Eventually $\mathcal{H}$-related sets and systems of equations over finite semigroups and rings

M. V. Sapir*

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Abstract

I prove that given a finite semigroup (finite associative ring) S and a system Σ of equations of the form $ax = b$ or $xa = b$ where $a, b \in S$, x is a unknown, it is algorithmically impossible to decide whether or not Σ is solvable over S, that is whether or not there exists a bigger finite semigroup (finite ring) $T > S$ such that Σ has a solution in T. The proof employs the unsolvability of the uniform word problem in the class of finite groups (Slobodskoii) and the so called split systems.

In this paper we consider mainly finite semigroups and finite associative rings given by their multiplication (addition) tables.

Let us recall some basic definitions from the theory of semigroups (see [1]). There are four important equivalence relations associated with every semigroup S. These relations were introduced by Green in the 40-s.

$$\begin{aligned}
&\mathcal{L}_S = \{(x, y) \mid zx = y,\ ty = x \text{ for some } z, t \in S^1\},\\
&\mathcal{R}_S = \{(x, y) \mid xz = y,\ yt = x \text{ for some } z, t \in S^1\},\\
&\mathcal{J}_S = \{(x, y) \mid z_1 x z_2 = y,\ t_1 y t_2 = x \text{ for some } z_1, z_2, t_1, t_2 \in S^1\},\\
&\mathcal{H}_S = \mathcal{L} \cap \mathcal{R},\\
&\mathcal{D}_S = \mathcal{L} \circ \mathcal{R}.
\end{aligned}$$

Here S^1 stands for the semigroup S with an identity element adjoint and $\circ$ is the operation of composition of relations. Similar relations can be defined on a finite associative ring.

In the case of finite semigroups relations $\mathcal{D}$ and $\mathcal{J}$ coincide [1]. A semigroup S is a group if and only if each of these relations is equal to the universal relation $S \times S$.

If S is a subsemigroup (subring) of a semigroup (ring) T then $\mathcal{L}_S$, $\mathcal{R}_S, \ldots$, $\mathcal{D}_S$ are subsets of the corresponding relations on T but do not necessarily coincide with the restrictions of these relations from T to S.

Let $\mathcal{X}$ be one of the letters $\mathcal{L}$, $\mathcal{R}, \ldots$, $\mathcal{D}$ and let $A \subseteq S \times S$. Then we say that A is *eventually $\mathcal{X}$-related* if $A \subseteq \mathcal{X}_T$ for some semigroup (ring) T containing S. The

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description of subsets of a semigroup which are eventually $\mathcal{L}$-related or eventually $\mathcal{R}$-related is well known (see Pastijn [7] or Fountain [4]). Let us define two relations $\mathcal{L}^*$ and $\mathcal{R}^*$ as follows

$$\begin{array}{lll} (x,y)\in\mathcal{L}^* & \text{iff for every } z,t\in S & xz=xt\leftrightarrow yz=yt; \\ (x,y)\in\mathcal{R}^* & \text{iff for every } z,t\in S & zx=tx\leftrightarrow zy=ty. \end{array} \tag{1}$$

Lemma 1 *(Pastijn, [7]). A subset $A\subseteq S\times S$ of a semigroup S is eventually $\mathcal{L}$-related (resp. $\mathcal{R}$-related) if and only if $A\subseteq\mathcal{L}^*$ (resp. $A\subseteq\mathcal{R}^*$).*

In particular there is an algorithm which given a finite semigroup S and a subset $A\subseteq S\times S$ determines whether or not A is eventually $\mathcal{L}$-related or $\mathcal{R}$-related. On the other hand a similar description of eventually $\mathcal{H}$-related sets or $\mathcal{J}$-related sets is not known.

The following Theorem 1 shows that there is no algorithm to check whether a set is eventually $\mathcal{H}$-related.

Theorem 1 *There is no algorithm which, given a finite semigroup (ring) S and two disjoint subsets $A,B\subseteq S$, decides whether or not there exists a finite semigroup (ring) $T>S$ such that $A\subseteq H_1$ and $B\subseteq H_2$ where H_1 and H_2 are $\mathcal{H}$-classes of T. In other words there is no algorithm to decide whether or not $A\times A\cup B\times B$ is eventually $\mathcal{H}$-related.*

Corollary 1 *There exists a finite semigroup S and a subset $A\subseteq S\times S$ such that A is eventually $\mathcal{L}$-related and eventually $\mathcal{R}$-related but not eventually $\mathcal{H}$-related.*

Proof. It is clear that if a set $A\subseteq S\times S$ is eventually $\mathcal{H}$-related then it is eventually $\mathcal{L}$-related and eventually $\mathcal{R}$-related. So if for every finite semigroup S and every subset $A\subseteq S\times S$ such that A is eventually $\mathcal{L}$-related and eventually $\mathcal{R}$-related we had that A is eventually $\mathcal{H}$-related, then we could recognize eventually $\mathcal{H}$-related sets by using Lemma 1.

Remark. From a result of M.Putcha [8], it follows that there exists an algorithm which given a finite semigroup S and a subset $A\subseteq S$, decides whether there is a bigger (not necessarily finite) semigroup T containing S such that $A\times A\subseteq\mathcal{H}_T$. Thus it is interesting to know the answers to the following questions.

Problem 1 *Is there an algorithm to decide whether a given subset of a finite semigroup S is contained in an $\mathcal{H}$-class of some bigger finite semigroup?*

Problem 2 *Is there an algorithm to decide, given a finite semigroup S and a subset $A\subseteq S\times S$, whether A is contained in $\mathcal{H}_T$ for some bigger (not necessarily finite) semigroup T.*

Theorem 1 can be reformulated in terms of equations over finite semigroups (rings). Let S be a finite universal algebra. An *equation* over S is a pair of terms in the alphabet $S\cup X$. Elements of X are called unknowns, elements of S are called constants. A system of equations $\{u_i = v_i \mid i \in I\}$ with the set of unknowns X is called *solvable* if there exists a finite universal algebra $T > S$ and a map $\phi : X \to T$ such that all equalities $u_i = v_i$ hold in T after the substitution of $\phi(x)$ for x ($x \in X$).

Let us remark that by a theorem of Kublanovsky [5] the problem of whether or not a finite semigroup is embeddable into a finite 0-simple[1] semigroup is undecidable. It is easy to prove that a finite semigroup S is embeddable into a finite 0-simple semigroup if and only if the set $(S\backslash\{0\}) \times (S\backslash\{0\})$ is eventually $\mathcal{J}$-related. Thus it is impossible to decide whether or not a subset of $S \times S$ is eventually $\mathcal{J}$-related. Notice that the property of being eventually $\mathcal{J}$-related can be expressed in terms of a system of quadratic equations of the form $x_1 a x_2 = b$. This implies that the solvability of systems of quadratic equations over finite semigroups is undecidable.

The following result shows that the problem of solvability of systems of linear equations over finite semigroups or algebras over $GF(p)$ is undecidable.

Theorem 2 *The following problem is undecidable.*

Input: *a finite semigroup (ring) S and a finite set Σ of equations of one of the following forms*

$$ax = b \tag{2}$$

$$xa = b \tag{3}$$

where $a, b \in S$.

Decide *whether or not Σ is solvable.*

Proof. Indeed, two elements a and b belong to the same $\mathcal{H}$-class of a semigroup T if and only if $a = x_1 b,\ b = x_2 a,\ a = b x_3,\ b = a x_4$ for some elements $x_1, x_2, x_3, x_4 \in T$.

This proof shows that in the formulation of the problem in Theorem 2 one can restrict oneself to such systems of equations where every two equations have different unknowns. From the proof of Theorem 1, it follows that in this corollary one can restrict oneself to finite 3-nilpotent semigroups (rings) [2].

It is interesting that if we consider equations of only one type (2) or (3) then there exists an algorithm to decide whether or not a given system of equations is solvable over a finite semigroup or a $GF(p)$-algebra. The following theorem is a generalization of the description of eventually $\mathcal{L}$- and $\mathcal{R}$-related sets mentioned above.

Theorem 3 *Let Σ be a system of equations of type (2) over a finite semigroup ($GF(p)$-algebra) S. This system is solvable if and only if the following condition AS (resp. AR) hold:*

AS. If $ax = b, cx = d \in \Sigma$ for some $a, b, c, d \in S,\ x \in X$ then for every $p, q \in S^1$ $pa = qc$ implies $pb = qd$.

AR. If $a_i x = b_i \in \Sigma$ for some $a_i, b_i \in S, x \in X,\ i = 1, \ldots, n$ then for every $p_i \in S^1$ $\sum_i p_i a_i = 0$ imples $\sum_i p_i b_i = 0$.

[1] A semigroup is called 0-*simple* if it does not have ideals except itself and possibly zero.

[2] A semigroup (ring) is called 3-nilpotent if the product of every 3 elements of it is zero.

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1 Proof of Theorem 1

It is clear that for every semigroup S a set $A \subseteq S \times S$ is eventually $\mathcal{H}$-related if and only if there exists a finite ring $T > GF(p)S$ such that $A \subseteq \mathcal{H}_T$. Thus it is enough to prove Theorem 1 in the case of semigroups.

In the proof, we shall use the so called split systems introduced by the author in 1978 (see [9]) and used in several of his papers on quasi-varieties and varieties of semigroups, groups and rings, (see [10], [11], [12], [13]).

A *split system* is a triple of disjoint non-empty sets (A, B, C) with an operation $A \times B \to C$.

A *morphism* from a split system (A, B, C) into a semigroup S is a triple of maps $f : A \to S, g : B \to S, h : C \to S$ such that $h(ab) = f(a)g(b)$ for every $a \in A, b \in B$.

We say that a morphism (f, g, h) is injective if each map f, g, h is injective. We say that a split system (A, B, C) is *embeddable* into a semigroup S if there exists an injective morphism from (A, B, C) into S.

Let S be a *partial group*, i.e. a set with a partial binary operation and an identity element 1 such that $x = 1 \cdot x = x \cdot 1$. Following Kublanovsky [5] let us call a partial group S' an *extension of rank* 2 of S if

1. the identity element of S is the identity element of S';
2. $S < S'$ and the operation of S is a restriction of the operation in S';
3. for every two elements $a, b \in S$ their product is defined in S';
4. $S' = S \cup S \cdot S$, that is every element in S' either belongs to S or is a product of two elements of S;
5. the product of $a, b \in S'$ is not defined whenever a or b does not belong to S.

As in [5] we shall use the following result which immediately follows from the undecidability of the uniform word problem in the class of all finite groups (Slobodskoii, [14]) and from Connection 2.2 in [6] (which is essentially Evans' result [2, 3]).

Lemma 2 *The problem of whether or not a given extension of rank 2 of a partial group is embeddable into a finite group is undecidable.*

With every partial group S' which is an extension of rank 2 of a partial group S one can associate the following split system:

$$F(S, S') = (S \times \{1\}, S \times \{2\}, S' \times \{3\})$$

with operation $(a,1)(b,2) = (ab,3)$. This map F is an extension from semigroups to partial groups of the functor F_{SYS} considered in [9], [10], [11], [12], [13]. The following lemma is a generalization of a result from [9], [10].

Lemma 3 *The split system $F(S,S')$ is embeddable into a group G if and only if S' is embeddable into G.*

Proof. Suppose that the split system $F(S,S')$ is embeddable into a group G and (f,g,h) is the corresponding embedding. We have that for every x in S

$$f(x)g(1) = h(x),\ \ f(1)g(x) = h(x).$$

Hence $f(x) = h(x)g(1)^{-1}$, $g(x) = f(1)^{-1}h(x)$. Therefore for every $x,y \in S$ we have: $h(xy) = f(x)g(y) = h(x)g(1)^{-1}f(1)^{-1}h(y)$. Therefore the map $\psi : S' \to G$ which takes x to $h(x)g(1)^{-1}f(1)^{-1}$ is a homomorphism on S. Since products xy do not exist if x or y is not in S, this map is a homomorphism of S'. It is easy to check that this homomorphism is one-to-one.

Now suppose that S' is embeddable into a group G and ψ is the embedding. Let $f : S \times \{1\} \to G$, $g : S \times \{2\} \to G$, $h : S' \times \{3\} \to G$ be maps defined as follows:

$$\psi(x) = f(x,1) = g(x,2) = h(x,3).$$

It is easy to see that this triple is a morphism from $F(S,S')$ into G and that each map f,g,h is one-to-one. Therefore $F(S,S')$ is embeddable into G.

The lemma is proved.

We shall say that a split system (A,B,C) has the *cancellation* property if $ax = ay$ implies $x = y$ for every $a \in A$, $x,y \in B$ and $xb = yb$ implies $x = y$ for every $x,y \in A$ and $b \in B$. It is clear that if a split system is embeddable into a group, it has the cancellation property.

Lemmas 3 and 2 imply the following result.

Lemma 4 *The problem of whether or not a given split system with the cancellation property is embeddable into a finite group is undecidable.*

We shall say that a split system (A,B,C) is $\mathcal{H}$-*embeddable* into a semigroup S if there exists such an embedding (f,g,h) of (A,B,C) into S such that the images of A and C are contained in some $\mathcal{H}$-classes H_1 and H_2 of S. This morphism (f,g,h) will be called an $\mathcal{H}$-*embedding.*

Lemma 5 *A split system (A,B,C) with the cancellation property is $\mathcal{H}$-embeddable into a finite regular semigroup S if and only if (A,B,C) is embeddable into a subgroup of S.*

Proof. The "if" part is trivial because a group is a regular semigroup with only one $\mathcal{H}$-class - itself.

Let us prove the "only if" part. Suppose that a split system (A,B,C) is $\mathcal{H}$-embeddable into a regular semigroup and (f,g,h) is the corresponding $\mathcal{H}$-embedding. Let us denote the $\mathcal{H}$-classes containing $f(A)$ and $h(C)$ by H_1 and H_2 respectively.

Since S is regular and finite, we can apply Green's lemma to the $\mathcal{J}$-class D_1 containing H_1 and conclude that there exist elements $p, q \in S$ such that pH_1 is contained in a subgroup G_1 of D_1 and $qpx = x$ for every $x \in H_1$.

Then $qph(xy) = qpf(x)g(y) = f(x)g(y) = h(xy)$ for every $x \in A,\ y \in B$. Therefore $ph(xy)$ and $h(xy)$ are in the same $\mathcal{J}$-class of S. Again by Green's lemma pH_2 is contained in an $\mathcal{H}$-class of S and for some $q' \in S$ we have that $q'px = x$ for every $x \in H_2$.

Consider three functions $f_1 : A \to S,\ g_1 : B \to S,\ h_1 : C \to S$ defined as follows:

$$f_1(x) = pf(x),\ g_1(x) = g(x),\ h_1(x) = ph(x).$$

Since $qf_1(x) = f(x)$ for every $x \in A$ and $q'h_1(x) = h(x)$ for every $x \in C$, functions f_1, g_1, h_1 are one-to-one. It is also easy to check that $h_1(xy) = f_1(x)g_1(y)$. Thus (f_1, g_1, h_1) is an embedding of (A, B, C) into S. This embedding is an $\mathcal{H}$-embedding and it satisfies the following additional property:

(1) The image of A is contained in a subgroup of S.

Thus without loss of generality we can assume that the original embedding satisfies this property, so H_1 is a group.

Let e be the identity element of H_1. Take any $a \in A$ and let $f_1(x) = f(a)^{-1}f(x)$ for every $x \in A$, $g_1(x) = g(x)$ for every $x \in B$, $h_1(x) = f(a)^{-1}h(x)$ for every $x \in C$. Green's lemma again tells us that these maps are one-to-one, $h_1(C)$ is contained in an $\mathcal{H}$-class. It is easy to check that (f_1, g_1, h_1) is an $\mathcal{H}$-embedding, it satisfies the property (1) and $f_1(a) = e$. Thus we can assume that the original triple (f, g, h) has the following property:

(2) $f(a) = e$.

Now we have that

$$h(xy) = f(x)g(y) = f(x)eg(y) = f(x)f(a)g(y) = f(x)h(ay)$$

for every $x \in A$, $y \in B$. Since h is one-to-one and (A, B, C) has the cancellation property, we can conclude that the map $g_1 : B \to S$ which maps every x from B to $h(ax)$ is one-to-one, so the triple (f, g_1, h) is an embedding. This $\mathcal{H}$-embedding has the same properties (1), (2) as (f, g, h) and the following additional property:

(3) The image of B is contained in the same $\mathcal{H}$-class as the image of C.

So we can assume that $g(B) \subseteq H_2$.

Now we can consider the $\mathcal{J}$-class D_2 which contains H_2. Since this $\mathcal{J}$-class is regular, there exist elements $p, q \in S$ such that H_2p is contained in a subgroup G_2 of D_2 and $xpq = x$ for every $x \in H_2$. Now consider three maps $f_1 = f$, $g_1(x) = g(x)p$ where $x \in B$ and $h_1(x) = h(x)p$. As above it is easy to check that the triple (f_1, g_1, h_1) is an $\mathcal{H}$-embedding which has the same properties (1)-(3) and the following property:

(4) The images of C and B are contained in a subgroup of S.

Again we can assume that (f, g, h) satisfies this property as well. So H_2 is a subgroup.

Let j be the identity element of H_2. The argument similar to that we have used above shows that we can assume that (f, g, h) satisfies the following property:

(5) $g(b) = j$ for some $b \in B$.

Therefore $h(xy) = f(x)g(y) = f(x)jg(y) = f(x)g(b)g(y) = h(xb)g(y)$ for every $x \in A$, $y \in B$. Since (A, B, C) satisfies the cancellation property, the function $f_1 : x \to h(xb)$ is one-to-one on A. So (f_1, g, h) is an $\mathcal{H}$-embedding of (A, B, C). The images of A, B, C are contained in the subgroup H_2. Thus (A, B, C) is embeddable into a subgroup of S.

The lemma is proved.

Now for every split system (A, B, C) consider the set $T(A, B, C) = A \cup B \cup C \cup \{0\}$ with operation $\cdot$ such that $x \cdot y = xy$ if $x \in A, y \in B$ and all other products are equal to 0. It is easy to see that $T(A, B, C)$ is a 3-nilpotent semigroup that is the product of any three elements of $T(A, B, C)$ is zero. Notice that the map T is a functor from the category of split systems into the category of semigroups. This functor has been considered in my previous papers cited above.

It is easy to see that the split system (A, B, C) is embeddable into the semigroup $T(A, B, C)$.

Let n be a natural number and G be a group. Recall that the *Brandt semigroup* $B_n(G)$ is the set of triples (i, x, j) where $1 \leq i, j \leq n$, $x \in G$ and 0 with the following operation:

$$(i, x, j)(k, y, \ell) = \begin{cases} (i, xy, \ell) & \text{if } j = k; \\ 0 & \text{if } j \neq k. \end{cases}$$

It is well known [1] and easy to check that $B_n(G)$ has n^2 $\mathcal{H}$-classes equal to $\{i\} \times G \times \{j\}$ where $1 \leq i, j \leq n$.

Lemma 6 *If (A, B, C) is embeddable into a group G then there exists an embedding of the semigroup $T(A, B, C)$ into the Brandt semigroup $B_3(G)$ which maps A and C into $\mathcal{H}$-classes of $B_3(G)$.*

Proof. Let (f, g, h) be the embedding of (A, B, C) into the group G. Consider the following map ϕ from $T(A, B, C)$ into $B_3(G)$:

$$\phi(x) = \begin{cases} (1, f(x), 2) & \text{if } x \in A; \\ (2, g(x), 3) & \text{if } x \in B; \\ (1, h(x), 3) & \text{if } x \in C; \\ 0 & \text{if } x = 0. \end{cases}$$

It is easy to verify that ϕ is an embedding of $T(A, B, C)$ into $B_3(G)$ and the images of A, B, C are in $\mathcal{H}$-classes of $B_3(G)$.

The lemma is proved.

Now we are ready to prove Theorem 1. Take any finite split system (A, B, C). By Lemma 6 if (A, B, C) is embeddable into a group then there exists an embedding of the semigroup $T(A, B, C)$ into a finite semigroup S such that the images of A and C are contained in $\mathcal{H}$-classes of S.

On the other hand suppose that (A, B, C) has the cancellation property and that there exists an embedding of $T(A, B, C)$ into a finite semigroup S which maps A and C into $\mathcal{H}$-classes of S. Every finite semigroup is embeddable into a finite regular semigroup (for example, the semigroup of all transformations on a finite set). Thus we can assume that S is regular. The embedding of the split system (A, B, C) into $T(A, B, C)$ induces an embedding of this split system into S. Since this embedding maps A and C into $\mathcal{H}$-classes of S, we can apply Lemma 5 and conclude that (A, B, C) is embeddable into a finite group. It remains to apply Lemma 4.

The theorem is proved.

2 Proof of Theorem 3

First let us consider the semigroup case.

Let S be a finite semigroup and let Σ be a system of equations over S of the form $ax = b$ where $a, b \in S$, x belongs to the set of unknowns X.

Suppose that condition AS of Theorem 2 holds. Consider the left regular representation of S by transformations on S^1 [1]. This representation takes every element a from S to the transformation ϕ_a on S^1 which maps every $s \in S^1$ to sa. Let T be the full transformation semigroup on the set S^1. We assume that for transformations ϕ and ψ from T the product $\phi\psi$ acts as follows: $\phi\psi(s) = \psi(\phi(s))$. Then the map $a \to \phi_a$ is an embedding of S into T. Thus we can consider T as an extension of S. Let us prove that the system of equations Σ has a solution in T. Indeed, let $x \in X$. Let us define a transformation ϕ_x of S^1. For every equation $ax = b$ that belongs to Σ let

$$\phi_x(\phi_a(s)) = \phi_x(sa) = \phi_b(s) = sb.$$

This rule defines ϕ_x on some elements of S^1, namely on the elements which are representable in the form sa for some $s \in S^1$, $a \in S$ such that ax is the left-hand side of an equation $ax = b$ from Σ. Let us define ϕ_x on all other elements of S^1 arbitrarily. In order to prove that the correspondence $x \to \phi_x$ is indeed a solution of the system Σ it is enough to check that ϕ_x is a transformation that is that ϕ_x is a single-valued function. Suppose that there exists an element $t \in S^1$ such that $\phi_x(t)$ has two different values. Then there exist two (possibly the same) equations $ax = b$ and $cx = d$ from Σ and two elements p and q from S^1 such that $t = pa = qc$ and $\phi_x(t) = pb \neq pd = \phi_x(t)$. But this contradicts condition AS.

Now suppose that the system Σ is solvable in some semigroup $T > S$. This means that there exists an assignment $X \to T$ which makes each of the equalities from Σ true. We can assume that $X \subseteq T$. Let $ax = b$ and $cx = d$ are two equations from Σ. Suppose that $pa = qc$ for some $p, q \in S^1$. Then $pb = pax = qcx = qd$. This proves condition AS.

Let us turn to the ring case. The proof that condition AR is necessary is the same the proof that condition AS is nacessary in the semigroup case. So let us prove that this condition is sufficient. Let S be a finite $GF(p)$-algebra and let Σ be a finite system of equations of the form $ax = b$ that satisfy condition AR. Without loss of generality we can assume that S has a unit. Then S is embeddable into the $GF(p)$-algebra of all endomorphisms R of S considered as a vector space over $GF(p)$. This embedding takes every $a \in S$ into ϕ_a where $\phi_a(s) = sa$ for every $s \in S$. For every $x \in X$ let us define an endomorphism ϕ_x from R. Let $\{a_i x = b_i \mid i = 1, \dots, n\}$ be the subset of equations from Σ which contain the unknown x. Let A be the set $\{a_i \mid i = 1, \dots, n\}$. Let $S_1 = SA$. Then S_1 is a subspace of S. Let S_2 be the complement of S_1 in S. For every $s \in S$ and every $t = \sum_i s_i a_i \in S_1$ let $\phi_x(\sum s_i a_i) = \sum_i s_i b_i$. It is easy to see that condition AR implies that this rule defines a single valued mapping from S_1 into S and that this mapping is a homomorphism. Let us extend ϕ_x to an endomorphism of S by setting $\phi_x(t) = 0$ for every $t \in S_2$. Then ϕ_x satisfies all the equalities $\phi_{a_i}\phi_x = \phi_{b_i}$. Thus our system Σ has a solution in R.

References

[1] A. H. Clifford and G. B. Preston. *The algebraic theory of semigroups.* Volume 1, 2, Amer. Math. Soc., Providence, 1961, 1967. MR 24 #A2627.

[2] T. Evans. On multiplicative systems defined by generators and relations. I. Normal form theorems. *Proc. Cambridge Phil. Soc*, 47:637–649, 1951. MR 13-312g.

[3] T. Evans. The word problem for abstract algebras. *J. London Math. Soc.*, 26:64–71, 1951. MR 12-475c.

[4] J. Fountain *Abundant semigroups*, Proc. London Math. Soc. 44 (1982), 103-129.

[5] T. Hall, S. Kublanovsky, S. Margolis, M. Sapir, and P. Trotter. Decidable and undecidable problems related to finite 0-simple semigroups. To appear.

[6] O.G. Kharlampovich and M.V. Sapir. Algorithmic problems in varieties. *Inter. J. Algebra Comput.*, 5, 1995. To appear.

[7] F.Pastijn, *A representation of a semigroup by a semigroup of matrices over a group with zero*, Semigroup Forum, 10 (1975), 238-249.

[8] M. Putcha, unpublished manuscript, 1983.

[9] M. V. Sapir. *Residually finite semigroups.* Ural state university. Diploma thesis. 1978. in Russian.

[10] M. V. Sapir. An implication characterization of prevarieties of semigroups and rings. *Research in modern algebra. Ural State University. Sverdlovsk.*, 120–134, 1982. in Russian.

[11] M. V. Sapir. *Quasivarieties of semigroups.* PhD thesis, Ural State University, Moscow State Pedagogical Institute, 1983. in Russian.

[12] M. V. Sapir. Lattices of quasivarieties of semigroups and groups. *Research in algebraic systems. Ural State University. Sverdlovsk*, 55–64, 1984. in Russian.

[13] M. V. Sapir. Residually finite semigroups in varieties. In J.C.Meakin T.E.Hall, P.R.Jones, editor, *Proceedings of the Monash international conference on semigroup theory. In honour of G.B.Preston. 1990.*, pages 258–268, World Scientific, 1991.

[14] A. M. Slobodskoii. Undecidability of the universal theory of finite groups. *Algebra i Logika*, 20(2):207–230, 1981.

Mark V. Sapir
Department of Mathematics and Statistics
Center for Communication and Information Science
University of Nebraska-Lincoln
Lincoln, NE 68588-0323.

mvs@unlinfo.unl.edu