ON THE JOINS OF SEMIGROUP VARIETIES WITH THE VARIETY OF COMMUTATIVE SEMIGROUPS

M.V.Sapir*and M.V.Volkov†

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Abstract

We show that the join of a variety of semigroup and the variety of all commutative semigroups is not finitely based provided some weak conditions.

Recall that the join of two varieties U and V is the minimal variety containing U and V. To obtain this variety one has to take all homomorphic images of subalgebras of direct products $U \times V$ where $U \in \mathcal{U}$, $V \in \mathcal{V}$. In [1], it was shown that joins of semigroup varieties with the variety of commutative semigroups **Comm** very often contain non-finitely based subvarieties. Here we will show that these joins themselves are often non-finitely based.

To formulate our result we need one notation. Let A stand for the well-known completely 0-simple semigroup over one-element group E={e} with sandwich matrix $\begin{bmatrix} e & e \\ 0 & e \end{bmatrix}$.

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THEOREM. *The join* $\mathbf{Comm} + \mathbf{V}$ *has no identity basis with finite number of variables whenever either* $\mathbf{V}$ *contains* A *or* $\mathbf{V}$ *contains all bands and is locally finite.*

COROLLARY. *Let* $\mathbf{B_{m,n}}$ *be the Burnside variety given by the identity* $x^m = x^{m+n}$. *Then* $\mathbf{B_{m,n}} + \mathbf{Comm}$ *has no identity basis with finite number of variables whenever either* $m > 1$ *or* $\mathbf{B_{m,n}}$ *is locally finite.*

For $m = n = 1$ this corollary gives us the result of T.Nordahl [2]: the join of the variety of bands and the variety of commutative semigroups is not finitely based. This is to say that this result served us an impuls to formulate the above theorem. We believe that in fact $\mathbf{B_{m,n}} + \mathbf{Comm}$ is always non-finitely based, but we are not able to prove it so far.

Proof of the theorem. Let us suppose first that $\mathbf{V}$ is a periodic variety containing A. For every integer $k \geq 3$ we consider the Rees matrix semigroup T_k over the infinite cyclic group $G = < g >$ with the sandwich $k \times k$-matrix P_k

$$\begin{bmatrix} g & e & 0 & 0 & \cdots & 0 & 0 \\ 0 & e & e & 0 & \cdots & 0 & 0 \\ 0 & 0 & e & e & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & e & e \\ e & 0 & 0 & 0 & \cdots & 0 & e \end{bmatrix}$$

where e denotes the identity element of G. (It should be mentioned that this construction has been used firstly by Mashevitski in [3].) We are going to prove that T_k does not belong to $\mathbf{Comm} + \mathbf{V}$ but each $(k-1)$-generated subsemigroup of T_k belongs to this join. Since V consists of periodic semigroups, it satisfies the identity $x^m = x^{m+n}$ for some positive integers m and n. Consider the identity

$$(x_1^2 \ldots x_k^2)^{m+n}(x_1 \ldots x_k)^m = (x_1^2 \ldots x_k^2)^m (x_1 \ldots x_k)^{m+2n} \quad (1)$$

This identity follows both from the commutativity and from the identity $x^m = x^{m+n}$ and therefore it holds in the variety $\mathbf{Comm} + \mathbf{V}$. Now we shall show that (1) fails in T_k. Indeed, substituting the matrix $[e; i, i] \in T_k$ for x_i, we calculate easily that the left part of (1) is equal to $[g^{m+n}; 1, k]$, but the right part of (1) equals $[g^m; 1, k]$. Since $g^{m+n} \neq g^m$ in G, T_k does not satisfy (1) and therefore does not belong to $\mathbf{Comm} + \mathbf{V}$.

For $\alpha = 1, \ldots, k$ consider the subsemigroup T_k^α of T_k consisting of 0 and all matrices $[h; i, j]$, $h \in G$, where $j \neq \alpha$. It is clear that each $(k-1)$-generated subsemigroup of T_k is contained in T_k^α for some α. Thus it remains to prove that T_k^α belongs to $\mathbf{Comm} + \mathbf{V}$for every α. We can identify T_k^α with Rees matrix semigroup over G with sandwich matrix P_k^α obtained from P_k by deleting its αth row. Let us multiply in succession the first row of P_k^α by g^{-1}, then the second colomn of the result by g, then the second row by g^{-1}, ..., the $(\alpha - 1)$th row by g^{-1}, and finally the αth colomn by g. We obtain the matrix Q_k consisting of 0 and e only. It is well known (see [4], Section 3.2) that the above described transformation of the matrix does not change the semigroup T_k^α, in other words, T_k^α is isomorphic to the Rees matrix semigroup R_k over G with sandwich matrix Q_k. Let U_k be the Rees matrix semigroup over the trivial group E with sandwich matrix Q_k. It is easy to check that the mapping π:

$$\begin{cases} G \times U_k \xrightarrow{\pi} P_k \\ (h, [e; i, j]) \mapsto [h; i, j] \\ (h, 0) \mapsto 0 \end{cases}$$

for all $h \in G$, i, $j = 1, \ldots, k$, $i \neq \alpha$, is a homomorphism onto P_k. Now we note that $G \in \mathbf{Comm}$ and any Rees matrix semigroup over E belongs to the variety generated by A (see section 20e of [5]). This yields $T_k^\alpha \cong R_k \in \mathbf{Comm} + \mathbf{V}$.

Let us suppose now that $\mathbf{Comm} + \mathbf{V}$ has an identity basis involving r variables. Then a semigroup S belongs to $\mathbf{Comm} + \mathbf{V}$whenever all its r-generated subsemigroups are in $\mathbf{Comm} + \mathbf{V}$. Taking $k \geq r$ and T_k as S we come to a contradiction.

Now let us suppose that $\mathbf{V}$ is locally finite and contains all bands. We shall use the following result from [6]. Recall that Zimin words Z_n are defined as follows: $Z_1 = x_1, \ldots, Z_n = Z_{n-1} x_n Z_{n-1}$.

PROPOSITION. *If* $\mathbf{V}$ *is a semigroup variety, all semigroups of* $\mathbf{V} \cap \mathbf{var}\{\mathbf{x^2 = 0}\}$ *are locally finite and* $\mathbf{V}$ *does not satisfy any non-trivial identity of the type* $Z_n = W$ *then* $\mathbf{V}$ *has no identity basis involving finite number of variables.*

To use this result we have to show that all semigroups in $\mathbf{Comm} + \mathbf{V}$with the additional identity $x^2 = 0$ are locally finite and that $\mathbf{Comm} + \mathbf{V}$does not satisfy an identity of the form $Z_n = W$.

Let S be a finitely generated semigroup in $\mathbf{Comm} + \mathbf{V}$satisfying the identity $x^2 = 0$. As a member of the join $\mathbf{Comm} + \mathbf{V}$, S is a factor of an appropriate direct product $G \times A$ where G is a finitely generated semigroup in $\mathbf{V}$ and A is a finitely generated commutative semigroup. Since $\mathbf{V}$ is locally finite, G is finite. Thus S is a homomorphic image of some subsemigroup R of $G \times A$. Since S satisfies $x^2 = 0$, it is an image of the factor-semigroup of R over the ideal I generated by all squares. So we may suppose $S \cong R/I$. Let σ be the corresponding congruence on R, and let γ be the congruence on R corresponding to the homomorphism of R into G. Denote $R/\sigma \cap \gamma$ by T and denote I/γ by J. J is an ideal in T and $T/J \cong S$. Since J is finite (γ has a finite index) and squares of all elements of T are in J, T is a periodic semigroup. Clearly, γ induces a congruence δ on T such that $T/\delta \cong R/\gamma$. Every δ-class which is a subsemigroup is a homomorphic of some γ-class which is a subsemigroup also. But every such γ-class is commutative since R is a subdirect product of finite G and commutative A. Thus every γ-class which is a subsemigroup is periodic and commutative, so it is locally finite. Therefore T has a congruence (δ) of finite index such that all classes of δ which are subsemigroups are locally finite. Hence we can apply a theorem of Brown [7] and conclude that T is locally finite itself. This imply S is locally finite too. Thus every semigroup in $\mathbf{Comm} + \mathbf{V}$satisfying $x^2 = 0$ is locally finite.

It remains to prove that $\mathbf{Comm} + \mathbf{V}$does not satisfy any non-trivial identity of type $Z_n = W$. We shall prove that even the join of the variety of bands and the variety of all commutative semigroups can not satisfy such an identity. Suppose this join satisfies identity $Z_n = W$ for some n and W. Recall that by the definition $Z_n = Z_{n-1}x_nZ_{n-1}$ and so x_n occurs in Z_n only one time. Since the identity $Z_n = W$ holds in the variety of all commutative semigroups it is a balanced identity. It means that every letter occurs the same number of times in its left and right parts. For instance, x_n must occur in W exactly one time. Hence W can be represented in the form $W_0x_nW_1$ for some words W_0 and W_1 which do not contain x_n. If U is a word, let, as usual [8], $\gamma(U)$ ($\delta(U)$) stands for the minimal left (right) part of U containing all variables occuring in U. If x is a variable then $l_x(U)$ stands for the number of occurrences of x in U. For example, $\gamma(Z_n) = \delta(Z_n) = Z_{n-1}$, $l_{x_n}(Z_n) = 1$. As is well known [8] if an identity $U = V$ holds in the variety of all bands then identities $\gamma(U) = \gamma(V)$ and $\delta(U) = \delta(V)$ also hold in this variety. Hence it must satisfy identities $\gamma(Z_n) = \gamma(W)$ and $\delta(Z_n) = \delta(W)$. But it is easy to see that $\gamma(W) = W_0$ and $\delta(W) = W_1$. The first one of these identities has the form $Z_{n-1} = W_0$, the second one has the form $Z_{n-1} = W_1$. Clearly, one of these identities is non-trivial. If one of them is trivial then the other one is balanced and we get the contradiction with the minimality of n. Assume both identities are non-trivial. Hence there is a number $k < n$ such that numbers $l_{x_k}(Z_{n-1})$, $l_{x_k}(W_0)$ and $l_{x_k}(W_1)$ are different. Take the maximal k satisfying this property. Without loss of generality we may suppose that $l_{x_k}(Z_{n-1}) > l_{x_k}(W_1)$ because $l_{x_k}(W_0) + l_{x_k}(W_1) = 2l_{x_k}(Z_{n-1})$. If $k = n-1$ then we have $1 = l_{x_k}(Z_{n-1}) > l_{x_k}(W_1)$ whence $l_{x_k}(W_1) = 0$. Therefore in this case the identity $Z_{n-1} = W_1$ is heterogeneous and the variety of all bands can not satisfy it (in fact even the variety of all semilattices can not satisfy such an identity). So $k < n-1$. We may suppose that the identity $Z_n = W$ was chosen in such a way that the difference between the number of variables in this identity and k is minimal. Again we have $Z_{n-1} = Z_{n-2}x_{n-1}Z_{n-2}$,$W_1 = W_{01}x_{n-2}W_{11}$ and words Z_{n-2}, W_{01}, W_{11} do not contain x_{n-2}, $\gamma(Z_{n-1}) = \delta(Z_{n-1}) = Z_{n-2}$, $\gamma(W_1) = W_{01}$, $\delta(W_1) = W_{11}$ and the variety of all bands satisfy identities $Z_{n-2} = W_{01}$ and $Z_{n-2} = W_{11}$. Of course, either $l_{x_k}(Z_{n-2}) > l_{x_k}(W_{01})$ or $l_{x_k}(Z_{n-2}) > l_{x_k}(W_{01})$. Let, say, the first possibility holds. It is easy to see that the difference between the number of variables in this identity and k is less than that for the identity $Z_{n-1} = W_1$ and so we get a contradiction. This contradiction completes the proof of the theorem.

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Mark V. Sapir
Department of Mathematics and Statistics
Center for Information and Communication Sciences
University of Nebraska
Lincoln, NE 68506

Michael V. Volkov
Russia, 620083, Ekaterinburg,
Lenina 51
Urals State University
Department of Mathematics and Mechanics