Idempotents, regular elements and sequences from finite semigroups

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Abstract

If n is the number of non-idempotent elements of a finite semigroup S, it is shown that each sequence of length 2^n of elements of S contains a consecutive subsequence whose product is an idempotent element; and that 2^n is best possible among all finite semigroups with n non-idempotent elements. The proof remains valid if "idempotent" is replaced by each of the words or phrases "regular", "group", "core", "regular and core" and "group and core". The best bound, among all semigroups S with $|S| = n$, is also found, for semigroups and for monoids with or without a zero.

1 Background

In 1972 (answering a question of P. Erdos), D.W.H. Gillam, T.E. Hall and N.H. Williams [4] proved that every sequence of length $n + 1$ from a finite semigroup S with n non-idempotent elements contained a subsequence (not necessarily consecutive) with idempotent product. And that $n+1$ is the best possible.

It seems more natural to consider consecutive subsequences, and indeed several proofs and algorithms concerning finite semigroups have involved an upper bound on the length of all sequences that do not contain a consecutive subsequence whose product is idempotent. Three relevant papers were independently published in 1984, by D. Easdown [3], P. Higgins [6] and I. Simon [9].

In [3] Theorem 12, D. Easdown uses the bound $(n + 1)^n$ in his algorithm for testing if a finite biordered set comes from a finite semigroup, where n is the number of non-idempotent elements. Our bound 2^n makes his algorithm more efficient.

P. Higgins, [6] Lemma 10, showed that an infinite sequence contained a segment with idempotent product, in proving that any finite permutative semigroup is saturated.

In [9], I. Simon announced without proof that the best bound $q = q(n)$, for all monoids of size n, satisfies

$$2^{n/2} \leq q \leq 2^n.$$

In Section 4 we find q exactly: it is a constant times $2^{[2n/3]}$ (where $[2n/3]$ denotes the integer part of $2n/3$, and where the constant depends on the remainder of $2n$ when divided by 3).

A little later, C.J. Ash [1] considered an upper bound $M = M(n)$ (where $n = |S|$) on the length of all sequences from a finite semigroup S that do not contain a consecutive subsequence whose product is a regular element. This formed part of his proof that every finite semigroup S with commuting idempotents divides (is a morphic image of a subsemigroup of) a finite inverse semigroup I. An important question in automata and language theory is, given $|S|$, what is the least possible order of I. The proof by Ash makes a contribution to this question, and both our bounds (Theorems 3.1 and 4.1) give improvements to $|I|$.

2 Preliminaries

By the *core* $\langle E \rangle$ of a semigroup S is meant the subsemigroup of S generated by the set $E = E(S)$ of idempotents of S, if E is nonempty. An element g in a subgroup of S is called a *group* element, and as usual a *regular* element is an element a with an inverse a' in S (meaning $aa'a = a$ and $a'aa' = a'$).

Lemma 2.1 *(Hall [5], Lemma 1 and proof). For any semigroup S and any regular element of S in the core, say $a = e_1e_2 \ldots e_n$ where $e_1, e_2, \ldots, e_n \in E(S)$, there exist idempotents $f_1, f_2, \ldots, f_n$ all $\mathcal{D}$-related to a, and such that $a = f_1f_2 \ldots f_n$.*

3 The best bound for all S such that $|S| - |E| = n$

Theorem 3.1 *Let α be one of the words or phrases "idempotent", "regular", "group", "core", "regular and core", and "group and core". Let S be any finite semigroup with n non-α elements. Let $x_1, x_2, \ldots, x_{2^n}$ be any sequence of elements of S of length 2^n. Then there exist i and j, $1 \leq i \leq j \leq 2^n$, such that the product $x_i x_{i+1} \cdots x_j$ is an α element of S. For every $n > 1$ there exists a finite semigroup S_n with n non-α elements and a sequence $x_1, x_2, \ldots, x_{2^n-1}$ of elements of S of length $2^n - 1$ such that for any i and j, with $1 \leq i \leq j \leq 2^n - 1$, the product $x_i x_{i+1} \cdots x_j$ is not an α-element of S.*

Corollary 3.1 *Let S be any finite semigroup with n non-α elements. Let ϕ be any homomorphism from a free semigroup F onto S. Then every word w from F has a representation $w = u_0 w_1 u_1 \ldots w_m u_m$ such that $\phi(w_i)$ is an α element of S and the word $u_0 u_1 \ldots u_m$ has length less than 2^n. For every $n > 1$, 2^n is the least number with this property.*

Proofs. It is easy to see that the corollary follows from the theorem, so our goal now is to prove the theorem. We start with the first part of the theorem.

Let S be a semigroup with n non-α elements. Since we can adjoin a zero element if necessary, there is no loss in generality if we assume S has a zero.

Case 1. Let S be null (that is, $S^2 = 0$) and $|S| > 1$. Then the only α element in S is 0. The product of any two elements in S is 0. Since $2^n \geq 2$, the statement of the theorem holds in this case.

Case 2. Let S be a completely 0-simple semigroup with n non-α elements. Notice that $2^n \geq n + 1$. Take any sequence of length $n + 1$ from S: $x_1, \ldots, x_{n+1}$. Consider products x_1, $x_1 x_2$, $\ldots$. Since there are only n non-α elements in S, either one of these products is an α element or two of these products coincide. In the first case, we have the required result, an α element. Consider the second case. Take i and j such that $1 \leq i < j \leq n+1$ and $x_1 x_2 \cdots x_i = x_1 x_2 \cdots x_j$. Since $a = x_1 x_2 \cdots x_i$ is not an α-element, it is not equal to 0. Let $b = x_{i+1} \cdots x_j$. Then $ab = a \neq 0$ and from the Rees Theorem [2] it is easy to check that b is an idempotent, and so an α element.

Case 3. Let S be neither null nor completely 0-simple. We shall proceed by induction on the number of elements of S. Let I be a 0-minimal ideal of

S. Then I is either null or completely 0-simple. Thus $S \neq I$. Of the n non-α elements of S, suppose that $S \setminus I$ contains p, and so I contains $n-p=q$ say. It is clear that the factor semigroup S/I has p non-α elements (the zero I is an α element). Now take any sequence $\mathbf{s}$ of length 2^n of elements in S. Since $2^n = 2^{p+q} = 2^p 2^q$, we can divide $\mathbf{s}$ into 2^q consecutive parts $\mathbf{s}_i$ each of length 2^p. Suppose that the product of the elements of some $\mathbf{s}_i$ in S/I is not equal to 0. Then each element of S from this $\mathbf{s}_i$ does not belong to I. Since S/I contains fewer elements than S, the product of some consecutive elements from $\mathbf{s}_i$ is equal to a non-zero α-element from S/I. Then the product of these elements in S is, of course, an α-element, as required. Suppose now that the product of elements in each $\mathbf{s}_i$ is equal to 0 in S/I. Let p_i be the product of $\mathbf{s}_i$ in S. We have $p_i \in I$. Then we have a sequence of 2^q elements $p_1, p_2, \ldots, p_{2^q}$ in I.

Case (a): α is one of the words or phrases "idempotent", "regular", "group", "regular and core" and "group and core". Then each α element of S within I is also an α element of the semigroup I (from Lemma 2.1, for "regular and core" and "group and core"), so the semigroup I has q non-α elements of I itself (and of S). From Cases 1 and 2, some segment of $p_1, p_2, \ldots, p_{2^q}$ has its product equal to an α element of I, and hence of S, as required.

Case (b): α is the word "core". If I is completely 0-simple (and hence regular) then from Lemma 2.1, each core element of S in I is also a core element of I, so I has q non-core elements (of I itself). Thus, from Case 1, some segment of the sequence $p_1, p_2, \ldots, p_{2^q}$ from I has product equal to a core element, which gives the required result.

So we suppose I is null. If $q = 0$ then p_1 (the product of $\mathbf{s}_1 = \mathbf{s}$) is an α element, since $p_1 \in I$ and I consists solely of α elements of S. If $q \geq 1$, then $p_1 p_2 = 0$, an α element of S.

This proves the first part of the theorem.

Now let us prove the second part.

For every $n \geq 1$ let S_n be the semigroup with elements $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ with the following multiplication: for all i, $a_i^2 = b_i^2 = a_i b_i = b_i a_i = b_i$, and for all $j > i$, $a_i b_j = b_j a_i = b_j$, $a_j b_i = b_i a_j = a_j$, $a_j a_i = a_i a_j = a_j$, $b_j b_i = b_i b_j = b_j$. It is easy to see that this binary operation is indeed associative. Observe that for every n the semigroup S_n contains exactly n non-α elements, whichever α we choose, namely the elements $a_1, \ldots, a_n$.

Now for every $m \geq 1$ let us consider the Zimin words Z_m, defined inductively by $Z_1 = a_1, \ldots, Z_{m+1} = Z_m a_{m+1} Z_m$. These words play crucial roles in many investigations in algebra (see the survey paper by Kharlampovich and Sapir [8]). The following properties of Z_m are well-known and are easy to prove by induction on m:

Z1 The length of Z_m is $2^m - 1$.

Z2 Let u be a subword of Z_m and let a_i be the (unique) letter in u with the largest index. Then a_i occurs in u exactly once.

Let us consider Z_n as a sequence of elements in S. By the property Z1 this sequence contains $2^n - 1$ elements. We claim that every product of consecutive elements in Z_n is equal to one of the a_i, and so is a non-α element. Indeed, let u be a subword of Z_n. Let a_i be the letter from this word with the largest index. Then the property Z2 implies that a_i occurs only once in this subword, and the definition of the multiplication in S_n implies that the product of the elements in u (considered as a sequence of elements in S_n) is equal to a_i.

The theorem is proved.

The semigroups S_n show that 2^n is the best bound for each class of finite semigroups containing each S_n, which gives us

Theorem 3.2 *Theorem 3.1 is also valid for finite semigroups of each, and each combination, of the following types: commutative; idempotent-commuting; aperiodic; $\mathcal{J}$-trivial; $\mathcal{L}$-trivial; $\mathcal{R}$-trivial.*

We now consider regular semigroups.

Theorem 3.3 *With α restricted to "idempotent", "core", "group and core", Theorem 3.1 is valid for finite semigroups of each of the following types: regular; inverse; semilattices of groups; commutative and regular.*

Proof. For $n \geq 1$, let $T_n = \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ have the following multiplication: for all i, $a_i^2 = b_i = b_i^2$, $a_i b_i = b_i a_i = a_i$ and for all $j > i$, $a_i b_j = b_j a_i = b_j$, $a_j b_i = b_i a_j = a_j$, $a_j a_i = a_i a_j = a_j$, $b_j b_i = b_i b_j = b_j$. Then T_n is a length n chain of the groups $\{a_i, b_i\}$. Again the non-α elements are $a_1, a_2, \ldots, a_n$ and the proof of the second statement in Theorem 3.1 is also valid with T_n replacing S_n.

4 The best bound for all S such that $|S| = n$

For $n = 1, 2, 3,$ ... we define integers $b(n)$ and $z(n)$. We prove $b(n)$ is the smallest of the integers m such that for every semigroup S of size n, every sequence of length m from S contains a consecutive subsequence whose product is idempotent. For all semigroups with zero and of size n, we prove that the corresponding smallest integer is $z(n)$. For monoids, we prove that the corresponding integers are $b^1(n)$ and $z^1(n)$, defined below.

Definition 4.1

n	*1*	*2*	$3k$	$3k+1$	$3k+2$	*(for* $k \geq 1$*)*
$b(n)$	*1*	*2*	4^k	$\frac{3}{2}4^k$	$(\frac{3}{2})^2 4^k$	
$b^1(n) = z(n)$	*1*	*2*	$3 \times 4^{k-1}$	$\frac{9}{2}4^{k-1}$	2×4^k	
	with the exception $z(4) = 4$					
$z^1(n)$	*1*	*1*	$9 \times 4^{k-2}$	4^k	$\frac{3}{2}4^k$	
	with the exception $z^1(3) = 2$					

We show some values of $b(n)$, $z(n)$, $b^1(n)$ and $z^1(n)$. Further values are easily obtained from the equation $f(n+3) = 4f(n)$ (for $n \geq 5$), where $f(n)$ is any of these four functions.

n	1	2	3	4	5	6	7	8	9
$b(n)$	1	2	4	6	9	16	24	36	64
$b^1(n) = z(n)$	1	2	3	4	8	12	18	32	48
$z^1(n)$	1	1	2	4	6	9	16	24	36

Lemma 4.1 *(a) For* $n \geq 4$, $z(n) = 2b(n-2)$.

(b) $z(4) < \frac{1}{2}b(5)$ *and for* $n \neq 4$, $z(n) = \frac{1}{2}b(n+1)$.

(c) For $n \geq 2$, $z^1(n) = b(n-1)$.

Proof. For the values given in the preceding table, it is easy to check (a), (b) and (c). Since all further values are obtained from $f(n+3) = 4f(n)$ (for $n \geq 5$), we have statements (a), (b) and (c), as required.

Now we construct semigroups A_n and $B_n = B_n^0$, and monoids M_n and $N_n = N_n^0$, all of size n, and for which $b(n)$, $z(n)$, $b^1(n)$ and $z^1(n)$ are the best bounds (these semigroups were the authors' source of these integers).

Of course $A_1 = B_1 = M_1 = N_1$ is the one element semigroup. Put $A_2 = B_2 = \{0, a\}$ the two element null semigroup ($a^2 = 0a = a0 = 00 = 0$). Let M_2 be the two element group and N_2 be the two element semilattice $\{0, 1\}$.

Put $A_3 = \{e, a, b\}$ where $\{e, a\}$ is an ideal and a subgroup with identity element e, and b acts like e, meaning that $b^2 = be = eb = e$, $ba = ab = a$. The sequence b, a, b of length 3 ($= b(3) - 1$) does not contain a consecutive subsequence whose product is idempotent.

Put $B_3 = \{0, a, a^2\}$, where $a^3 = 0$, $a^2 \neq 0$. Let M_3 be the three element group and put $N_3 = M_2^0$, the two element group with an adjoined zero.

Put $A_4 = \{e, a, a^2, b\}$ where $\{e, a, a^2\}$ is an ideal and a subgroup with identity element e (so $a^3 = e$) and b acts like e ($b^2 = be = eb = e$, $ba = ab = a$). The sequence b, a, b, a, b of length 5 ($= b(4) - 1$) has no consecutive subsequence with idempotent product.

For any semigroups $U_1, U_2, ..., U_k$ we define a semigroup $T = U_1 \circ U_2 \circ ... \circ U_k$, called the *tower of* $U_1, U_2, ..., U_k$, as follows: T is the disjoint union of $U_1, U_2, ..., U_k$ (disjoint copies of $U_1, U_2, ..., U_k$ can be made if necessary), with each U_i as a subsemigroup of T, and with each element of U_i acting as an identity element for U_j if $i < j$ (that is, $u_i u_j = u_j u_i = u_j$ for all $u_i \in U_i$, $u_j \in U_j$, if $i < j$). It is clear that the binary operation of T is associative and that also $\circ$ is associative. Notice that this construction was first introduced by Kaufman [7] in 1949. Kaufman called it the sequentially annihilating sum. Later this construction appeared in many other papers under different names. Notice also that the semigroup S_n of Section 3 is a tower of n copies of the two element null semigroup. The semigroup T_n of Section 3 is a tower of n copies of the two element group.

Lemma 4.2 *For any semigroups U, V, if the sequences $\mathbf{u} = u_1, u_2, ..., u_{p-1}$, $\mathbf{v} = v_1, v_2, ..., v_{q-1}$, from U, V respectively, have no consecutive subsequences* *with idempotent products, then in $U \circ V$ the sequence*

$$\mathbf{u}, v_1, \mathbf{u}, v_2, \mathbf{u}, v_3, ..., \mathbf{u}, v_{q-1}, \mathbf{u}$$

of length $p(q-1) + (p-1) = pq - 1$ has no consecutive subsequence with idempotent product.

Proof. Take any consecutive subsequence $\mathbf{w}$. If $\mathbf{w}$ is a consecutive subsequence of $\mathbf{u}$, then the required conclusion is obvious. If not, then the product of $\mathbf{w}$ equals the product of its subsequence obtained by deleting all u_i from $\mathbf{w}$, which is then a consecutive subsequence of $\mathbf{v}$, and so the product of $\mathbf{w}$ is not an idempotent, as required.

Put $B_4 = M_4 = N_4 = M_2 \circ B_2$, a semigroup with 0 and 1.

Put $A_5 = \{e, a, a^2, b, b^2\}$ where $\{e, a, a^2\}$ is an ideal and a subgroup with identity element $e = a^3$, and both b, b^2 act on $\{e, a, a^2\}$ like e does ($xy = yx = y$ for $x \in \{b, b^2\}$, $y \in \{e, a, a^2\}$). The sequence b, b, a, b, b, a, b, b of length 8 ($= b(5) - 1$)) has no consecutive subsequence with idempotent product.

For all remaining values of n, we define semigroups A_n, B_n, M_n, N_n of size n, to be towers as shown ($A_3^r = A_3 \circ A_3 \circ ... \circ A_3$, a tower of r copies of A_3, and in particular $A_3^r = \emptyset$ if $r = 0$):

$n(\geq 3)$	$3k$	$3k+1$	$3k+2$
A_n	A_3^k	$A_3^{k-1} \circ A_4$	$A_3^{k-1} \circ A_5$
B_n	$A_3^{k-1} \circ B_3$	$A_3^{k-2} \circ A_5 \circ B_2$ ($k \geq 2$)	$A_3^k \circ B_2$
M_n	$M_3 \circ A_3^{k-1}$	$M_2 \circ A_3^{k-2} \circ A_5$ ($k \geq 2$)	$M_2 \circ A_3^k$
N_n	$M_3 \circ A_3^{k-2} \circ B_3$ ($k \geq 2$)	$M_2 \circ A_3^{k-1} \circ B_2$	$M_2 \circ A_3^{k-1} \circ B_3$

It is routine to check, by repeated use of Lemma 4.2, that A_n, B_n, M_n and N_n contain a sequence of length $b(n)-1$, $z(n)-1$, $b^1(n)-1$ and $z^1(n)-1$ respectively, with no consecutive subsequence having an idempotent product.

Lemma 4.3 *For $l, m \geq 1$,*

(a) $b(l+m) \geq b(l)b(m)$

(b) $z(l+m) \geq b(l)z(m)$

(c) $b^1(l+m) \geq b^1(l)b(m)$

(d) $z^1(l+m) \geq b^1(l)z(m)$.

Proof.

(a) From Definition 4.1 for $b(n)$, statement (a) is routine, but lengthy, to prove, by consideration of the cases

(i) $l = 1$

(ii) $l = 2 \leq m$

(iii) $l = 3k, m \in \{3p, 3p+1, 3p+2\}$

(iv) $l = 3k+1, m \in \{3p+1, 3p+2\}$

(v) $l = 3k+2,\ m = 3p+2,$ where $p, k \geq 1$.

The symmetry between l and m implies that further cases need not be considered. As an example, we treat case (v):

$$b(l+m) = b(3(k+p+1)+1) = \tfrac{3}{2}4^{k+p+1}$$

$$b(l)b(m) = (\tfrac{3}{2})^2 4^k\ (\tfrac{3}{2})^2 4^p = \tfrac{3}{2}\ .\tfrac{27}{8}.4^{k+p} \leq b(l+m).$$

(b) For $m \geq 4$, from Lemma 4.1 we have

$$\begin{aligned} z(l+m) &= 2b(l+m-2) \\ &\geq 2b(l)b(m-2) && \text{(from (a))} \\ &= b(l)z(m) && \text{(Lemma 4.1).} \end{aligned}$$

For $m = 1, 2, 3$ we want $z(l+1) \geq b(l)$, $z(l+2) \geq 2b(l)$ and $z(l+3) \geq 3b(l)$ respectively, all of which are easy to check from the table of values and from $f(n+3) = 4f(n)$ $(n \geq 5)$.

(c) This is a restatement of (b), with l and m interchanged.

(d) For $l = m = 1$ we have $z^1(1+1) = 1$ and $b^1(1)z(1) = 1 \leq z^1(1+1)$.

For $l + m \geq 3$, we have

$$\begin{array}{rcll} z^1(l+m) & = & b(l+m-1) & \text{(Lemma 4.1(c))} \\ & = & \frac{1}{2}z(l+m+1) & \text{(Lemma 4.1(a))} \\ & \geq & \frac{1}{2}b(l+1)z(m) & \text{(part (b))} \\ & \geq & z(l)z(m) & \text{(Lemma 4.1(b))} \\ & = & b^1(l)z(m), & \text{as required.} \end{array}$$

Theorem 4.1 *(a) From all semigroups [monoids] with* n *elements, every sequence of length* $b(n)$ $[b^1(n)]$ *has a consecutive subsequence with idempotent product. From all semigroups [monoids] with zero and with* n *elements, every sequence of length* $z(n)$ $[z^1(n)]$ *has such a subsequence.*

(b) The integers $b(n)$ *and* $z(n)$ $[b^1(n)$ *and* $z^1(n)]$ *are the smallest such integers.*

Proof. Our proof for semigroups is easily modified to cover monoids: the modifications are shown in square brackets.

(a) For $n = 1$, the statement is trivial, so we take any positive integer $n > 1$ and assume both statements in part (a) are true of all semigroups [monoids] of size less than n. Take any semigroup [monoid] S of size n.

Since $n \leq z(n) \leq b(n)$ $[n \leq b^1(n)]$, the cases S is null, completely simple, or completely 0-simple can be dealt with as in Cases 1 and 2 in the proof of Theorem 3.1. So we assume S is otherwise.

Take an ideal I of S which is 0-minimal if S has a zero, and is the least ideal otherwise. Then $2 \leq |I| < n$ and $|S/I| < n$, so both statements in part (a) are true of I and S/I. Put $p = |I|$, $q = |S/I|$; then $p + q = n + 1$.

Case 1: S has no zero; equivalently I is completely simple.

Take any sequence from S of length $b(n)$ $[b^1(n)]$, say $\mathbf{s}$. Denote the integer part of $b(n)/z(q)$ $[b^1(n)/z^1(q)]$ by t and break $\mathbf{s}$ into segments $\mathbf{s} = \mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_t, \mathbf{r}$ where each $\mathbf{s}_i$ has length $z(q)$ $[z^1(q)]$. If the product of some $\mathbf{s}_i$ is in $S \setminus I$, then some consecutive subsequence has idempotent product, as required, so we assume the product c_i of each $\mathbf{s}_i$ is in I. We consider the sequence $c_1, c_2, ..., c_t$ from the completely simple semigroup I, and we show that $t \geq |I|$.

From Lemma 4.3(a) [(c)],

$$b(n) = b(q + p - 1) \geq b(q)b(p - 1) \geq z(q)b(p - 1)$$

$$[b^1(n) = b^1(q + p - 1) \geq b^1(q)b(p - 1) \geq z^1(q)b(p - 1)],$$

so $b(n)/z(q) \geq b(p - 1)$ $[b^1(n)/z^1(q) \geq b(p - 1)]$, which gives $t \geq b(p - 1)$. From the table of values (and from $f(n + 3) = 4f(n)$ for $n \geq 5$) we deduce $t \geq b(p - 1) \geq p$ for $p \geq 4$. Thus $t \geq |I|$ for $|I| \geq 4$.

For $p = 2$ and $q \geq 1$ we deduce from the table that

$$b(n) = b(q + 1) \geq 2z(q) \ \ [b^1(n) = b^1(q + 1) \geq 2z^1(q)],$$

so again $t \geq 2 = |I|$.

For $p = 3$ and $q \geq 1$ we deduce from the table that

$$b(n) = b(q + 2) \geq 3z(q) \ \ [b^1(n) = b^1(q + 2) \geq 3z^1(q)],$$

so again $t \geq 3 = |I|$.

In all cases $t \geq |I|$. From Case 2, in the proof of Theorem 3.1, we have that $c_1, c_2, ..., c_t$ has a consecutive subsequence with idempotent product, as required.

Case 2: S has a zero; equivalently, I is completely 0-simple or null.

Take any sequence from S of length $z(n)[z^1(n)]$, say $\mathbf{s}$. Denote the integer part of $z(n)/z(q)$ $[z^1(n)/z^1(q)]$ by t and again break $\mathbf{s}$ into segments $\mathbf{s} = \mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_t, \mathbf{r}$ where each $\mathbf{s}_i$ has length $z(q)$. As in Case 1 we can assume the product c_i of each $\mathbf{s}_i$ is in I and we consider the sequence $c_1, c_2, ..., c_t$ from I.

From Lemma 4.3 (b) [(d)],

$$z(n) = z(p - 1 + q) \geq b(p - 1)z(q)$$

$$[z^1(n) = z^1(p - 1 + q) \geq b^1(p - 1)z(q) \geq b^1(p - 1)z^1(q)],$$

which gives $z(n)/z(q) \geq b(p - 1)$ $[z^1(n)/z^1(q) \geq b^1(p - 1)]$, whence

$$t \geq b(p - 1)[t \geq b^1(p - 1)] \geq p - 1 = |I| - 1$$

from the table.

If I is completely 0-simple then by Case 2 in the proof of Theorem 3.1, any sequence of length $t \geq |I|-1$ has a consecutive subsequence with idempotent product, as required.

If I is null and $|I| \geq 3$ then $t \geq 2$ and $c_1c_2 = 0$: there remains the case I is null and of size 2 (whence $t = 1$ in most subcases).

Denote by a the nonzero element of I, and put

$$J = \{x \in S : \ ax = 0\}, \ \ T = \{x \in S : \ ax = a\}.$$

Then $S = T \cup J$, $T \cap J = \emptyset$, and T (if nonempty) is a subsemigroup of S and J is an ideal of S.

Case 2(a): $T \neq \emptyset$. Then $|T| < n$, $|J| < n$, so the two statements of part (a) are true of T and J. Break the sequence $\mathbf{s}$ into segments $\mathbf{s} = \mathbf{d}_1, \mathbf{d}_2, ..., \mathbf{d}_l, \mathbf{r}$, where each $\mathbf{d}_i$ has length $b(|T|)$ [$b^1(|T|)$] and l is the integer part of $z(n)/b(|T|)$ [$z^1(n)/b^1(|T|)$]. From $n = |T| + |J|$ and Lemma 4.2(b) [(d)] we have $z(n) \geq b(|T|)z(|J|)$ [$z^1(n) \geq b^1(|T|)z(|J|)$] so $l \geq z(|J|)$. Again, if the product q_i of some $\mathbf{d}_i$ is in T, then some consecutive subsequence of $\mathbf{d}_i$ has idempotent product, and so we can assume the product q_i of each $\mathbf{d}_i$ is in J, and we consider the sequence $q_1, q_2, ..., q_l$ from J. Since $l \geq z(|J|)$, some consecutive subsequence of $q_1, q_2, ..., q_l$, and thus also of $\mathbf{s}$, has idempotent product.

Case 2(b): $T = \emptyset$; that is, $ax = 0$ for all $x \in S$.

We return to the segmentation $\mathbf{s} = \mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_t, \mathbf{r}$ of $\mathbf{s}$. Then $\mathbf{s}$ and $\mathbf{s}_1$ have lengths $z(n)$ and $z(n-1)$ respectively, and $z(n) > z(n-1)$, so $\mathbf{s} = \mathbf{s}_1, x, \mathbf{u}$ for some $x \in S$ and (possibly empty) sequence $\mathbf{u}$ from S. Now c_1, the product of $\mathbf{s}_1$, is in $I = \{0, a\}$, so $c_1x = 0$ (an idempotent).

By induction, we have proved Theorem 4.1 (a)

(b) Part (b) now follows from the examples A_n and B_n [M_n and N_n] given just prior to the theorem.

Remark. Since these examples A_n, B_n [M_n, N_n] are commutative, Theorem 4.1 is also valid for the class of commutative semigroups [monoids].

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