On Dehn functions of free products of groups

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Abstract

In this article we show that the Dehn function of a nontrivial free product of groups is equivalent to its subnegative closure.

Let us recall the definition of the isoperimetric function of a group presentation. Let Σ be a group alphabet and let $\mathcal{R}$ be a set of words over Σ. Denote by G a group given by presentation $\langle\, \Sigma \mid \mathcal{R}\,\rangle$. Let F be the free group with free basis Σ and let N be the normal closure of the set $\mathcal{R}$ in F. Then $G \cong F/N$.

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For any word w which is equal to 1 in G, there exist an integer $k \geq 0$ and words $R_i \in \mathcal{R}$, X_i $(1 \leq i \leq k)$ such that

$$w = X_1^{-1} R_1^{\pm 1} X_1 \dots X_k^{-1} R_k^{\pm 1} R_k \qquad (1)$$

in F. The least k depending on w such that (1) exists will be denoted by $k(w)$. We may considered k as a function from $N \setminus \{1\}$ into the set $\mathbf{N}$ of positive integers.

Let f be a function from $\mathbf{N}$ into itself. Given a group presentation $\langle\, \Sigma \mid \mathcal{R}\,\rangle$, we say that $f(n)$ is an *isoperimetric function* of this presentation, if $k(w) \leq f(n)$ provided w is a word of length $\leq n$. If the presentation is finite then isoperimetric functions always exist. The least isoperimetric function is called the *Dehn function* of the presentation.

Consider any two functions f, g from $\mathbf{N}$ to $\mathbf{N}$. By definition $f \preceq g$ means that there are integers c_1, c_2, c_3 such that $f(n) \leq c_1 g(c_2 n) + c_3 n$ for all n. This order induces an equivalence relation $\simeq$ on the set of functions from $\mathbf{N}$ into itself: $f \simeq g$ if and only if $f \preceq g$ and $g \preceq f$.

The following result is well-known (see [6, 1, 3]): if H is a finitely presented group and f, g are Dehn functions for two finite presentations of H, then $f \simeq g$. Thus for a finitely presented group H there exists a unique Dehn function $f(n)$ (modulo the equivalence relation $\simeq$).

Thus the Dehn function (modulo $\simeq$) may be considered as a characteristic of a finitely presented group. For instance, word hyperbolic groups are exactly groups that have linear Dehn functions [4]. For other information about Dehn functions see [].

Let $f\colon \mathbf{N} \to \mathbf{N}$ be a function. Following Brick [2], we say that f is *subnegative* if $f(m) + f(n) \leq f(m+n)$ for all $m, n \in \mathbf{N}$. Let us define a function $\bar{f}(n)$ by the following formula:

$$\bar{f}(n) = \max\left(f(n_1) + f(n_2) + \cdots f(n_r)\right) \qquad (2)$$

where maximum is taken for all $r \geq 1$, $n_1, \dots, n_r \in \mathbf{N}$, $n_1 + \cdots + n_r = n$. It is easy to see that $\bar{f}(n)$ is the least subnegative function which is greater than or equals $f(n)$. We say that $\bar{f}$ is the *subnegative closure* of f.

It is easy to see that if $f(n)$ is the Dehn function of a finite presentation then f may not be subnegative. For example, for the presentation $\langle\, a \mid a^2 = 1, a^3 = 1\,\rangle$ one has $f(1) = 2$, $f(2) = 2$. But the following question may be asked:

Is it true that the Dehn function of any finite presentation is equivalent to a subnegative function?

Let us note that if $f \preceq g$ then $\bar{f} \preceq \bar{g}$. Therefore $f \simeq g$ implies $\bar{f} \simeq \bar{g}$. We obtain the following equivalent question:

Is every Dehn function of a finite presentation equivalent to its subnegative closure?

This question seems not to be very easy. The aim of this article is to prove the following.

Theorem 1 *Let G_1, G_2 be nontrivial finitely presented groups. Then the Dehn function of their free product $G_1 * G_2$ is equivalent to its subnegative closure.*

It follows from this fact that, for any finitely presented group G, the Dehn function of $G * \mathbf{Z}$ is equivalent to its subnegative closure. So our question is equivalent to the following one:

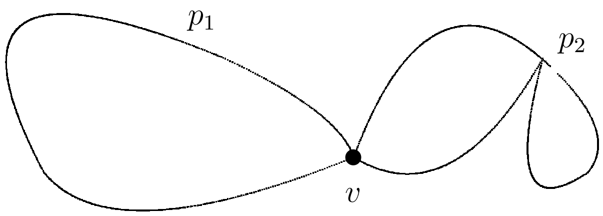
Is it true that, for any finitely presented group G, the Dehn functions of G and $G * \mathbf{Z}$ are equivalent?

Indeed, let f be the Dehn function of a finite presentation of G. Adding a new generator x to this presentation, we obtain a finite presentation for $G * \mathbf{Z}$. Denote by g its Dehn function.

It is easy to see that $f \preceq g$ and so $\bar{f} \preceq \bar{g}$. Let us show that $\bar{g} \preceq \bar{f}$. For any n there exists a diagram Δ over the presentation of $G * \mathbf{Z}$ with $g(n)$ cells such that its perimeter is not greater than n. We may assume that no diagrams with the same boundary label have fewer cells. This diagram is obtained from several diagrams $\Delta_1, \ldots, \Delta_r$ over the presentation of G joined by edges labelled by $x^{\pm 1}$. Denote by n_j the perimeter of Δ_j. Clearly, Δ_j has not greater than $f(n_j)$ cells $(1 \le j \le r)$. Since $n_1 + \cdots + n_r \le n$, applying (2) we obtain $g(n) \le f(n_1) + \cdots + f(n_r) \le \bar{f}(n_1 + \cdots + n_r) \le \bar{f}(n)$. We proved that $g \preceq \bar{f}$. From that follows that $\bar{g} \preceq \bar{f}$. Thus $\bar{f} \simeq \bar{g} \simeq g$. This shows that f is equivalent to its subnegative closure if and only if it is equivalent to g.

To prove our theorem, we use diagrams over group presentations. All needed information about diagrams can be found in [5]. Let us recall that $\varphi(e)$ denotes the label of an edge e of a diagram. All contours of diagrams

Figure 1:



are read clockwise, all contours of cells are read counterclockwise. By $\iota(e)$, $\tau(e)$ we denote the initial and the terminal vertex of an edge e respectively.

We need the following auxiliary definition.

Let Δ be a diagram over a group presentation and let v be a vertex in the boundary of Δ. We say that v is a *double vertex* of Δ if Δ has a contour of the form p_1p_2 where p_1, p_2 are nontrivial loops at v (see Fig. 1).

Let $\langle \Sigma_i \mid \mathcal{R}_i \rangle$ be a finite presentation of a group G_i $(i = 1, 2)$. We assume that Σ_1 and Σ_2 do not intersect. It is obvious that $\langle \Sigma \mid \mathcal{R} \rangle$ is a finite presentation for $G_1 * G_2$ where $\Sigma = \Sigma_1 \cup \Sigma_2$, $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$. We keep these definitions throughout the paper.

Lemma 1 *Let Δ be a diagram over $\langle \Sigma \mid \mathcal{R} \rangle$ with a contour e_1pe_2 where e_1, e_2 are edges. Suppose that $\varphi(e_i) \in \Sigma_i^{\pm 1}$ $(i = 1, 2)$. Then $v = \iota(e_1) = \tau(e_2)$ is a double vertex of Δ.*

Proof. Let $h_0, h_1, \ldots, h_r$ $(r \geq 1)$ where $h_0 = e_1$, $h_1 = e_2^{-1}$ be the sequence of all edges coming out from v in the clockwise order. (Note that this sequence may have occurrences of the form h, h^{-1} if h forms a loop.) Each of edges h_j $(0 \leq j \leq r)$ has label either from $\Sigma_1^{\pm 1}$ or from $\Sigma_2^{\pm 1}$. Since h_0 and h_r have labels from different alphabets, there exist two neighbour edges h_j and h_{j+1} $(0 \leq j < r)$ such that their labels belong to different alphabets.

Since there are no defining relations from $\mathcal{R}$ that contain letters from both of the two alphabets, there are no cells "between" h_j and h_{j+1} (i.e. no contours of cells of Δ have subpaths $h_j^{-1}h_{j+1}$). It is clear that the edge h_j^{-1} cannot belong to a boundary contour of any cell since otherwise there are edges between h_j and h_{j+1}. The same argument applies for the edge h_{j+1}.

This implies that h_j^{-1}, h_{j+1} belong to a boundary contour of Δ and moreover, they are consecutive edges of the cyclic contour. We obtain that the contour of Δ at v has a form p_1p_2 where $p_1 = e_1 \dots h_j^{-1}$, $p_2 = h_{j+1} \dots e_2$ are nontrivial loops at v. Thus v is a double vertex. The proof is complete.

Lemma 2 *Let Δ be a diagram over $\langle \Sigma \mid \mathcal{R} \rangle$. Suppose that G_1 and G_2 are nontrivial. Choose $a_i \in \Sigma_i$ such that $a_i \neq 1$ in G_i ($i = 1, 2$). Then there exists a vertex v in the boundary of Δ and there exists $i = 1, 2$ such that for any decomposition pq of the contour of Δ as a loop at v, we have $\varphi(p) \neq a_i$ in G.*

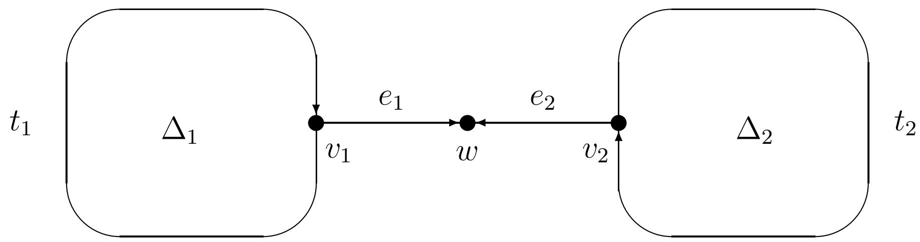
Proof. Suppose that the conclusion is not true. Let v_0 be an arbitrary vertex in the boundary of Δ. Let us define an infinite sequence v_m of vertices in the boundary of Δ and an infinite sequence p_m of subpaths of the cyclic contour of Δ by induction ($m \geq 1$). Let $m \geq 1$ and let v_j, p_j be defined for all $0 \leq j < m$. By our assumption, for any $i = 1, 2$ there exists a decomposition $p(i)q(i)$ of the contour of Δ at v_{m-1} such that $\varphi(p(i)) = a_i$ in G. Let p_m be $p(1)$ for odd m and $p(2)$ for even m. By definition, v_m is the terminal vertex of p_m. Note that all p_m are nontrivial since a_1, a_2 are not equal to 1 in G.

Since the set of the boundary vertices of Δ is finite, we have $v_m = v_{m+s}$ for some $m \geq 0$, $s \geq 1$. Thus the path $p = p_{m+1} \dots p_{m+s}$ is a nontrivial power of a boundary label of Δ. Since the boundary label of any diagram is equal to 1 in the corresponding group, this implies that $\varphi(p) = 1$ in G. But $\varphi(p)$ is a nontrivial product of letters a_1, a_2 such that all neighbour letters are different. This shows that $\varphi(p)$ presents a normal form of a free product $G_1 * G_2$. But a nonempty normal form presents a nontrivial element of a free product (see [5]). We have a contradiction. This completes the proof.

Now we are able to prove our Theorem.

Let f be the Dehn function of $\langle \Sigma \mid \mathcal{R} \rangle$ where G_1, G_2 are nontrivial. We shall prove the inequality $f(n_1) + f(n_2) \leq f(n_1 + n_2 + 6)$ for all n_1, n_2. Recall that any isoperimetric function is increasing. It follows easily from these facts that $f(n_1) + \dots + f(n_r) \leq f(n_1 + \dots + n_r + 6(r-1))$. Now if any integer n is presented as a sum of positive integers $n_1, \dots, n_r$ then we have $r \leq n$ and $f(n_1) + \dots + f(n_r) \leq f(n + 6(r-1)) \leq f(7n)$. Using (2) we obtain $\bar{f}(n) \leq f(7n)$ for all n. This means that $\bar{f} \preceq f$. Since $f(n) \leq \bar{f}(n)$ for all n, we always have $f \preceq \bar{f}$. Now it is clear that $f \simeq \bar{f}$. So it remains to prove the inequality.

Figure 2:



By definition of f, there exist diagrams Δ_1, Δ_2 over $\langle\, \Sigma \mid \mathcal{R}\, \rangle$ with boundary labels W_i that have $f(n_i)$ cells and the length of W_i does not exceed n_i $(i = 1, 2)$. Moreover, no diagram with boundary label W_i may have less than $f(n_i)$ cells $(i = 1, 2)$.

Starting from Δ_1, Δ_2, we shall construct a new diagram Δ with boundary label W and the following properties:

- The perimeter of Δ does not exceed $n_1 + n_2 + 6$.
- Δ has exactly $f(n_1) + f(n_2)$ cells.
- No diagram with the same boundary label W may have fewer cells.

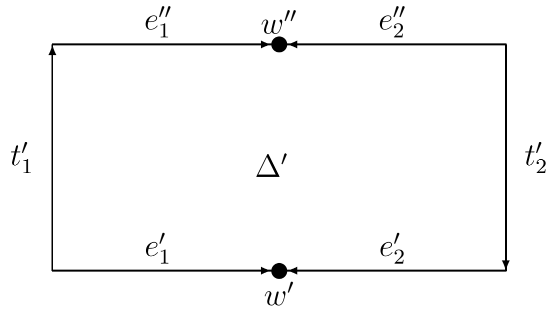
Our inequality follows from these properties by definition of Dehn functions. So it remains to construct such a diagram Δ.

First of all, we apply Lemma 2 to Δ_1. We find a boundary vertex v_1 of Δ_1 and an integer $i_1 = 1, 2$ such that no initial subpaths of the boundary contour of Δ_1 at v_1 may have labels equal to a_{i_1} in G. After that, applying Lemma 2 to the mirror copy of Δ_2, we find a boundary vertex v_2 of Δ_2 and an integer $i_2 = 1, 2$ such that no terminal subpaths of the boundary contour of Δ_2 at v_2 may have labels equal to $a_{i_2}^{-1}$.

Denote by t_i the boundary contour of Δ_i at v_i $(i = 1, 2)$. We distinguish two cases: $i_1 \neq i_2$ and $i_1 = i_2$.

Case 1. For the simplicity, let us assume that $i_1 = 1$, $i_2 = 2$. Let us construct a diagram Δ taking Δ_1, Δ_2 and joining them by a "bridge" of two edges e_1, e_2 (see Fig. 2).

Figure 3:



Here $\varphi(e_i) = a_i$ for $i = 1, 2$ and Δ has a contour $e_1^{-1}t_1e_1e_2^{-1}t_2e_2$ at $w = \tau(e_1) = \tau(e_2)$. It is clear from our construction that Δ has exactly $f(n_1) + f(n_2)$ cells, and its perimeter does not exceed $n_1 + n_2 + 4$. We shall prove that no diagram with the same boundary label may have fewer cells.

Suppose that a diagram Δ' has the same boundary label. Thus its contour may be presented as $(e_1')^{-1}t_1'e_1''(e_2'')^{-1}t_2'e_2'$ where $W_i = \varphi(t_i) = \varphi(t_i')$, $\varphi(e_i) = \varphi(e_i') = \varphi(e_i'') = a_i$ $(i = 1, 2)$. (See Fig. 3.)

Let $w' = \tau(e_1') = \tau(e_2')$, $w'' = \tau(e_1'') = \tau(e_2'')$. Applying Lemma 1 we conclude that w' is a double vertex of Δ'. Thus w' coincides with some other boundary vertex.

Suppose that $w' = w''$. In this case we may consider diagrams Δ_1' and Δ_2' with contours $(e_1')^{-1}t_1'e_1''$ and $(e_2'')^{-1}t_2'e_2'$ respectively. In each of these two diagrams we may glue edges e_i' and e_i'' (they have the same label) obtaining two diagrams Δ_i'' $(i = 1, 2)$. Perhaps we will need to cut off some spherical subdiagrams. As a result, we get two diagrams that have boundary labels W_1 and W_2. By the remark above, Δ_i'' has at least $f(n_i)$ cells $(i = 1, 2)$. Therefore, Δ' has at least $f(n_1) + f(n_2)$ cells as desired.

Now suppose that $w' = w''$. Thus w' coincides with one of vertices that belongs to either t_1' or t_2'. In the first case we may find a subdiagram in Δ' with contour $e_1't_1''$ where t_1'' is an initial subpath of t_1'. Thus $\varphi(t_1'') = \varphi(e_1') = a_1$ in G. This is a contradiction since t_1 has an initial subpath whose label is equal to a_1 in G. In the second case we have a subdiagram in Δ' with boundary contour $t_2''e_2'$ where t_2'' is a terminal subpath of t_2'. This implies that $\varphi(t_2'') = \varphi(e_2')^{-1} = a_2^{-1}$ in G. We have also a contradiction since t_2 cannot have a terminal subpath whose label is equal to a_2^{-1} in G.

Figure 4:

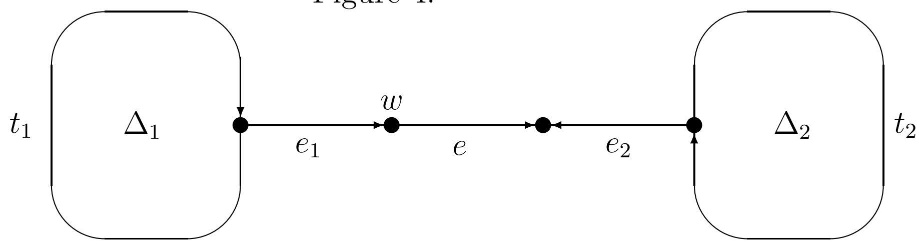
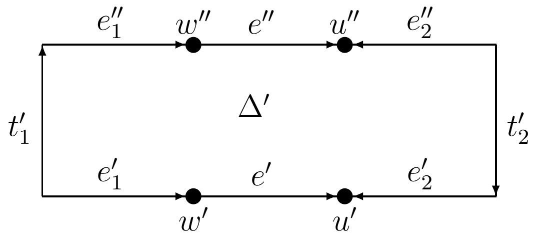


Figure 5:



Case 2. Now without loss of generality assume that $i_1 = i_2 = 1$. We also construct a diagram Δ taking Δ_1, Δ_2 and joining them by a "bridge" of three edges e_1, e, e_2 (see Fig. 4).

Here $\varphi(e_1) = \varphi(e_2) = a_1$, $\varphi(e) = a_2$ and Δ has a contour $e_1^{-1}t_1e_1ee_2^{-1}t_2e_2e^{-1}$ at $w = \tau(e_1) = \iota(e)$. It is clear from our construction that Δ has exactly $f(n_1) + f(n_2)$ cells, and its perimeter does not exceed $n_1 + n_2 + 6$. We shall prove that no diagram with the same boundary label may have fewer cells.

Suppose that a diagram Δ' has the same boundary label. Thus its contour may be presented as $(e_1')^{-1}t_1'e_1''e''(e_2'')^{-1}t_2'e_2'(e')^{-1}$ where $W_i = \varphi(t_i) = \varphi(t_i')$, $\varphi(e_i) = \varphi(e_i') = \varphi(e_i'') = a_1$ $(i = 1, 2)$, $\varphi(e) = \varphi(e') = \varphi(e'') = a_2$. (See Fig. 5.)

Let $w' = \tau(e_1') = \iota(e')$, $w'' = \tau(e_1'') = \iota(e'')$, $u' = \tau(e') = \tau(e_2')$, $u'' = \tau(e'') = \tau(e_2'')$. Applying Lemma 1 we conclude that w' is a double vertex of Δ'. Thus w' coincides with some other boundary vertex.

Suppose that $w' = w''$. In this case we have two subdiagrams in Δ' with contours $(e_1')^{-1}t_1'e_1''$ and $e''(e_2'')^{-1}t_2'e_2'(e')^{-1}$. Gluing in these subdiagrams e_1' and e_1'', e' and e'', e_2' and e_2'' and cutting off spherical subdiagrams if necessary, we obtain diagrams Δ_i'' with boundary labels W_i $(i = 1, 2)$. These diagrams have at least $f(n_1)$, $f(n_2)$ cells respectively, so Δ' has at least $f(n_1)+f(n_2)$ cells, as desired.

If w' coincides with a vertex in t_1', we finish the proof similarly to Case 1. Note that w' does not coincide with u' since otherwise $a_2=\varphi(e')=1$ in G. Therefore w' coincides with a vertex in $(e_2'')^{-1}t_2'$. By Lemma 1, u' is also a double vertex. If w' coincides with a vertex in t_2' then u' also coincides with a vertex in t_2'. If $w'=u''$ then u' cannot coincide with u'' so it coincides with a vertex in t_2' anyway. Now we finish the proof similarly to Case 1.

Theorem 1 is proved.

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