Potential divisibility in finite semigroups is undecidable

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May 28, 1996

Abstract

We prove that there is no algorithm to decide, given a finite semigroup S and two elements $a, b \in S$, whether there exists a bigger finite semigroup $T > S$ where a divides b and b divides a. This solves a thirty years old problem by John Rhodes.

Let us recall some basic definitions from the theory of semigroups (see [1]). There are four important equivalence relations associated with every semigroup S. These relations were formally introduced by Green in 1951.

$$\begin{aligned}
&(x,y) \in \mathcal{L}_S \text{ iff } zx = y,\ ty = x \text{ for some } z,t \in S^1,\\
&(x,y) \in \mathcal{R}_S \text{ iff } xz = y,\ yt = x \text{ for some } z,t \in S^1,\\
&(x,y) \in \mathcal{J}_S \text{ iff } z_1xz_2 = y,\ t_1yt_2 = x \text{ for some } z_1,z_2,t_1,t_2 \in S^1,\\
&\mathcal{H}_S = \mathcal{L} \cap \mathcal{R},
\end{aligned}$$

Here S^1 stands for the semigroup S with an identity element adjoint. potentiallySimilar relations can be defined on a finite associative ring.

If S is a subsemigroup (subring) of a finite semigroup (ring) T then $\mathcal{L}_S$, $\mathcal{R}_S$,$\mathcal{J}$, $\mathcal{H}$ are subsets of the corresponding relations on T but do not necessarily coincide with the restrictions of these relations from T to S.

Let $\mathcal{X}$ be one of the letters $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{H}$ and let $A \subseteq S \times S$. Then we say that A is *potentially $\mathcal{X}$-related in the class of finite semigroups* if $A \subseteq \mathcal{X}_T$ for some finite semigroup (resp. ring) T containing S. In this paper we will omit the words "in the class of finite semigroups" because we consider only finite semigroups here. The description of subsets of a semigroup which are potentially $\mathcal{L}$-related or potentially $\mathcal{R}$-related is well known. Lyapin [7] was probably the first to describe them. Let us define two relations $\mathcal{L}^*$ and $\mathcal{R}^*$ as follows

$$\begin{aligned}
&(x,y) \in \mathcal{L}^* \quad \text{iff for every } z,t \in S \quad xz = xt \leftrightarrow yz = yt;\\
&(x,y) \in \mathcal{R}^* \quad \text{iff for every } z,t \in S \quad zx = tx \leftrightarrow zy = ty.
\end{aligned} \tag{1}$$

Similar relations can be defined for rings.

Lemma 1 *(Lyapin, [7]). A subset $A \subseteq S \times S$ of a finite semigroup S is potentially $\mathcal{L}$-related (resp. $\mathcal{R}$-related) if and only if $A \subseteq \mathcal{L}^*$ (resp. $A \subseteq \mathcal{R}^*$).*

Similar statement holds for rings.

In particular there is an algorithm which, given a finite semigroup (ring) S and a subset $A \subseteq S \times S$, determines whether or not A is potentially $\mathcal{L}$-related or $\mathcal{R}$-related.

On the other hand, the first author proved [4] that there is no algorithm to decide whether a subset of $S \times S$ is potentially $\mathcal{J}$-related. The second author proved in [9] that there is no algorithm to decide whether a subset of $S \times S$ is potentially $\mathcal{H}$-related.

In both cases the subsets of $S \times S$ had unbounded (finite) cardinalities depending on S. In around 1966 John Rhodes asked for an algorithm to decide when one pair of elements of a finite semigroup is potentially $\mathcal{J}$-related. In other words the problem asks for an algorithm to decide, given a finite semigroup S and a pair of elements $a, b \in S$, whether a divides b and b divides a in some bigger finite semigroup T.

This problem is obviously equivalent to the problem of algorithmic solvability of the system of two identities $a = xby, b = x'ay'$ over a finite semigroup.

Recall that a system of identities $u_i = v_i$ where u_i and v_i are words in the alphabet of unknowns and constants from a finite semigroup (ring) S is called solvable over S if there exists a finite semigroup $T > S$ where this system has a solution. The main result of [4] can be interpreted as the algorithmic unsolvability of systems of quadratic equations $a_i = x_i b_i y_i$ (with unbounded number of equations) over finite semigroups. The result of [9] implies that the solvability of systems of linear equations of the form $a = bx$ or $a = xb$ is also undecidable (again the number of equations is unbounded).

The problem of Rhodes asks whether the solvability of the system of two quadratic equations over a finite semigroup is decidable.

A partial result was obtained by T. Hall and M. Putcha [5]. They proved that if a and b are regular elements of a finite semigroup S then a and b are $\mathcal{J}$-related in a bigger finite semigroup T if and only if either $(a, b) \in \mathcal{J}$ or $a \notin S^1bS^1$ and $b \notin S^1aS^1$. Thus Rhodes' problem is decidable for regular semigroups and even in the case when S is arbitrary but a and b are regular.

The main result of this paper proves that Rhodes' problem is undecidable and moreover the solvability of one equation $a = xyb$ over a finite semigroup is undecidable. The same results are true for finite associative rings.

Theorem 1 *There are no algorithms to decide, given a finite 4-nilpotent semigroup (associative ring) S and two elements a, b in S, whether there exists a bigger finite semigroup (associative ring) $T > S$ such that:*
(i) $a = xby$ for some $x, y \in T$.
(ii) $a = xby$ and $b = x'ay'$ for some $x, x', y, y' \in T$.

It is clear that for every semigroup S and any two elements a, b in S one of the properties (i), (ii) holds if and only if there exists a finite ring $T > GF(p)S$ such that this property holds in T. Thus it is enough to prove Theorem 1 in the case of semigroups.

A set A with a partial binary operation $\cdot$ on it and a distinguished element 1 such that $1 \cdot a = a \cdot 1 = a$ for all $a \in A$ will be called a *partial group* if it satisfies the *cancellation property*: $ac = bc \rightarrow a = b$ and $ca = cb \rightarrow a = b$.

By a theorem of T. Evans [2, 3] (see also Connection 2.2 in [6]), the problem of embeddability of finite partial algebras into algebras of a pseudo-variety is decidable if and only if the universal theory of this pseudo-variety is decidable. Since by a result of Slobodskoj [10] the universal theory of finite groups is undecidable, the following problem is undecidable:

> Given a finite partial group A, decide whether or not A is embeddable into a group from $\mathcal{V}$.

Let us call a partial group A *symmetric* if for every $a \in A$ there exists a unique element $a' \in A$ such that $aa' = a'a = 1$. A partial group $B > A$ is called a symmetric extension of A if B is symmetric and for every element $b \in B$ either b or b' belongs to A. Thus the order of every symmetric extension of A does not exceed $2|A|$ so every finite partial group has only finitely many symmetric extensions and all of them may be effectively listed. It is clear that a partial group A is embeddable into a group if and only if one of its symmetric extensions is embeddable into this group. Therefore we have the following result.

Lemma 2 *There is no algorithm to decide whether a given a finite symmetric partial group is embeddable into a finite group.*

Let B be a partial group and let A be embedded into B. For every $i = 1, 2\ldots$ let us define a subset A^i of B. Let $A^0 = \{1\}$, $A^1 = A$ and for every $i > 1$ let $A^{i+1} = A^i \cdot A$. We shall call a partial group B an *extension* of A *or rank* k if:

- $B = \bigcup_{s=0}^{k} A^s$.
- for every numbers i and j between 0 and k such that $i + j \leq k$ and for every pair of elements $x \in A^i$ and $y \in A^j$ the product xy exists in B and belongs to A^{i+j}.
- all products $x \cdot y$ where $x \in A^i \backslash A^{i-1}$, $y \in A^j \backslash A^{j-1}$ and $i + j > k$ are undefined.
- for every $x \in A^i, y \in A^j, z \in A^m$ such that $i + j + m \leq k$ both $(xy)z$ and $x(yz)$ are defined and $(xy)z = x(yz)$.

It is clear that there are only finitely many extensions of rank k of any finite partial group A and all of them may be effectively listed.

It is also clear that a partial group A is embeddable into a group G if and only if some extension of A of rank k is embeddable into G.

In the proof, we shall use the so called split systems introduced in [8].

A *split system* is a triple of disjoint non-empty sets (L, R, Z) with an operation $L \times R \to Z$.

A *morphism* from a split system (L, R, Z) into a semigroup S is a triple of maps $f : L \to S, g : R \to S, h : Z \to S$ such that $h(ab) = f(a)g(b)$ for every $a \in L, b \in R$.

We say that a morphism (f, g, h) is injective if each map f, g, h is injective. We say that a split system (L, R, Z) is *embeddable* into a semigroup S if there exists an injective morphism from (L, R, Z) into S.

With every partial group A' which is an extension of rank 2 of a partial group A one can associate the following split system:

$$F(A, A') = (A \times \{1\}, A \times \{2\}, A^2 \times \{3\})$$

with operation $(a, 1)(b, 2) = (ab, 3)$. This map F is an extension from semigroups to partial groups of the functor F_{SYS} considered in [8], [9] and several other papers (see references in [9]). The following two lemmas are proved in [9]. For the sake of completeness we reproduce the proofs here.

Lemma 3 *The split system $F(A, A')$ is embeddable into a group G if and only if the partial group A' is embeddable into G.*

Proof. Suppose that the split system $F(A, A')$ is embeddable into a group G and (f, g, h) is the corresponding embedding. We have that for every x in S

$$f(x)g(1) = h(x),\ f(1)g(x) = h(x).$$

Hence $f(x) = h(x)g(1)^{-1}$, $g(x) = f(1)^{-1}h(x)$. Therefore for every $x, y \in S$ we have: $h(xy) = f(x)g(y) = h(x)g(1)^{-1}f(1)^{-1}h(y)$. Therefore the map $\psi : A' \to G$ which takes x to $h(x)g(1)^{-1}f(1)^{-1}$ is a homomorphism on A. Since products xy do not exist if x or y is not in A, this map is a homomorphism of A'. It is easy to check that this homomorphism is one-to-one.

Now suppose that A' is embeddable into a group G and ψ is the embedding. Let $f : A \times \{1\} \to G$, $g : A \times \{2\} \to G$, $h : S' \times \{3\} \to G$ be maps defined as follows:

$$\psi(x) = f(x, 1) = g(x, 2) = h(x, 3).$$

It is easy to see that this triple is a morphism from $F(A, A')$ into G and that each map f, g, h is one-to-one. Therefore $F(A, A')$ is embeddable into G.

The lemma is proved.

We say that a split system (L, R, Z) has the *cancellation property* if $\ell_1 r = \ell_2 r \to \ell_1 = \ell_2$ and $\ell r_1 = \ell r_2 \to r_1 = r_2$ where $\ell_i \in L, r_i \in R$. It is easy to see that if $F(A, A')$ always has the cancellation property if A is a partial group and A' is an extension of A of rank 2.

We say that a split system (L, R, Z) is *$\mathcal{H}$-embeddable* into a semigroup S if there exists an embedding (f, g, h) such that $f(L)$ is contained in an $\mathcal{H}$-class of S and $h(Z)$ is contained in an $\mathcal{H}$-class of S.

Lemma 4 *A split system (L, R, Z) with the cancellation property is $\mathcal{H}$-embeddable into a regular semigroup S if and only if (L, R, Z) is embeddable into a subgroup of S.*

Proof. The "if" part is trivial because a group is a regular semigroup with only one $\mathcal{H}$-class - itself.

Let us prove the "only if" part. Suppose that a split system (L, R, Z) is $\mathcal{H}$-embeddable into a regular semigroup and (f, g, h) is the corresponding $\mathcal{H}$-embedding. Let us denote the $\mathcal{H}$-classes containing $f(L)$ and $h(Z)$ by H_1 and H_2 respectively.

Since S is regular, every $\mathcal{L}$-class and every $\mathcal{R}$-class of S contains a subgroup [1]. Now we can apply Green's lemma [1] to the $\mathcal{J}$-class M_1 containing H_1 and conclude that there exist elements $p, q \in S$ such that pH_1 is contained in a subgroup G_1 of M_1 and $qpx = x$ for every $x \in H_1$.

Then $qph(xy) = qpf(x)g(y) = f(x)g(y) = h(xy)$ for every $x \in L,\ y \in R$. Therefore $ph(xy)$ and $h(xy)$ are in the same $\mathcal{J}$-class of S. Again by Green's lemma pH_2 is contained in an $\mathcal{H}$-class of S and for some $q' \in S$ we have that $q'px = x$ for every $x \in H_2$.

Consider three functions $f_1 : L \to S,\ g_1 : R \to S,\ h_1 : Z \to S$ defined as follows:

$$f_1(x) = pf(x),\ g_1(x) = g(x),\ h_1(x) = ph(x).$$

Since $qf_1(x) = f(x)$ for every $x \in L$ and $q'h_1(x) = h(x)$ for every $x \in Z$, functions f_1, g_1, h_1 are one-to-one. It is also easy to check that $h_1(xy) = f_1(x)g_1(y)$. Thus (f_1, g_1, h_1) is an embedding of (L, R, Z) into S. This embedding is an $\mathcal{H}$-embedding and it satisfies the following additional property:

(1) The image of L is contained in a subgroup of S.

Thus without loss of generality we can assume that the original embedding satisfies this property, so H_1 is a group.

Let e be the identity element of H_1. Take any $a \in L$ and let $f_1(x) = f(a)^{-1}f(x)$ for every $x \in L$, $g_1(x) = g(x)$ for every $x \in R$, $h_1(x) = f(a)^{-1}h(x)$ for every $x \in Z$. Green's lemma again tells us that these maps are one-to-one, $h_1(Z)$ is contained in an $\mathcal{H}$-class. It is easy to check that (f_1, g_1, h_1) is an $\mathcal{H}$-embedding, it satisfies the property (1) and $f_1(a) = e$. Thus we can assume that the original triple (f, g, h) has the following property:

(2) $f(a) = e$.

Now we have that

$$h(xy) = f(x)g(y) = f(x)eg(y) = f(x)f(a)g(y) = f(x)h(ay)$$

for every $x \in L, y \in Z$. Since h is one-to-one and (L, R, Z) has the cancellation property, we can conclude that the map $g_1 : R \to S$ which maps every x from R to $h(ax)$ is one-to-one, so the triple (f, g_1, h) is an embedding. This $\mathcal{H}$-embedding has the same properties (1), (2) as (f, g, h) and the following additional property:

(3) The image of R is contained in the same $\mathcal{H}$-class as the image of Z.

So we can assume that $g(R) \subseteq H_2$.

Now we can consider the $\mathcal{J}$-class M_2 which contains H_2. Since S is regular, there exist elements $p, q \in S$ such that H_2p is contained in a subgroup G_2 of M_2 and $xpq = x$ for every $x \in H_2$. Now consider three maps $f_1 = f,\ g_1(x) = g(x)p$ where $x \in R$ and $h_1(x) = h(x)p$. As above it is easy to check that the triple (f_1, g_1, h_1) is an $\mathcal{H}$-embedding which has the same properties (1)-(3) and the following property:

(4) The images of Z and R are contained in a subgroup of S.

Again we can assume that (f, g, h) satisfies this property as well. So H_2 is a subgroup.

Let j be the identity element of H_2. The argument similar to that we have used above shows that we can assume that (f, g, h) satisfies the following property:

(5) $g(b) = j$ for some $b \in R$.

Therefore $h(xy) = f(x)g(y) = f(x)jg(y) = f(x)g(b)g(y) = h(xb)g(y)$ for every $x \in L$, $y \in R$. Since (L, R, Z) satisfies the cancellation property, the function $f_1 : x \to h(xb)$ is one-to-one on L. So (f_1, g, h) is an $\mathcal{H}$-embedding of (L, R, Z). The images of L, R, Z are contained in the subgroup H_2. Thus (L, R, Z) is embeddable into a subgroup of S.

The lemma is proved.

Now take any extension C of rank 3 of a finite symmetric partial group A.

With every such C we associate the following algebra

$$
\begin{aligned}
&S(C) = (\{1\} \times A \times \{2\}) \cup (\{2\} \times A \times \{3\}) \cup (\{3\} \times A \times \{4\}) \\
&\cup (\{1\} \times (A^2 \cup A) \times \{3\}) \cup (\{2\} \times (A^2 \cup A) \times \{4\}) \cup (\{1\} \times C \times \{4\}) \cup \{0\}.
\end{aligned}
$$

with one binary operation: $(i, a, j)(j, b, k) = (i, ab, k)$ for every $a, b \in A$, $1 \leq i < j < k \leq 4$, other products are equal to 0.

It is easy to check that $S(C)$ is a 4-nilpotent semigroup. This construction first appeared in [4].

For every $i < j, i, j = 1, ..., 4$, let $e_{i,j} = (i, 1, j) \in S(C)$.

Lemma 5 *For every finite symmetric partial group A the following conditions are equivalent:*

1. *A is embeddable into a finite group.*
2. *There exists an extension C of A of rank 3 such that the semigroup $S(C)$ is embeddable into a finite 0-simple semigroup.*
3. *There exists an extension C of A of rank 3 such that the semigroup $S(C)$ is embeddable into a finite semigroup T where $e_{2,3} = xe_{1,4}y$ for some $x, y \in T$.*
4. *There exists an extension C of A of rank 3 such that the semigroup $S(C)$ is embeddable into a finite semigroup T where $e_{1,4} = xe_{2,3}y$ and $e_{2,3} = x'e_{1,4}y'$ for some $x, x', y.y' \in T$, that is $e_{1,4}$ and $e_{2,3}$ are in the same $\mathcal{J}$-class of T.*

Proof. $1 \to 2$. It is clear that if C is embeddable into a group G then $S(C)$ is embeddable into $S(G)$ which is a subsemigroup of the 0-simple Brandt semigroup $B(G) = \{1, 2, 3, 4\} \times G \times \{1, 2, 3, 4\} \cup \{0\}$ with the following operation: $(i, g, j)(j, h, k) = (i, gh, k)$ and all other products are equal to 0 [1].

Implications $2 \to 3$ is obvious. The implication $3 \to 4$ follows from the fact that $e_{1,4} = e_{1,2}e_{2,3}e_{3,4}$.

It remains to prove $4 \to 1$. Suppose that $S(C)$ is embedded into a finite semigroup T such that $e_{2,3}$ and $e_{1,4}$ are in the same $\mathcal{J}$-class M of T. Since every finite semigroup is embeddable into a finite regular semigroup (say, the semigroup of all transformations of a finite set), we can assume without loss of generality that T is a regular semigroup. The principal factor M^0 of T corresponding to M is a finite 0-simple semigroup [1]. Then by the Sushkevich theorem [1] M^0 is isomorphic to the Sushkevich semigroup $M^0(G; L, R; P)$ where G is a finite group, L, R are finite sets, and P is a $|L| \times |R|$-matrix over G^0. This semigroup is equal as a set to $L \times G \times R \cup \{0\}$ and the operation is defined as follows: $(\ell, g, r)(\ell', g', r') = (\ell, gP(\ell', r)g', r')$ if $P(\ell', r) \neq 0$; all other products are equal to 0. We shall identify the principal factor M^0 with $M^0(G; L, R; P)$. If $a = (\ell, g, r)$ then we shall say that ℓ is the left index of a and r is the right index of a. To avoid confusion, notice that although elements $e_{2,3} = (2, 1, 3)$ and $e_{1,4} = (1, 1, 4)$ are in M^0 the left index of $e_{2,3}$ is not necessarily 2, the right index of $e_{2,3}$ is not necessarily 3, etc. Recall [1] that two elements in M^0 are in the same $\mathcal{H}$-class if and only if they have the same left and right indexes. Recall also that every element in a finite 0-simple semigroup has a left unity and a right unity elements.

For every $a \in A$ we have:

$$(1, a, 3) = (1, a, 2)e_{2,3}, (1, a, 3)(3, a', 4) = e_{1,4}.$$

These equalities imply that the set $H_{1,3} = \{1\} \times A \times \{3\}$ is contained in M. Let e be the left unity of $e_{2,3}$ and u_a be the right unity of $(1, a, 3)$. Then

$$(1, a, 3) = ((1, a, 2)e)e_{2,3}, (1, a, 3)(u_a(3, a', 4)) = e_{1,4}.$$

These equalities imply that $(1, a, 2)e$ and $u_a(3, a', 4)$ belong to M. This, in turn, implies that for every $a \in A$ the element $(1, a, 3)$ has the same right index as $e_{2,3}$ and the same left index as $e_{1,4}$. Therefore $H_{1,3}$ is contained in an $\mathcal{H}$-class of T.

Similarly equalities

$$(2, a, 4) = e_{2,3}(3, a, 4), (1, a', 2)(2, a, 4) = e_{1,4}$$

imply that $H_{2,4} = \{2\} \times A \times \{4\}$ is contained in an $\mathcal{H}$-class of T.

Let $b \in A$. Let f be the right unity of $e_{2,3}$. Then $f(3, b, 4) \in M$ because $e_{2,3}f(3, b, 4) \in H_{2,4} \subseteq M$ and $f \in M$. Let ℓ_b be the left index of $f(3, b, 4)$ and let r be the right index of elements in $H_{1,3}$. Since

$$(1, a', 3)(f(3, a, 4)) = ((1, a', 2)e_{2,3})(f(3, a, 4)) = e_{1,4},$$

by the definition of a Sushkevich semigroup, $P(\ell, r) \neq 0$. Then the definition of a Sushkevich semigroup implies that $H_{1,3}f(3, a, 4)$ is contained in M. But it is easy to see that $H_{1,3}f(3, a, 4) = \{(1, ba, 4) \mid b \in A\}$. Therefore the set $H_{1,4} = \{1\} \times A^2 \times \{4\}$ is contained in M. Notice that $H_{1,4} = H_{1,3}(f\{3\} \times A \times \{4\}) = ((\{1\} \times A \times \{2\}e)H_{2,4}$. Therefore every element of $H_{1,4}$ has the same left index as elements of $H_{1,3}$ and the same right index as elements of $H_{2,4}$. Therefore $H_{1,4}$ is contained in an $\mathcal{H}$-class of T.

Let $B \subseteq C$ be the extension of A of rank 2 defined as follows. As a set, B is equal to $A \cup A^2$; if $a, b \in A$ then $a \cdot b$ exists and is defined as in C, $1 \cdot b = b \cdot 1 = b$ for every $b \in B$. Other products of elements of B are not defined. It is easy to check that B is an extension of A of rank 2. Consider the split system $F(A, B)$. Consider three maps $\alpha : A \times \{1\} \to T$, $\beta : A \times \{2\} \to T$ and $\tau : A^2 \times \{3\} \to T$ defined as follows:

$$\alpha(a, 1) = (1, a, 3)$$
$$\beta(a, 2) = (3, a, 4)$$
$$\tau(a, 3) = (1, a, 4)$$

It is easy to check that the triple (α, β, τ) is an embedding of $F(A, B)$ into the semigroup T. We proved above that $\alpha(A \times \{1\})$ is contained in an $\mathcal{H}$-class of T and $\tau(A \times \{3\})$ is contained in an $\mathcal{H}$-class of T. Thus (α, β, τ) is an $\mathcal{H}$-embedding of the split system $F(A, B)$ into a regular semigroup T. By Lemma 4 this split system is embeddable into a subgroup of T. Now by Lemma 3 A is embeddable into a finite group. This completes the proof of our lemma.

Theorem 1 immediately follows from Lemmas 2 and 5.

Remark 1. Recall that Hall and Putcha [5] proved that Rhodes' problem is solvable in the case when both a and b are regular. By modifying a little the proof of Theorem 1, the authors can prove that Rhodes' problem is undecidable if one of the elements a or b is regular and even in the case when a or b is an idempotent.

Remark 2. One can easily adapt the proof of our theorem to show that the solvability of the system of two linear equations $a = xb, c = ay$ over finite semigroups (rings) is undecidable. Lyapin's result (Lemma 1) shows that the solvability of one linear equation of this form is decidable.

In general let us call a system of equations $u_i = v_i$ $i = 1, ...,$, where u_i, v_i are words in the alphabet of unknowns and constants, *decidable* if there exists an algorithm which, given a finite semigroup S and values of the constants in S, decides whether this system has a solution in some bigger finite semigroup T. It is an interesting problem, to describe all decidable equations or systems of equations. Since, as our theorem shows, even simple equations can be undecidable, this problem does not seem too wild.

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