+1}(x_1^{-1}x_0^{i+1}x_1^{-1})x_0^{-2i}x_1x_0^{2i} \to x_0^{2i+2}x_1^{-1}x_0^{-i}x_1^{-1}x_0^{-i}x_1x_0^{2i};\\
&XQ \equiv (x_1x_0^{2i+2}x_1^{-1})x_0^{-i}x_1^{-1}x_0^i \to x_0^{2i+2}x_1^{-1}x_0^{-2i-1}x_1x_0^{2i+1}x_0^{-i}x_1^{-1}x_0^i \Longrightarrow\\
&x_0^{2i+2}x_1^{-1}x_0^{-2i-1}(x_1x_0^{i+1}x_1^{-1})x_0^i \to x_0^{2i+2}x_1^{-1}x_0^{-2i-1}x_0^{i+1}x_1^{-1}x_0^{-i}x_1x_0^{2i} \Longrightarrow\\
&x_0^{2i+2}x_1^{-1}x_0^{-i}x_1^{-1}x_0^{-i}x_1x_0^{2i}.
\end{aligned}$$

The last word can be regarded as $W$.

Now let us prove that our rewriting system is terminating. Every word $W$ in $x_0^{\pm1}, x_1^{\pm1}$ has a unique decomposition of the form $W = v_0x_1^{e_1}v_1\ldots x_1^{e_n}v_n$, where $e_i$ are $\pm1$ and all $v_i$ are products of $x_0$ and $x_0^{-1}$. Denote by $a_i$ the sum of exponents for $v_i$. Now we associate with $W$ the following vector of length $n$: $h(W) = (h_1, \ldots, h_n)$, where $h_j$ is the maximum of the integers $0, a_j, a_j+a_{j+1}, \ldots, a_j+a_{j+1}+\ldots+a_n$. Thus all $h_j$ are nonnegative. Now let us introduce the (partial) order $<$ on the set of words in $x_0^{\pm1}, x_1^{\pm1}$. We write $X < Y$ if (a) $h(X) < h(Y)$ in the ShortLex order or (b) $h(X) = h(Y)$, but $|X| < |Y|$. It is obvious that this partial order satisfies the descending chain condition.

Now let us find out what happens when we apply one of the rules of our rewriting system and go from $W$ to $W'$. If we apply one of the two rules which delete $x_1^{\pm1}x_1^{\mp1}$, then we decrease the length of the vector $h$. Suppose we delete a word of the form $x_0^{\pm1}x_0^{\mp1}$. In this case the vector $h$ will be the same, but the length of the word decreases. In both cases $W' < W$. Finally, let us apply one of the rules $5-8$. Suppose that the rule touches the $j$-th and the $j+1$-th occurrences of the letter $x_1^{\pm1}$ in $W$. Then the word $v_j$ must be equal to $x_0^{a_j}$ where $a_j > 0$ if we apply one of the rules 5 or 6 and $a_j > 1$ if we apply one of the rules 7 or 8. Then it is easy to check that $h(W)$ and $h(W')$ will have the same components except for $j$-th and $j+1$-th. Recall that the $j$-th component of $h(W)$ is equal to the maximum of the numbers $0, a_j, a_j + a_{j+1}, \ldots, a_j + \ldots + a_n$. Suppose that we applied one of the rules 5 or 6. Then the $j$-th component of $W'$ will be equal to the maximal of the numbers $0, -a_j-1, a_{j+1}, a_{j+1}+a_{j+2}, \ldots, a_{j+1}+\ldots+a_n$ which is strictly smaller than the $j$-th component of $W$ because $a_j > 0$. Now suppose that we applied one of the rules 7 or 8. Then the $j$-th component of $h(W')$ is equal to the maximal of the numbers $0, -a_{j+1}, a_{j+1}, \ldots, a_{j+1} + \ldots + a_n$ and again it is strictly smaller than the $j$-th component of $h(W)$. Thus $h(W')$ is smaller than $h(W)$ in the ShortLex order. We can conclude that $W'$ is always smaller than $W$ in our order $<$. This completes the proof of Theorem 2.

# 3 Proof of Theorem 1

First of all we shall prove that the isoperimetric function for the presentation (1) is quadratic.

**Lemma 4** *Let $W$ be a word over the alphabet $X^{\pm 1} = \{x_0, x_0^{-1}, x_1, x_1^{-1}, \ldots\}$ and let $n$ be the length of $W$. We need no more than $(n-1)n/2$ applications of defining relations from (1) to reduce $W$ to its normal form (in the sense of Lemma 3). In particular, the presentation (1) has an isoperimetric function $f(n)$, which satisfies the inequality $f(n) < n^2$.*

**Proof.** We use induction on $n$. If $W$ is empty or has length 1, the result is obvious. Suppose that we have proved the result for all words $W$ of length $n$. Let us prove this fact for all words of the form $Wx_i^\delta$, where $\delta = \pm 1$. The word $W$ can be reduced to its normal form $V$ in $\leq (n-1)n/2$ steps. Therefore we will reduce the word $Wx_i^\delta$ to $Vx_i^\delta$ in the same number of steps. The word $V$ has the form

$$x_{i_1}^{\delta_1} x_{i_2}^{\delta_2} \ldots x_{i_m}^{\delta_m},$$

where $m \geq 0$, $i_1, \ldots i_m \geq 0$, $\delta_1, \ldots, \delta_m = \pm 1$ such that for any $j$, $1 \leq j < m$, one of the following three conditions holds: $(a)$ $i_j < i_{j+1}$ or $(b)$ $i_j = i_{j+1}$ & $\delta_j = \delta_{j+1}$ or $(c)$ $i_j = i_{j+1} + 1$ & $\delta_{j+1} = -1$. Note that $m \leq n$ because the reduction of a word to its normal form does not increase the length.

Suppose that $\delta = 1$. Then we choose the minimal number $k$ between 0 and $m$ such that all subscripts $i_{k+1}, \ldots, i_m$ are greater than $i$. Applying rewriting rules of the form $x_j^{\pm 1} x_i \to x_i x_{j+1}^{\pm 1}$ (here $i < j$) $m - k$ times we obtain the following word:

$$U = x_{i_1}^{\delta_1} \ldots x_{i_k}^{\delta_k} x_i x_{i_{k+1}+1}^{\delta_{k+1}} \ldots x_{i_m+1}^{\delta_m}.$$

Notice that $m - k \leq n$. Now $i_k \leq i$ and the word $U$ is not a normal form only if $i_k = i$ and $\delta_k = -1$. If this is the case we delete the subword $x_{i_k}^{\delta_k} x_i = x_i^{-1} x_i$ from $U$. This is a reduction of the word in the free group so we do not apply defining relations of $F$. The result will be a normal form since $i_{k-1} \leq i_k + 1 = i + 1 < i_{k+1} + 1$.

Now suppose that $\delta = -1$. Choose the minimal number $k$ between 0 and $m$ such that all subscripts $i_{k+1}, \ldots, i_m$ are greater than $i + 1$. Applying rewriting rules of the form $x_{j+1}^{\pm 1} x_i^{-1} \to x_i^{-1} x_j^{\pm 1}$, $(i < j)$, $m - k$ times we obtain the word

$$U = x_{i_1}^{\delta_1} \ldots x_{i_k}^{\delta_k} x_i^{-1} x_{i_{k+1}-1}^{\delta_{k+1}} \ldots x_{i_m-1}^{\delta_m}.$$

Again notice that $m - k$ does not exceed $n$. Since $i_k \leq i + 1$, the word $U$ will not be a normal form only if $i_k = i$ and $\delta_k = 1$. In this case we delete from $U$ the subword $x_{i_k}^{\delta_k} x_i^{-1} = x_i x_i^{-1}$. The result will be a normal form since $i_{k-1} \neq i + 1$, otherwise we have a subword $x_{i+1} x_i$ in the reduced word $V$. Thus, $i_{k-1} \leq i < i_{k+1} - 1$.

In both cases we need no more than $(n-1)n/2+n = n(n+1)/2$ steps, as desired. This completes the proof.

Now let us consider the Dehn function of a finite presentation of $F$.

Take two formal group generators $y_0, y_1$. Let us define a word $y_m$ by induction on $m \geq 2$:

$$y_m = y_{m-1}^{y_0}. \tag{5}$$

Now all $y_i, i \geq 0$, are words in $y_0^{\pm 1}, y_1^{\pm 1}$. Consider the following presentation:

$$< \; y_0, y_1 \,|\, y_2^{y_1} = y_3, y_3^{y_1} = y_4 \; > . \tag{6}$$

It is well known that this is a presentation for the group $F$ (see [BG], [CFP]). The fact that the group given by (6) is isomorphic to $F$ follows from the fact that for any $0 < i < j$ the relation

$$y_j^{y_i} = y_{j+1} \tag{7}$$

holds in the group given by (6). Indeed this fact implies that the correspondence $x_i \leftrightarrow y_i$ induces an isomorphism between groups given by (1) and (6).

The following lemma gives us some upper bounds for the number of applications of relations (6) required in order to deduce a relation from (7).

**Lemma 5** *Let us define the sequence $f(n)$, where*

$$f(1) = f(2) = 1; \tag{8}$$

$$f(2n+1) = f(2n) + 4f(n), \quad n \geq 1; \tag{9}$$

$$f(2n) = f(2n-1) + 2(f(n) + f(n-1)), \quad n \geq 2. \tag{10}$$

*Then for any $0 < i < j$ the relation $y_j^{y_i} = y_{j+1}$ can be deduced from the two defining relations of presentation (6) in $f(j-i)$ steps.*

**Proof.** Let us denote the relation $y_{m+1}^{y_1} = y_{m+2}$ by $R_m$. We will show by the induction on $m$ that this relation may be deduced in $f(m)$ steps. We shall often use the fact that the number of steps needed to deduce the relation $y_{m+k+1}^{y_{k+1}} = y_{m+k+2}$ for any $k \geq 0$ is the same as the number of steps to deduce the relation $R_m$ because this relation is obtained by conjugating $R_m$ by $y_0^k$. This implies, in particular, that it is enough to prove that one can deduce $R_m$ in $f(m)$ steps. This fact is obvious for $m = 1, 2$ because $R_1, R_2$ are our defining relations. Now let us consider two cases.

Case 1: $m = 2n+1, \quad n \geq 1$.

Here is the deduction of $R_m$ with some commentaries and the calculation of the number of steps.

$$y_{m+1}^{y_1} = y_{2n+2}^{y_1} = y_{2n+1}^{y_{n+1}y_1}$$

here we used the fact that $y_{2n+2} = y_{2n+1}^{y_{n+1}}$; by the induction hypothesis this transition requires no more than $f(n)$ steps

$$= y_{2n+1}^{y_1 y_{n+1}^{y_1}}$$

we used the relation $y_{n+1}y_1 = y_1 y_{n+1}^{y_1}$ which holds in the free group:

$$= y_{2n+1}^{y_1 y_{n+2}} = (y_{2n+1}^{y_1})^{y_{n+2}}$$

we used the fact that $y_{n+2} = y_{n+1}^{y_1}$; this transition requires at most $2f(n)$ steps

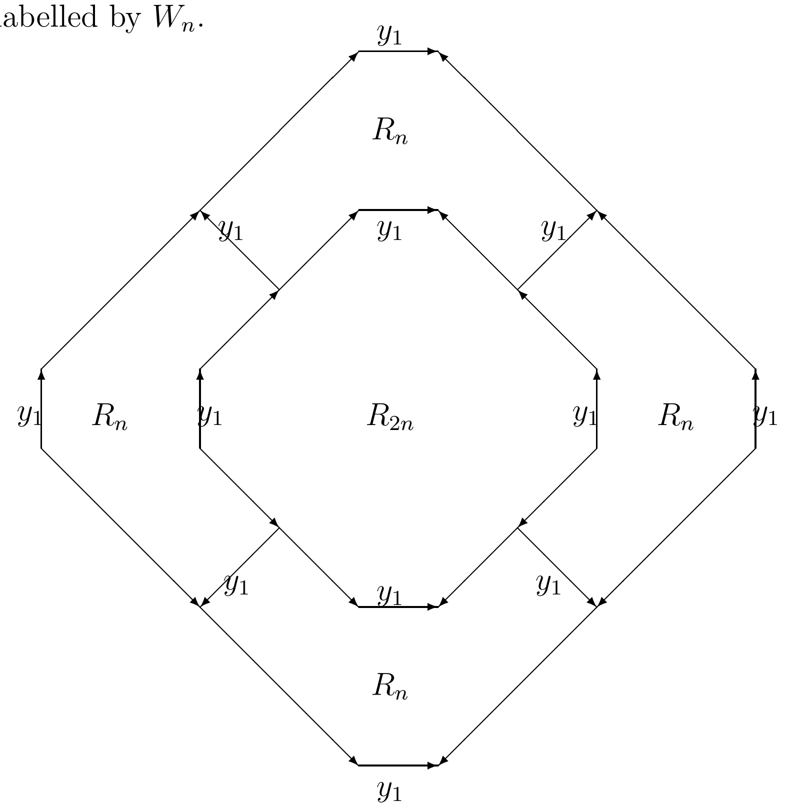
$$= y_{2n+2}^{y_{n+2}}$$

we used the fact that $y_{2n+1}^{y_1} = y_{2n+2}$; this transition needs at most $f(2n)$ steps

$$= y_{2n+3} = y_{m+2}$$

this transition needs at most $f(n)$ steps.

Here is the van Kampen diagram which shows the inductive process of deduction of $R_m$ in this case. The boundary of the big polygon is labelled by $y_{m+1}^{y_1} y_{m+2}^{-1}$, that is by the word $W_m = y_1^{-1} y_0^{-2n} y_1 y_0^{2n} y_1 y_0^{-(2n+1)} y_1^{-1} y_0^{2n+1}$. The boundary of the polygon in the middle is labelled by $W_{m-1} = W_{2n}$ and the boundaries of the polygons on the sides are labelled by $W_n$.




Thus we can deduce $R_m$ in $f(2n) + 4f(n) = f(2n+1) = f(m)$ steps, as required.

Case 2: $m = 2n, \quad n \geq 2$.

Again here is a deduction of $R_m$:

$$y_{m+1}^{y_1} = y_{2n+1}^{y_1} = y_{2n}^{y_n y_1}$$

we used the relation $y_{2n+1} = y_{2n}^{y_n}$; this transition needs at most $f(n)$ steps

$$= y_{2n}^{y_1 y_n^{y_1}}$$

we used a relation of the free group

$$= y_{2n}^{y_1 y_{n+1}} = (y_{2n}^{y_1})^{y_{n+1}}$$

we used the relation $y_{n+1} = y_n^{y_1}$; this transition needs at most $2f(n-1)$ steps

$$= y_{2n+1}^{y_{n+1}}$$

we used the relation $y_{2n}^{y_1} = y_{2n+1}$; this transition needs $f(2n-1)$ steps

$$= y_{2n+2} = y_{m+2}.$$

This transition needs at most $f(n)$ steps.

Thus we may deduce $R_m$ in $f(2n-1) + 2f(n) + 2f(n-1) = f(2n) = f(m)$ steps, as required. The proof is complete.

Now we are able to complete the proof of Theorem 1. We shall prove that for any $\lambda > 1/2$ the Dehn function $\Phi(n)$ of the group $F$ satisfies the following inequality:

$$\Phi(n) \preceq 2^{\lambda(\log_2 n)^2} = n^{\lambda \log_2 n}. \tag{11}$$

Let us take any $\lambda > 1/2$. Given a word of length $n$ in variables $y_0^{\pm 1}, y_1^{\pm 1}$ that is equal to 1 in $F$, we have to prove that one can reduce it to 1 in the number of steps not exceeding the right side of (11).

First of all we show that for any $\mu > 1/2$ the sequence $f(n)$ defined in (8)–(10), satisfies the following inequality:

$$f(n) < n^{\mu \log_2 n}, \tag{12}$$

for all sufficiently large $n$.

We claim that for every $n \geq 1$ we have the following inequality

$$f(2n) < 8nf(n). \tag{13}$$

This is obviously true for $n = 1$. If (13) holds for a given $n$ then, using (9) and (10), we obtain that $f(2(n+1)) = f(2n+2) = f(2n+1) + 2f(n+1) + 2f(n) = f(2n) + 4f(n) + 2f(n+1) + 2f(n) = f(2n) + 2f(n+1) + 6f(n) < (8n+6)f(n) + 2f(n+1) \leq 8(n+1)f(n+1)$, so (13) always holds. Now it is easy to use (13) in order to prove that for any $k \geq 1$ the following inequality holds:

$$f(2^k) < 2^{k(k+5)/2} \tag{14}$$

Since (14) is true for $k = 1$, suppose it is true for a given $k$. We have: $f(2^{k+1}) < 8 \cdot 2^k f(2^k) < 2^{3+k+k(k+5)/2} = 2^{(k+1)(k+6)/2}$, as desired.

Now let us take any $n \geq 2$ and suppose that $2^k \leq n < 2^{k+1}$. It follows immediately from (14) that $f(n) < f(2^{k+1}) < 2^{(k+1)(k+6)/2} < 2^{\mu k^2} \leq n^{\mu \log_2 n}$, if $k$ is sufficiently large.

Thus (12) is proved. Now fix a number $\lambda > 1/2$ and take a number $\mu$ such that $1/2 < \mu < \lambda$. If $W$ is a word of length $n$ in $x_0, x_1$ which is equal to 1 in $F$, then according to Lemma 4 we can reduce this word to its normal form (which is 1) in $< n^2$

steps of applying defining relations of (1). It follows from the proof of Lemma 4, that in this reduction process we get no subscripts greater than $n$ (in each of the $n$ steps, when we adjoin the next letter to the reduced form of the previous word we increase each subscript at most by 1). So for each of the relations $x_j^{x_i} = x_{j+1}$ that we use $i, j \leq n$. By Lemma 5 this relation can be substituted by a sequence of at most $f(j-i)$ relations (6). Since $f(j-i) < f(n)$, we can deduce any relation of length $n$ from the defining relations $R_1, R_2$ in a number of steps which does not exceed $n^2 \cdot n^{\mu \log_2 n} < n^{\lambda \log_2 n}$, if $n$ is sufficiently large. This means that the Dehn function satisfies (11).

**Remark.** It would be interesting to study the Dehn function of another R. Thompson's groups $T$ (see the definition in [CFP]). This group is finitely presented infinite and simple. It contains $F$ and is generated by $F$ and an element of order 3. In the preprint version of this paper we stated without proof that the Dehn function of $T$ is also subexponential. In fact, this can be considered only as a conjecture: the method similar to that used in this paper does not work for the group $T$.

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