Rigidity properties of diagram groups

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Abstract

In this paper we establish a rigid connection between two classical objects: the R.Thompson group (the group of all piece-wise linear homeomorphisms of the unit interval with finitely many dyadic break points and all slopes powers of 2) and the Dunce hat (the topological space obtained from the triangle ABC by gluing AB, BC and AC). We prove that a diagram group of a directed 2-complex contains a copy of the R.Thompson group if and only if the 2-complex contains a copy of the Dunce hut.

1 Introduction

The class of diagram groups was introduced by Meakin and Sapir in 1993. Kilibarda obtained the first results about diagram groups in [6], [7]. The theory was further developed in [3], [4]. It turned out that many important groups (including the R. Thompson group F) are diagram groups. On the other hand, diagram groups satisfy some interesting properties, and there exists a deep similarity between combinatorics on diagrams and combinatorics on words. Recent results by D. Farley [2] show that diagram groups act by isometries on CAT(0)-spaces. This allowed him to prove that the R. Thompson group satisfies the rational Novikov conjecture.

The first (and still very useful) definition of diagram groups (see [7],[3]) was algebraic. From this point of view, every diagram group $\mathcal{D}(\mathcal{P}, w)$ is determined by a semigroup presentation $\mathcal{P}$ and a distinguished word w. One can give an equivalent topological definition of diagram groups [5]. From

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the topological point of view, diagram groups are determined by a directed 2-complex K (all edges have directions, every cell is bounded by two positive paths, the top and the bottom), and a distinguished positive path p. Diagram groups are similar to second relative homotopy groups of 2-complexes, only one needs to consider directed 2-complexes and homotopies consisting of positive paths only (we call them *directed homotopies*).

Here is an informal definition of diagram groups (see [3] for details).

Let $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ be a semigroup presentation where Σ is an alphabet and $\mathcal{R}$ is the set of defining relations.

Any diagram over $\mathcal{P}$ is obtained as follows. Start with a positive (horizontal) path p on the plane labeled by some word w over Σ (that is, a linear oriented labeled graph with $|w|$ edges which form a path, whose label is w). This is a *trivial* (w, w)*-diagram*, and p is the *top* and the *bottom* path of this diagram.

Next find a subword in w which is equal to u (or v) for some relation $u = v$ in $\mathcal{R}$: $p = p'qp''$ where the label of q is u (resp. v). Below p, draw a path q' labeled by v (resp. u) whose initial and terminal vertices coincide with the initial and terminal vertices of q. The path $q(q')^{-1}$ must bound a region on the plane (called a *cell*). The result of this operation is a one-cell diagram whose top path is labeled by w and the bottom path is labeled by the word obtained from w by replacing u by v (resp. v by u). Attaching a new cell to the bottom path of the diagram, we get a diagram with two cells, etc. Every diagram Δ is a plane labeled oriented graph which tesselates a region of the plane between two positive paths $\mathbf{top}(\Delta)$ and $\mathbf{bot}(\Delta)$. If w is the label of $\mathbf{top}(\Delta)$ and w' is the label of $\mathbf{bot}(\Delta)$ then Δ is called a (w, w')-diagram.

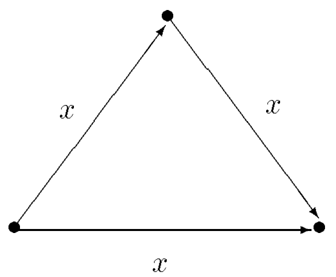
Two diagrams are called *equal* if there exists an isotopy of the plane which takes one of the diagrams to the other one.

A diagram is called *reduced* if it does not contain dipoles. A *dipole* is a pair of cells such that the bottom path of one of them coincides with the top path of the other one and these cells are mirror images of each other. If a diagram contains a dipole, the two cells forming the dipole can be removed. So every diagram can be reduced. By the theorem of Kilibarda [7] the reduced form of every diagram is unique.

Fix a word w and consider the set $\mathcal{D}(\mathcal{P}, w)$ of all (w, w)-diagrams over $\mathcal{P}$. One can multiply two diagrams Δ_1 and Δ_2 in $\mathcal{D}(\mathcal{P}, w)$ by gluing together $\mathbf{bot}(\Delta_1)$ and $\mathbf{top}(\Delta_2)$ and reducing the resulting diagram. This operation is associative, the trivial (w, w)-diagram plays the role of the identity element, and every diagram Δ has an inverse, the mirror image of Δ. Thus $\mathcal{D}(\mathcal{P}, w)$ is a group which is called the *diagram group over the presentation $\mathcal{P}$ with base word w.*

Since we are going to use only the algebraic definition of diagram groups, we do not give here a precise topological definition. Let us only mention that the directed complex corresponding to a semigroup presentation is similar to the standard 2-complex of a group presentation. It has one vertex, one oriented edge for each generator and one oriented cell for each relation $u = v$ with bottom path u and the top path v. Then the word w in the algebraic definition of a diagram group turns into a positive path w in the directed 2-complex, and every (w, w)-diagram is a planar representative of a directed homotopy from w to w. Conversely, every directed (w, w)-homotopy is represented by a (w, w)-diagram. The product of homotopies corresponds to the product of diagrams. Equivalent diagrams correspond to equivalent (isotopic) homotopies. This allows one to translate every statement about diagram groups from the algebraic language to the topological language and back.

The relation between diagram groups and semigroup presentations (directed complexes) is not rigid. For example, if the presentation $\mathcal{P}$ is aspherical, then the diagram groups are trivial (regardless of the base). On the other hand, presentations of finite semigroups may correspond to "large" diagram groups. In particular, the diagram group corresponding to the presentation $\langle\, x \mid x^2 = x \,\rangle$ of the trivial semigroup is the well known R. Thompson group F (for every base). The directed complex corresponding to this presentation is the well known *Dunce hat* [8] which can be obtained from the triangle



by gluing all three sides according to their direction. It is easy to construct other semigroup presentations (directed complexes) with diagram groups isomorphic to F. Nevertheless in this paper we show that if F appears in a diagram group of a directed complex (resp. presentation of a semigroup) then Dunce hat maps into the complex (the semigroup contains an idempotent). Thus there is a rigid relationship between F and the Dunce hat (the presentation $\langle\, x \mid x^2 = x\,\rangle$).

Recall that the group F can be given by the following infinite presentation:

$$\langle\, x_0, x_1, \dots \mid x_j^{x_i} = x_{j+1}\ (j > i)\,\rangle.$$

It has also a finite presentation

$$\langle\, x_0, x_1 \mid x_2^{x_1} = x_3, x_3^{x_1} = x_4\,\rangle, \tag{1}$$

where $x_2 = x_1^{x_0}$, $x_3 = x_2^{x_0}$, $x_4 = x_3^{x_0}$ by definition.

Theorem 1. *The following conditions are equivalent.*

1. *For some word w, the diagram group $\mathcal{D}(\mathcal{P}, w)$ contains an isomorphic copy of the R. Thompson group F.*

2. *The semigroup given by $\langle\, \Sigma \mid \mathcal{R}\,\rangle$ contains an idempotent.*

A part of this theorem, namely the implication $2 \Longrightarrow 1$, has been proved in [4, Theorem 25]. We asked [4, Problem 2] whether the converse is true. Theorem 1 gives an affirmative answer to this question.

The topological formulation of Theorem 1 is the following

Theorem 2. *Let K be a directed complex. Then the following conditions are equivalent.*

1. *A diagram group corresponding to K contains an isomorphic copy of the R. Thompson group F.*

2. *The complex K contains a positive non-empty path t which is directly homotopic to its square.*

3. *There exists a directed morphism from the Dunce hat to K.*

Clearly Theorem 1 and 2 are equivalent We shall prove the theorem in the first formulation.

Recall also that in [4, Theorem 24], we have proved a similar rigidity theorem for the restricted wreath product $\mathbf{Z}$ wr $\mathbf{Z}$. Similar rigidity theorems might be true for other diagram groups as well.

2 Proof of the rigidity theorem

We need one auxiliary geometric fact. Let $\mathcal{P}$ be a semigroup presentation and let Δ be a diagram over $\mathcal{P}$. For any two vertices o', o'' in Δ we put $o' \le o''$ whenever there exists a positive path in Δ from o' to o''. It is easy to see that the labels of any two positive paths from o' to o'' are equal modulo $\mathcal{P}$ (see [3]). So one can define the element $\mu(o', o'')$ in the monoid M presented by $\mathcal{P}$. This element is represented in M by the label of any positive path from o' to o''.

Recall [3] also that for every (u, v)-diagram Δ and (u', v')-diagram Δ' one can define the *sum* $\Delta + \Delta'$ by gluing the terminal vertex of $\mathbf{top}(\Delta)$ with the initial vertex of $\mathbf{top}(\Delta')$. The result is a (uu', vv')-diagram.

Lemma 3. *Let $\mathcal{P}$ be a semigroup presentation. Let M denote the monoid presented by $\mathcal{P}$. Suppose that Δ is a (uv, uv)-diagram over $\mathcal{P}$. Let o_1 (resp. o_2) be the vertex in the top (bottom) path of Δ that subdivides it into a product of two paths labeled by u and v. Suppose that o is a vertex in Δ, where $o \le o_1$, $o \le o_2$. If Δ is equivalent to a sum of a (u, u)-diagram and a (v, v)-diagram, then $\mu(o, o_1) = \mu(o, o_2)$. (It would be more precise to write μ_Δ but it will always be clear what diagram we refer to).*

Proof. Obviously, the reduced form of Δ is a sum of a (u, u)-diagram and a (v, v)-diagram. Suppose that we need to cancel $m \ge 0$ pairs of dipoles in order to reduce Δ. We prove the claim by induction on m. If $m = 0$ then the conclusion is obvious since in this case $o_1 = o_2$. Let $m > 0$. Cancel a dipole that consists of two cells π_1 and π_2, where the bottom path of π_1 coincides with the top path of π_2. As a result, we get a diagram Δ' that can be reduced in $m - 1$ step. Let p_1 be the top path of π_1, p_2 be the bottom path of π_2, and let p be the common boundary of π_1 and π_2.

Suppose first that o is a vertex that does not disappear in Δ', that is, o does not belong to p as an inner point. In this case, for any positive path from o to o_1 in Δ, we can find a positive path from o to o_1, which does not contain p as a subpath (just replace p by p_1). The same is true for positive paths from o to o_2. The vertices o, o_1, o_2 still exist in Δ', and the elements $\mu(o, o_1)$, $\mu(o, o_2)$ do not change when we replace Δ by Δ'. Applying our inductive assumption to Δ', we see that these elements are equal there. Thus they are equal for Δ, too.

Now suppose that o disappears in Δ'. Thus p is subdivided by o into two paths, say, q and r. Let $\bar{o}$ be the terminal point of r. Obviously, any positive path from o to o_1 or o_2 begins with r. By the previous paragraph, $\mu(\bar{o}, o_1) = \mu(\bar{o}, o_2)$. Now it remains to notice that $\mu(o, o_i) = \nu\mu(\bar{o}, o_i)$, where $\nu \in M$ is the element represented by the label of r $(i = 1, 2)$. This completes the proof.

Let H be a group, $y_0, y_1 \in H$. Suppose that y_0, y_1 do not commute in H and satisfy relations (1), that is, $y_1^{y_0 y_1} = y_1^{y_0^2}$, $y_1^{y_0^2 y_1} = y_1^{y_0^3}$. Since all proper homomorphic images of F are abelian [1], it is clear that y_0, y_1 generate F as a subgroup of H. In this case, we say that an ordered pair y_0, y_1 *generates F canonically.* We also introduce elements y_i for $i \geq 2$ by $y_i = y_1^{y_0^{i-1}}$.

Let us recall some definitions. We refer to [3, Section 15] for details. Let $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ be any semigroup presentation. A (w, w)-diagram Δ over $\mathcal{P}$ is called *absolutely reduced* provided Δ^n is reduced for every $n \geq 1$. For any (w, w)-diagram Δ over $\mathcal{P}$, where $w \in \Sigma^+$, there exists a word $v \in \Sigma^+$, a (w, v)-diagram Ψ and an absolutely reduced (v, v)-diagram $\bar{\Delta}$ such that $\Delta = \Psi\bar{\Delta}\Psi^{-1}$. One can decompose $\bar{\Delta}$ into a sum $A_1 + \cdots + A_m$ of spherical diagrams. Here each nontrivial summand cannot be decomposed into a sum of spherical diagrams. We also assume that for any i $(1 \leq i < m)$ at least one of the diagrams A_i, A_{i+1} is nontrivial. The summands A_i $(1 \leq i \leq m)$ are called *components* of $\bar{\Delta}$. The number of nontrivial components does not depend on the choice of $\bar{\Delta}$. So it can be denoted by $\mathbf{comp}(\Delta)$.

Let $G = \mathcal{D}(\mathcal{P}, w)$ be a diagram group. For any (w, v)-diagram Ψ over $\mathcal{P}$, where $v \in \Sigma^+$, we have an isomorphism $\psi\colon G \to H = \mathcal{D}(\mathcal{P}, v)$ that takes any diagram $\Delta \in G$ to $\Psi^{-1}\Delta\Psi$. For any (w, w)-diagram Δ over $\mathcal{P}$ we can construct an isomorphism ψ defined above such that the diagram $\psi(\Delta) = \Psi^{-1}\Delta\Psi$ will be absolutely reduced. In this case we will often assume without loss of generality that Δ is absolutely reduced up to changing the base of our diagram group.

Suppose that $A = A_1 + \cdots + A_m$ and $B = B_1 + \cdots + B_n$ are absolutely reduced diagrams each decomposed into a sum of components. Let A_i $(1 \leq i \leq m)$ and B_j $(1 \leq j \leq n)$ be (v_i, v_i)- and (w_j, w_j)-diagrams, respectively. If there exists a (w, v)-diagram Γ such that $A = \Gamma^{-1}B\Gamma$, where $v = v_1 \ldots v_m$, $w = w_1 \ldots w_n$, then $m = n$ and Γ can be decomposed into a sum $\Gamma_1 + \cdots + \Gamma_m$ of (w_i, v_i)-diagrams Γ_i such that $A_i = \Gamma_i^{-1}B_i\Gamma_i$ $(1 \leq i \leq m)$. Any element C in the centralizer of A in $\mathcal{D}(\mathcal{P}, v)$ can be decomposed into a sum

$C = C_1 + \cdots + C_m$, where C_i is a (v_i, v_i)-diagram that commutes with A_i $(1 \le i \le m)$. If A_i is nontrivial then its centralizer is cyclic so A_i and C_i belong to the same cyclic subgroup. It is easy to see that one can change the base in such a way that both diagrams A and C become cyclically reduced (see [4, Theorem 17]).

The following theorem is stronger than the implication $1 \Longrightarrow 2$ in Theorem 1.

Theorem 4. *Let $\mathcal{P} = \langle\, \Sigma \mid \mathcal{R} \,\rangle$ be a semigroup presentation, $w \in \Sigma^+$. If the diagram group $G = \mathcal{D}(\mathcal{P}, w)$ contains an isomorphic copy of R. Thompson's group F, then the semigroup S presented by $\mathcal{P}$ contains an idempotent. Moreover, G contains a copy of F if and only if there exist words $w_1, w_2 \in \Sigma^*$, $e \in \Sigma^+$ such that equalities $w = w_1 e w_2$, $e^2 = e$ hold modulo $\mathcal{P}$.*

Proof. Suppose that $G = \mathcal{D}(\mathcal{P}, w)$ contains an isomorphic copy of F. Then there exist (w, w)-diagrams Y_0, Y_1 over $\mathcal{P}$ that generate F canonically. We assume that the total number of their components, that is, $\mathbf{comp}\,(Y_0) + \mathbf{comp}\,(Y_1)$, is minimal possible. Note that this number does not change if we replace Y_0, Y_1 by their conjugates Y_0^D, Y_1^D for any (w, v)-diagram D over $\mathcal{P}$, where v is a nonempty word over Σ.

It is easy to see that the element $x_2 x_3 x_2^{-2} \in F$ commutes with x_i for all $i \ge 3$. So it also commutes with $x_3 x_4 x_3^{-2}$. Changing the base w, we can assume without loss of generality that $D_2 = Y_2 Y_3 Y_2^{-2}$ is a cyclically reduced diagram over $\mathcal{P}$ decomposed into the sum of components $A_1 + \cdots + A_m$, where A_i is a (v_i, v_i)-diagram $(1 \le i \le m)$. Obviously, D_2 is nontrivial. (Otherwise $Y_2 = Y_3$ and Y_0 commutes with Y_1 .) Since $D_3 = Y_3 Y_4 Y_3^{-2}$ is in the centralizer of D_2, we can assume that both diagrams D_2, D_3 are absolutely reduced and $D_3 = B_1 + \cdots + B_m$, where B_i commutes with A_i for all $1 \le i \le m$. (Note that the summands B_i are not necessarily components of D_3.)

Suppose that A_i is nontrivial for some i. Let it be the jth nontrivial component of D_2 counting from left to right. It is clear that $D_3 = D_2^{Y_0} = D_2^{Y_1}$. Thus D_2 and D_3 conjugate and so they have the same structure of components. Let B' be the jth nontrivial component of D_3 counting from left to right. There are three possible cases: B' is contained in either 1) B_i, or 2) $B_1 + \cdots + B_{i-1}$, or 3) $B_{i+1} + \cdots + B_m$. Clearly, the third case is symmetric to the second one. So we consider only the first two cases.

Case 1. It is obvious that $B' = B_i$. Hence the conjugation of D_2 by each of Y_0, Y_1 takes A_i to B_i. This implies that each of the diagrams Y_0,

Y_1 can be decomposed into a sum of three spherical diagrams with bases $u_1 = v_1 \dots v_{i-1}$, $u_2 = v_i$, $u_3 = v_{i+1} \dots v_m$, respectively. So we have an injective homomorphism ϕ from the Thompson group F (generated by Y_0, Y_1) to the direct product $\mathcal{D}(\mathcal{P}, u_1) \times \mathcal{D}(\mathcal{P}, u_2) \times \mathcal{D}(\mathcal{P}, u_3)$. Denote by H_k the projection of F onto kth factor and let ψ_k be the homomorphism from F onto H_k ($k = 1, 2, 3$). The group F embeds into $H_1 \times H_2 \times H_3$. Therefore, at least one of the three groups H_k is not abelian. Then it must be isomorphic to F because all proper homomorphic images of F are abelian. So let H_k be non-abelian. Let us show that $k = 1$ or $k = 3$.

We know that the diagrams A_i, B_i belong to the same cyclic subgroup. By [3, Theorem 15.30], we may assume that they belong to the maximal cyclic subgroup K of the diagram group $\mathcal{D}(\mathcal{P}, v_i)$. Let us establish that any (v_i, v_i)-diagram D over $\mathcal{P}$ such that $A_i^D = B_i$, also belongs to K. Let C be the generator of K. By definition, A_i is nontrivial. So B_i is also nontrivial and so we have $A_i = C^r$, $B_i = C^s$, where r, s are non-zero integers. We now have $(C^D)^r = C^s$. So we can apply [3, Corollary 15.28] to conclude that there is a diagram C_0 and some integers p, q such that $C^D = C_0^p$, $C = C_0^q$ and $pr = qs$. Since C generates maximal cyclic subgroup, we have $|p| = |q| = 1$. Thus $C^D = C^{\pm 1}$. If $C^D = C^{-1}$, then $(CD)^2 = D^2$. Using the fact that diagram groups have the unique extraction of roots property ([3, Section 15]), we deduce that C is trivial. This is a contradiction. So $C^D = C$. Hence D belongs to K because K coincides with its centralizer. Now we can conclude that the images of Y_0, Y_1 under ψ_2 belong to the same cyclic subgroup. So $H_2 = \psi_2(F)$ is abelian.

We have proved that either H_1 or H_3 is isomorphic to F. It is obvious that for any diagram Δ from the subgroup generated by Y_0, Y_1, one has $\sum_{k=1}^{3} \mathbf{comp}\,(\psi_k(\Delta)) = \mathbf{comp}\,(\Delta)$. Since $\psi_2(F)$ is nontrivial, we see that $\mathbf{comp}\,(\psi_2(Y_0)) + \mathbf{comp}\,(\psi_2(Y_1)) > 0$. So for any $k = 1, 3$ we have $\mathbf{comp}\,(\psi_k(Y_0)) + \mathbf{comp}\,(\psi_k(Y_1)) < \mathbf{comp}\,(Y_0) + \mathbf{comp}\,(Y_1)$. Now we can take the value of k such that $\psi_k(F) \cong F$ and replace the elements of our canonical generating pair Y_0, Y_1 by their images under ψ_k. We get another canonical generating pair with smaller total number of components. This is a contradiction, so Case 1 is impossible.

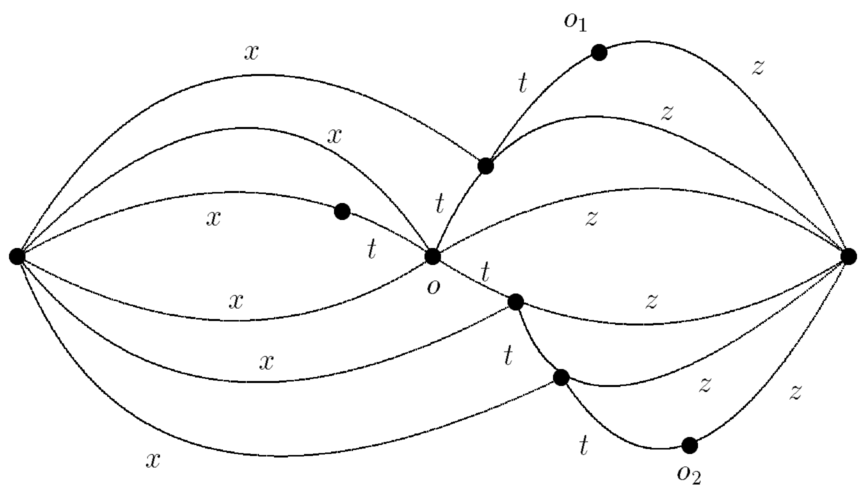
Case 2. Let B' be contained in $B_1 + \dots + B_{i-1}$ as a subdiagram. We have $B_1 + \dots + B_{i-1} = \Xi_1 + B' + \Xi_2$ for some spherical diagrams Ξ_1, Ξ_2. Let z be the base of the diagram $B_{i+1} + \dots + B_m$ and let t be the base of $\Xi_2 + B_i$. Obviously, t is nonempty because it has a terminal segment v_i. We will show that $t^2 = t^3$ modulo $\mathcal{P}$ so $e = t^2$ represents an idempotent in S. It will be also clear that w belongs to the two-sided ideal in M generated by e, where $M = S^1$ is the monoid presented by $\mathcal{P}$.

Let D be Y_0 or Y_1. We use the fact that $D_2^D = D_3$. Each of the diagrams D_2, D_3 is a sum of components. According to the above description, D can be naturally decomposed into a sum of m diagrams (not necessarily spherical) such that the conjugation by the kth summand ($1 \le k \le m$) takes A_k (the kth component of D_2) to the kth component of D_3 (recall that this component may not coincide with B_k). Then A_i, the jth nontrivial component of D_2, is taken to B', the jth nontrivial component of D_3. The bases of diagrams to the right of A_i, B' in D_2 and D_3, respectively, are z and tz. This means that D is a sum of an (xt, x)-diagram and a (z, tz)-diagram, where x is the base of Ξ_1.

Note that $x_2 x_3 x_2^{-2}$ commutes with x_3. So Y_3 belongs to the centralizer of D_2. Hence Y_3 is a sum of an (xt, xt)-diagram and a (z, z)-diagram. The diagram

$$\Delta \equiv Y_0^{-1} \circ Y_0^{-1} \circ Y_1 \circ Y_0 \circ Y_0,$$

equivalent to Y_3, has the following structure:



Here o is the vertex in Y_1 that subdivides it into the sum of an (xt, x)- and a (z, zt)-diagrams. By o_1 (o_2) we denote the vertex on the top (bottom) path of Δ that subdivides this path into a product of paths with labels xt and z. Clearly, there is a path in Δ from o to o_1 labeled by t^2 and there is a path in Δ from o to o_2 labeled by t^3. Applying Lemma 3, we conclude that $t^2 = t^3$ modulo $\mathcal{P}$. (It is obvious that w belongs to Mt^2M as an element in S.)

The converse is proved in [4, Theorem 25].

The proof is complete.

Remark 5. Given a finite semigroup presentation $\mathcal{P}$ and a word $w \in \Sigma^+$, we cannot decide algorithmically whether the diagram group $\mathcal{D}(\mathcal{P}, w)$ contains F as a subgroup. Indeed, the property of a finitely presented semigroup not to have an idempotent, is a Markov property. Let a, b be new letters that do not belong to Σ. Adding them to Σ and adding relations of the form $ax = a$, $xb = b$ ($x \in \Sigma$), we get a new semigroup presentation $\mathcal{Q}$. The diagram group $\mathcal{D}(\mathcal{Q}, ab)$ contains F as a subgroup if and only if S has an idempotent, where S is the semigroup presented by $\mathcal{P}$. This is clear because all idempotents in the semigroup presented by $\mathcal{Q}$ are represented by words over Σ and ab belongs to the two-sided ideal generated by any word over Σ.

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