LEAST SQUARES APPROXIMATION USING RADIAL-BASIS FUNCTIONS

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(joint work with Narcowich and Wendland)

Set Up

Discrete $X \subset$ compact Ω ;

Data vector $f|_X$, $\qquad f \in W_p^k(\Omega)$.

Wish to approximate f by some procedure:

I. Interpolation

$V_X = \mathrm{sp}\{\phi(\cdot - x_j)\colon\ x_j \in X\}$, ϕ an RBF. Find $I_X(f) \in V_X\colon\ I_X(f)|_X = f|_X$—always possible!

<u>Basic convergence result</u>:

If

$$f \in \mathcal{N}_\phi := \left\{ g\colon\ \int_{R^n} |\hat{g}(\xi)|^2 / \hat{\phi}(\xi) d\xi < \infty \right\},$$

then

$$\|f - I_X(f)\|_{L^\infty(\Omega)} \le c h^{k-n/2} \|f\|_\phi \,.$$

II. Least Squares

$$V_Y = \mathrm{sp}\{\phi(\cdot - y_j) \colon \; y_j \in Y\}, \qquad Y \subset X$$

Continuous Least Squares:

$$\min_{v \in V_Y} \|f - v\|_{L^2(\Omega)}, \qquad f \in W_2^k(\Omega)$$

Discrete Least Squares:

$$\min_{v \in V_Y} \|(f - v)|_X\|_{\tilde{\ell}^2(X)}, \qquad f \in W_2^k(\Omega)$$

Two main papers concerning continuous least squares

A. de Boor, DeVore, Ron ($\sim$ 1990)

Setting: $f \in W_2^k(R^n), Y = hZ^n$

Result: optimal error estimates (depending on ϕ) of form

$$\|f - b.a.\|_{L^2(R^n)} \sim \mathcal{O}(h^k)\|f\|_{W_2^k} .$$

B. Duchon ($\sim$ 1978)

Setting :

$\Phi_{n,\ell} := \phi_{n,\ell}(\|\cdot\|_2) \quad \ell > n/2$

$$\phi_{n,\ell}(r) := \begin{cases} C_1 r^{2\ell-n} & n \text{ odd} \\ C_2 r^{2\ell-n}\log r & n \text{ even} \end{cases}$$

$f \in \mathcal{N}_\phi \sim |f|_{W_2^\ell}$ semi-norm

Ω cpt. with Lipschitz boundary; $h =$ mesh norm of X

Result :

$\|f - I_X(f)\|_{L^\infty(\Omega)} \sim \mathcal{O}(h^{\ell-n/2})|f|_{W_2^\ell}$;

$\|f - I_X(f)\|_{L^2(\Omega)} \sim \mathcal{O}(h^{\ell})|f|_{W_2^\ell}$.

Goal

1. Extend Duchon's results to compactly supported Wendland functions:

$$\mathcal{N}_\phi \sim W_2^{k+1/2}(R^n) \text{ or } W_2^k(R^n)$$

2. Extend results to $f \in C^k$ "outside" native space.

3. Extend to Discrete Least Squares.

4. Provide "intrinsic proof" of these results.

Outline of Approach

- L_2 error obtained via interpolation.
- Interpolation produces error function with “lots” of zeros.
- $f \in W_2^k(\Omega)$ with “lots” of zeros has “small” norm controlled by h (mesh norm of zero set).

Outline of technical details

Definition. A domain D is said to be star-shaped with respect to the ball

$$B(x_C, r) := \{x \in R^n : \ |x - x_C| < r\}$$

if, for every $x \in D$, the closed convex hull of $\{x\} \cup B$ is contained in D.

(see Brenner–Scott).

Proposition. *If D is bounded, star-shaped with respect to $B(x_C, r)$, and contained in $B(x_C, R)$, then every $x \in D$ is the vertex of a cone $C \subset D$ having radius r and angle $\theta = 2 \text{ arc } \sin(r/2R)$.*

A Bernstein Inequality and its Consequence

Proposition. *Let D be a star-shaped domain described above. If $p \in \pi_\ell(R^n)$ and α is a multi-index, $|\alpha| \le \ell$, then*

$$\|D^\alpha p\|_{L^\infty(D)} \le \left(\frac{2\ell^2}{r \sin\theta}\right)^{|\alpha|} \|p\|_{L^\infty(D)}.$$

The preceding result, combined with an argument involving the notion of 'norming sets' yields:

Proposition. *Let $p \in \pi_\ell(R^n)$ and let D be as above. If the mesh norm h for $X = \{x_1, \ldots, x_N\}$ in D satisfies*

$$h \le \frac{r \sin\theta}{4(1+\sin\theta)\ell^2},$$

then there exist scalars $a_j^\alpha(x)$ such that the following hold for any multi-index α with $|\alpha| \le \ell$:

$$D^\alpha p(x) = \Sigma a_j^\alpha(x) p(x_j),$$

and

$$\Sigma |a_j^\alpha(x)| \le 2 \left(\frac{2\ell^2}{r \sin\theta} \right)^{|\alpha|}.$$

Remark: Thus $D^\alpha p(x)$ can be estimated in terms of $p(x_j)$.

Approximation with Averaged Taylor Polynomials

Assume D is star-shaped relative to B_ρ, $\rho \geq \frac{\rho_{\max}}{2}$.

Define

$$Q^k u(x) := \sum_{|\alpha|<k} \frac{1}{\alpha!} \int_{B_\rho} D^\alpha u(y)(x-y)^\alpha \psi(y) dy$$

$$\in \pi_{k-1}(R^n),$$

and

$$R^k u := u - Q^k u.$$

Proposition (Brenner–Scott). *If* $u \in W_p^k(D)$ *and* D *is as above, then*

$$\|R^k u\|_{L^\infty(D)} \le C_{D,n}\, d_D^{k-n/p}\, |u|_{W_p^k(D)}\,,$$

provided $1 < p < \infty$, $k > n/p$ *or* $p = 1$, $k \ge n$.

Proposition (NWW). *Suppose* $0 < s \le 1$. *If* $p \in (1,\infty)$, k *is as above, and* $u \in W_p^{k+s}(D)$, *then*

$$\|R^{k+1} u\|_{L^\infty(D)} \le C_{D,n,p}\, d_D^{k+s-n/p}\, |u|_{W_p^{k+s}(D)}.$$

Note: Direct proof and/or interpolation theory.

Sobolev bounds for functions with scattered zeroes

MAIN IDEA:

Suppose $u|_X = 0$, $X \subset D$, and $u \in W_p^k(D)$. Then

$$u = \underbrace{u - Q^{k+1}u}_{R^{k+1}u} + Q^{k+1}u$$

and

$$(Q^{k+1}u)(x) = -\sum_{x_j \in X} a_j(x)(R^{k+1}u)(x_j).$$

Earlier remark allows norm estimates on u !

Proposition (NWW). *Let k be a positive integer, $1 \le p < \infty$, $0 < s \le 1$ and let α be a multi-index satisfying $k > |\alpha| + n/p$, or, for $p = 1$, $k \ge |\alpha| + n$. Let $X \subset D$ be a discrete set with mesh norm h. If $u \in W_p^{k+s}(D)$ satisfies $u|_X = 0$, then*

$$\|D^\alpha u\|_{L^\infty(D)} \le C d_D^{k+s-|\alpha|-n/p} \, |u|_{W_p^{k+s}(D)}.$$

Corollary. The following holds:

$$|u|_{W_p^{|\alpha|}(D)} \le \widetilde{C} d_D^{k+s-|\alpha|} \, |u|_{W_p^{k+s}(D)}.$$

Big Domain as Union of Small Domains

Suppose Ω is a compact set with Lipschitz boundary and satisfying the interior cone condition. Let $X \subset \Omega$ with mesh norm h.

Given k, set

$$\rho = \frac{k^2 h}{Q(\varphi)}, \text{ where } 0 < Q(\varphi) < 1$$

depends on the geometry of Ω.

Lemma (modification of a result of Duchon). *There exists a covering $\{D_t\}_{t \in T}$ of Ω such that:*

A. Each $D_t \subseteq \Omega \cap B(t, \rho)$, $d_{D_t} < 2\rho < ch$.

B. $\Omega = \bigcup_{t \in T} D_t$, each D_t star-shaped w.r.t. ball.

C. $\sum_{t \in T} \chi_{D_t}(x) \leq M_1(\varphi, n), x \in R^n$.

Theorem. *Let k, s, p, and α be as before. Suppose $X \subset \Omega$ is a discrete set with mesh norm h. If $u \in W_p^{k+s}(\Omega)$ satisfies $u|_X = 0$, then*

$$|u|_{W_p^{|\alpha|}(\Omega)} \leq ch^{k+s-|\alpha|}|u|_{W_p^{k+s}(\Omega)}.$$

Proof.

$$|u|^p_{W_p^{|\alpha|}(\Omega)} \leq \sum_{t \in T} |u|^p_{W_p^{|\alpha|}(D_t)}.$$

Use $d_{D_t} \leq ch$ and apply previous Corollary. $\square$

Sobolev bounds for discrete ℓ_p norms

$Y = \{y_1, \dots, y_M\} \subset \Omega$; given q_Y and h_Y. For $u \in C(\Omega)$,

$$\|u\|_{\ell_p(Y)} := \left(\frac{1}{M} \sum_{1}^{M} |u(y_j)|^p \right)^{1/p};$$

$$|u|_{w_p^k(Y)} := \left(\sum_{|\alpha|=k} \|D^\alpha u\|_{\ell_p(Y)}^p \right)^{1/p};$$

$$\|u\|_{w_p^k(Y)} := \left(\sum_{|\alpha|\le k} \|D^\alpha u\|_{\ell_p(Y)}^p \right)^{1/p}.$$

Corollary. Let k, s, p, and α be as before. Let $X \subset \Omega$ be a discrete set with mesh norm $h = h_X$ "sufficiently small". Let $Y = \{y_1, \ldots, y_M\} \subset \Omega$ be a second discrete set with $h_Y \leq h$ and a constant $c > 0$ such that $q_Y \geq ch$. If $u \in W_p^{k+s}(\Omega)$ satisfies $u|_X = 0$, then

$$|u|_{w_p^{|\alpha|}(Y)} \leq ch^{k+s-|\alpha|} \, |u|_{W_p^{k+s}(\Omega)} .$$

In particular, if $|\alpha| = 0$, then

$$\|u\|_{\ell_p(Y)} \leq ch^{k+s} |u|_{W_p^{k+s}(\Omega)} .$$

Proof.

Use L^∞ bounds and conditions on D_t :

$$\sum_{y_j \in D_t} |D^\alpha u(y_j)|^p \leq$$

$$ch^{(k+s-|\alpha|)p-n} \,\text{card}(D_t \cap Y) |u|^p_{W_p^{k+s}(D_t)}.$$

$\square$

Wrapping Things Up

In the case of Thin Plate Splines and Wendland's compactly supported RBF's,

$$\|f\|_{\mathcal{N}_\phi(R^n)} \sim \|f\|_{W_2^\tau(R^n)} \quad \text{for some} \quad \tau.$$

<u>RBF theory</u>:

$$\min_{v \in V_{X,\phi}} \|f - v\|_{\mathcal{N}_\phi} = \|f - I_X f\|_{\mathcal{N}_\phi} \leq \|f\|_{\mathcal{N}_\phi}.$$

Lemma. *Let* $\tau \geq 0$, $f \in W_2^\tau(\Omega)$, $X = \{x_1, \ldots, x_N\} \subset \Omega$. *Let* $I_X f \in V_{X,\phi}$ *be the unique function that interpolates* f *on* X. *If* ϕ *has native space* $W_2^\tau(R^d)$, *then there is a constant* $C = C_{\Omega,\Phi}$ *such that*

$$\|f - I_X f\|_{W_2^\tau(\Omega)} \leq C\|f\|_{W_2^\tau(\Omega)}.$$

Proof. $\mathcal{E}f = f$ on Ω implies $I_X\mathcal{E}f = I_X f$. Thus

$$\begin{aligned}
\|f - I_X f\|_{W_2^\tau(\Omega)} &= \|\mathcal{E}f - I_X\mathcal{E}f\|_{W_2^\tau(\Omega)} \\
&\leq \|\mathcal{E}f - I_X\mathcal{E}f\|_{W_2^\tau(R^n)} \\
&\leq C_1\|\mathcal{E}f - I_X\mathcal{E}f\|_{\mathcal{N}_\phi} \\
&\leq C_1\|\mathcal{E}f\|_{\mathcal{N}_\phi} \\
&\leq C_1C_2\|\mathcal{E}f\|_{W_2^\tau(R^n)} \\
&\leq C_1C_2\|\mathcal{E}\|\|f\|_{W_2^\tau(\Omega)}
\end{aligned}$$

Now set $C_{\Omega,\phi} := C_1C_2\|\mathcal{E}\|$. $\square$

Proposition. *Let k, s, and α be given as before. Let $X \subset \Omega$ be a discrete set with mesh norm h "small". If $f \in W_2^{k+s}(\Omega)$, then*

$$|f - I_X f|_{W_2^{|\alpha|}(\Omega)} \le ch^{k+s-|\alpha|} \|f - I_X f\|_{W_2^{k+s}(\Omega)}$$
$$\le ch^{k+s-|\alpha|} \|f\|_{W_2^{k+s}(\Omega)}.$$

Therefore

$$\|f - I_X f\|_{L^2(\Omega)} \le ch^{k+s} \|f\|_{W_2^{k+s}(\Omega)}.$$

Escaping The Native Space

If $f \in W_2^k(R^n) \cap C_0^k$ and ϕ is a "smoother" Thin Plate Spline or Wendland function, then one has the following estimates:

$$\|f - I_X f\|_{L^\infty(\Omega)} \le \tilde{c} h^{k-n/2} \|f\|_{C_0^k \cap W_2^k(R^n)};$$

$$\|f - I_X f\|_{L^2(\Omega)} \le c h^k \|f\|_{C_0^k \cap W_2^k(R^n)}.$$

<u>Relevant material</u>:

Brownlee and Light

Narcowich and Ward

$$|u|_{W_p^k(\Omega)} = \left(\sum_{|\alpha|=k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}$$

and

$$\|u\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}.$$

$$|u|_{W_p^{k+s}(\Omega)} =$$

$$\left(\sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+ps}} dxdy \right)^{1/p}$$

and

$$\|u\|_{W_p^{k+s}(\Omega)}^p = \|u\|_{W_p^k(\Omega)}^p + |u|_{W_p^{k+s}(\Omega)}^p.$$

Definition. Let $0 < c \leq 1$ be given. A set $X_c \subset \Omega$ is said to be a norming set for $\pi_\ell(\Omega)$ if for every $p \in \pi_\ell(\Omega)$,

$$\|p|_{X_c}\|_\infty \geq c\|p\|_\infty.$$

Corollary. *Given $x \in \Omega$, there exist coefficients $\{a_j(x)\}$ so that* $\forall p \in \pi_\ell(\Omega)$

$$p(x) = \sum_{x_j \in X_c} a_j(x) p(x_j), \quad \sum |a_j(x)| \leq 1/c.$$

Remark: X_c is unisolvent for $\pi_\ell(\Omega)$.