Shape Preserving Approximation: the Final Frontier???

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Research supported by NSERC of Canada

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"There has been an alarming increase in the number of things we know nothing about."

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Overview

- Smoothness vs. Approximation Orders: the “right” estimates, the “right” moduli of smoothness, characterization of approximation spaces, etc.
- Shape Preserving Approximation: definitions, approximation orders, comparison of the errors of shape preserving and unconstrained approximation, relative n-width with constraints, recent developments, open problems, etc.
- Remarks and Final Conclusions

Smoothness vs. Approximation Order: classics

Smoothness vs. Approx. Order: trigonometric polynomials

Function f is "smooth enough"

$\iff$ f can be approximated well enough

Theorem (Bernstein [1912]) *A continuous* 2π*-periodic function* f *belongs to* $\mathrm{Lip}\,\alpha$ *class, i.e., is such that* $\omega(f,t) = O(t^\alpha)$, $0 < \alpha < 1$, *if and only if*

$$E_n^*(f) \le C n^{-\alpha}\,.$$

Smoothness vs. Approx. Order: trigonometric polynomials

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Theorem (Zygmund [1945]) *A continuous* 2π*-periodic function* f *is such that* $\omega_2(f,t) = O(t)$ *if and only if*

$$E_n^*(f) \le Cn^{-1}\,.$$

Smoothness vs. Approx. Order: algebraic polynomials

$f \in \mathbb{C}[-1,1]$ *belongs to* $\mathrm{Lip}\,\alpha$, $0 < \alpha < 1$, *class*

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Smoothness vs. Approx. Order: algebraic polynomials

Theorem (**Nikolskii [1946]**, **Timan [1951]**, **Dzyadyk [1956]**)
$f \in \mathbb{C}[-1,1]$ *belongs to* $\mathrm{Lip}\,\alpha$, $0 < \alpha < 1$, *if and only if there exists a sequence of polynomials* $p_n(x)$ *such that*

$$|f(x) - p_n(x)| \le C \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2} \right)^{\alpha}, \quad x \in [-1,1].$$

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$\Delta_n(x)$

Spaces $\mathrm{Lip}(\alpha, p)$ **and** $\mathrm{Lip}^*(\alpha, p)$**:** $0 < p \le \infty$

Let $\alpha = r + \beta$, where $r \in \mathbb{N}_0$ and $0 < \beta \le 1$.

Lipschitz space:

$$\mathrm{Lip}(\alpha, p) := \left\{ f | \omega(f^{(r)}, t)_p \le Ct^{\beta} \right\}$$

Generalized Lipschitz space:

$$\mathrm{Lip}^*(\alpha, p) \; := \left\{ f | \omega_{\lfloor \alpha \rfloor + 1}(f, t)_p \le Ct^{\alpha} \right\}$$

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Pointwise estimates by algebraic polynomials: $p = \infty$

Theorem (Timan, Dzyadyk, Freud, Brudnyi)
$f \in \mathrm{Lip}^*(\alpha, \infty)$, $\alpha > 0$, *if and only if there exists a sequence of polynomials* $p_n(x)$ *such that*

$$|f(x) - p_n(x)| \le C\Delta_n(x)^\alpha\,, \quad x \in [-1, 1]\,.$$

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Question: $E_n(f) = O(n^{-\alpha}) \quad \Longleftrightarrow \quad ???$

Uniform estimates by algebraic polynomials

Theorem (Ditzian, Totik, Ivanov, Tachev)

Let $0 < p \leq \infty$ *and* $0 < \alpha < k$. *For* $f \in \mathbb{L}_p$ *we have*

$$E_n(f)_p = O(n^{-\alpha}) \quad \Longleftrightarrow \quad \begin{cases} \omega_k^{\varphi}(f,t)_p = O(t^{\alpha}) \\ \tau_k(f,1,\Delta_n(x))_{p,p} = O(t^{\alpha}) \end{cases}$$

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Ditzian-Totik modulus of smoothness:

$$\omega_k^{\varphi}(f,t)_p := \sup_{0<h\le t} \|\Delta_{h\varphi(\cdot)}^k(f,\cdot)\|_p \, ,$$

where $\varphi(x) := \sqrt{1-x^2}$.

Uniform estimates by algebraic polynomials

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Ivanov modulus of smoothness (Sendov $q = \infty$):

$$\tau_k(f,\psi,\delta)_{q,p} = \left\| \psi(\cdot)\, \omega_k(f,\cdot,\delta(\cdot))_q \right\|_p ,$$

where

$$\omega_k(f,x,\delta(x))_q^q = \frac{1}{2\delta(x)} \int_{-\delta(x)}^{\delta(x)} \left| \Delta_\nu^k(f,x) \right|^q \, d\nu .$$

Spaces $\mathrm{Lip}^*_\varphi(\alpha,p)$**:** $0 < p \le \infty$

Let $\alpha = r + \beta$, where $r \in \mathbb{N}_0$ and $0 < \beta \le 1$, and $1 \le p \le \infty$.

$\mathbb{B}^r H^{t^\beta}_{\lfloor\beta\rfloor+1}(p)$ or $\hat{H}^\alpha_p$ or $\mathrm{Lip}^*_\varphi(\alpha,p)$ space :

$$\mathrm{Lip}^*_\varphi(\alpha,p) \quad := \left\{ f | \omega^\varphi_{\lfloor\alpha\rfloor+1}(f,t)_p \le Ct^\alpha \right\}$$

Spaces $\mathrm{Lip}^*_\varphi(\alpha,p)$: $0<p\le\infty$

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$$\|$$

$$\mathrm{Lip}^\star_\varphi(\alpha,p) := \left\{ f \middle| \omega^\varphi_{\lfloor\beta\rfloor+1,r}(f^{(r)},t)_p \le Ct^\beta \right\}$$

The Ditzian-Totik weighted modulus of smoothness:

$$\omega^\varphi_{k,r}(f,t)_p := \sup_{0<h\le t} \left\| \varphi^r_{kh}(\cdot)\Delta^k_{h\varphi(\cdot)}(f,\cdot)\right\|_p ,$$

where $\varphi_\delta(x) := \sqrt{\left(1-x-\frac{\delta}{2}\varphi(x)\right)\left(1+x-\frac{\delta}{2}\varphi(x)\right)}$

Uniform estimates by algebraic polynomials: $\mathrm{Lip}^*_\varphi(\alpha, p)$ spaces

Theorem *Let* $1 \le p \le \infty$ *and* $\alpha > 0$. *Then*

$$E_n(f)_p = O(n^{-\alpha}) \quad \Longleftrightarrow \quad f \in \mathrm{Lip}^*_\varphi(\alpha, p)\,.$$

Theorem *Let* $1 \le p \le \infty$ *and* $\alpha > 0$. *Then*

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For example,

$$E_n(f)_p = O(n^{-3}) \quad \Longleftrightarrow \quad \omega^\varphi_{2,2}(f'', t)_p = O(t)$$

$$E_n(f)_p = O(n^{-2.5}) \quad \Longleftrightarrow \quad \omega^\varphi_{1,2}(f'', t)_p = O(t^{1/2})$$

$$E_n(f)_p = O(n^{-10}) \quad \Longleftrightarrow \quad \omega^\varphi_{2,9}(f^{(9)}, t)_p = O(t)$$

Some conclusions

The “right estimates” for approximation by algebraic polynomials should be in terms of:

$p = \infty$: $\omega_k(f, \Delta_n(x))$ or $\omega_k(f^{(r)}, \Delta_n(x))$

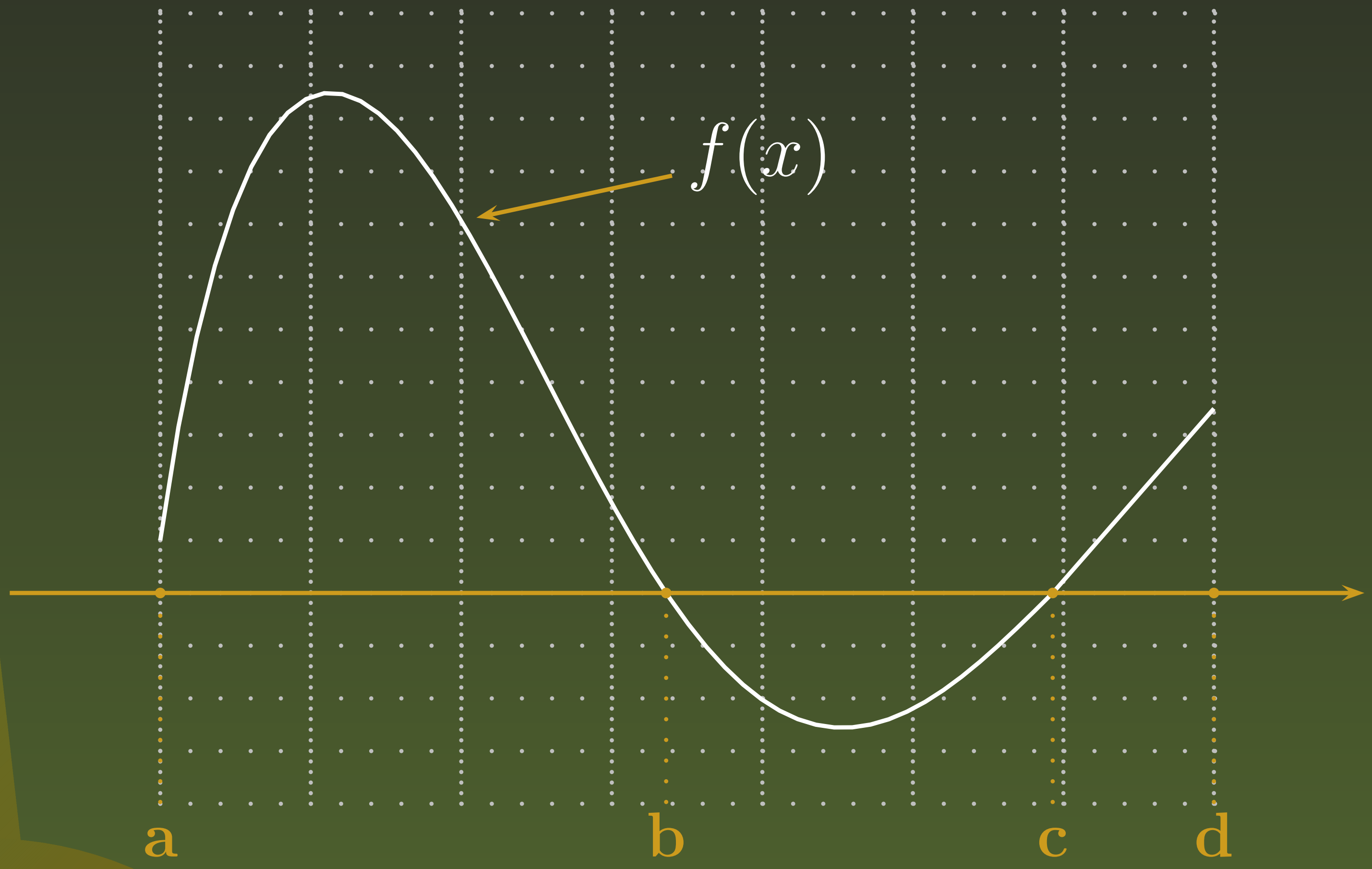
$p \le \infty$: $\omega_k^{\varphi}(f, n^{-1})_p$ or $\omega_{k,r}^{\varphi}(f^{(r)}, n^{-1})_p$ or

$\tau_k(f, 1, \Delta_n(x))_{p,p}$

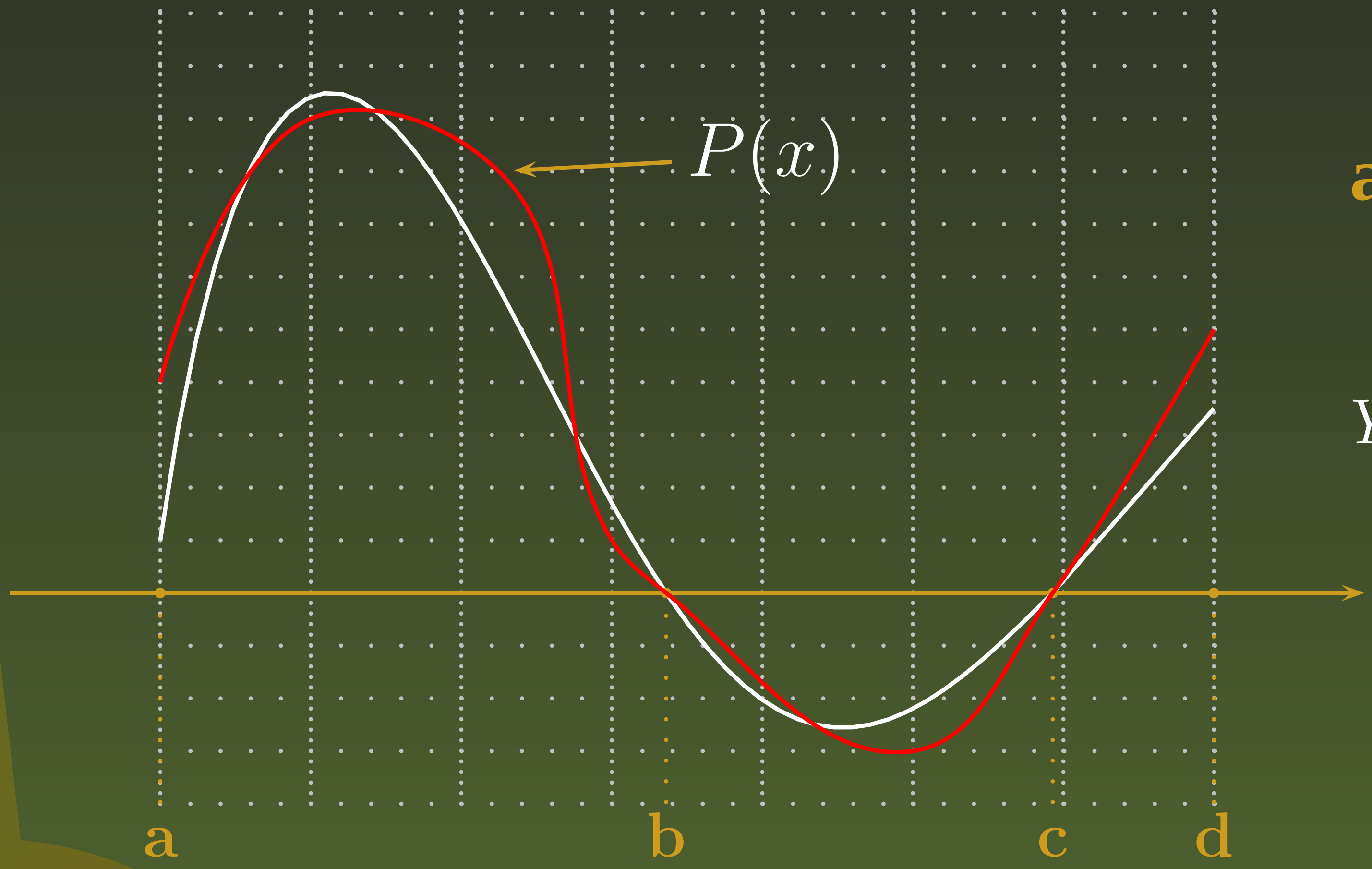
They allow characterization of classes of functions with prescribed order of approximation, and so are exact in this sense.

Shape Preserving Approximation (SPA)

Shape Preserving Approximation (SPA): examples



Shape Preserving Approximation (SPA): examples



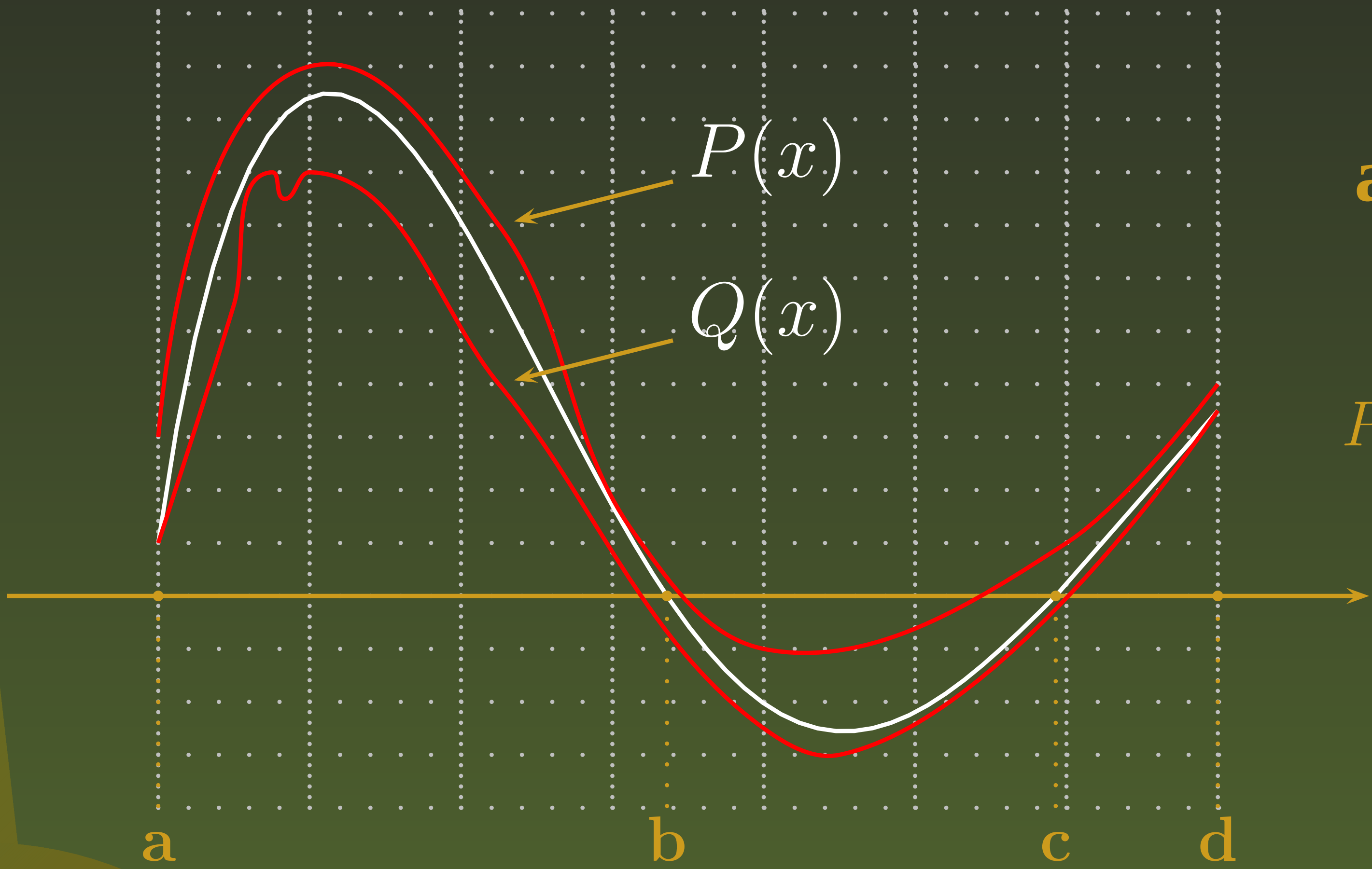
Copositive approximation

$Y_s = \{b, c\}$, $s = 2$

$f, P \in \Delta^0(Y_s)$

$$E_n^{(0)}(f, Y_s)_p := \inf\{\|f - P\|_p \mid f(x) \cdot P(x) \geq 0\}$$

Shape Preserving Approximation (SPA): examples

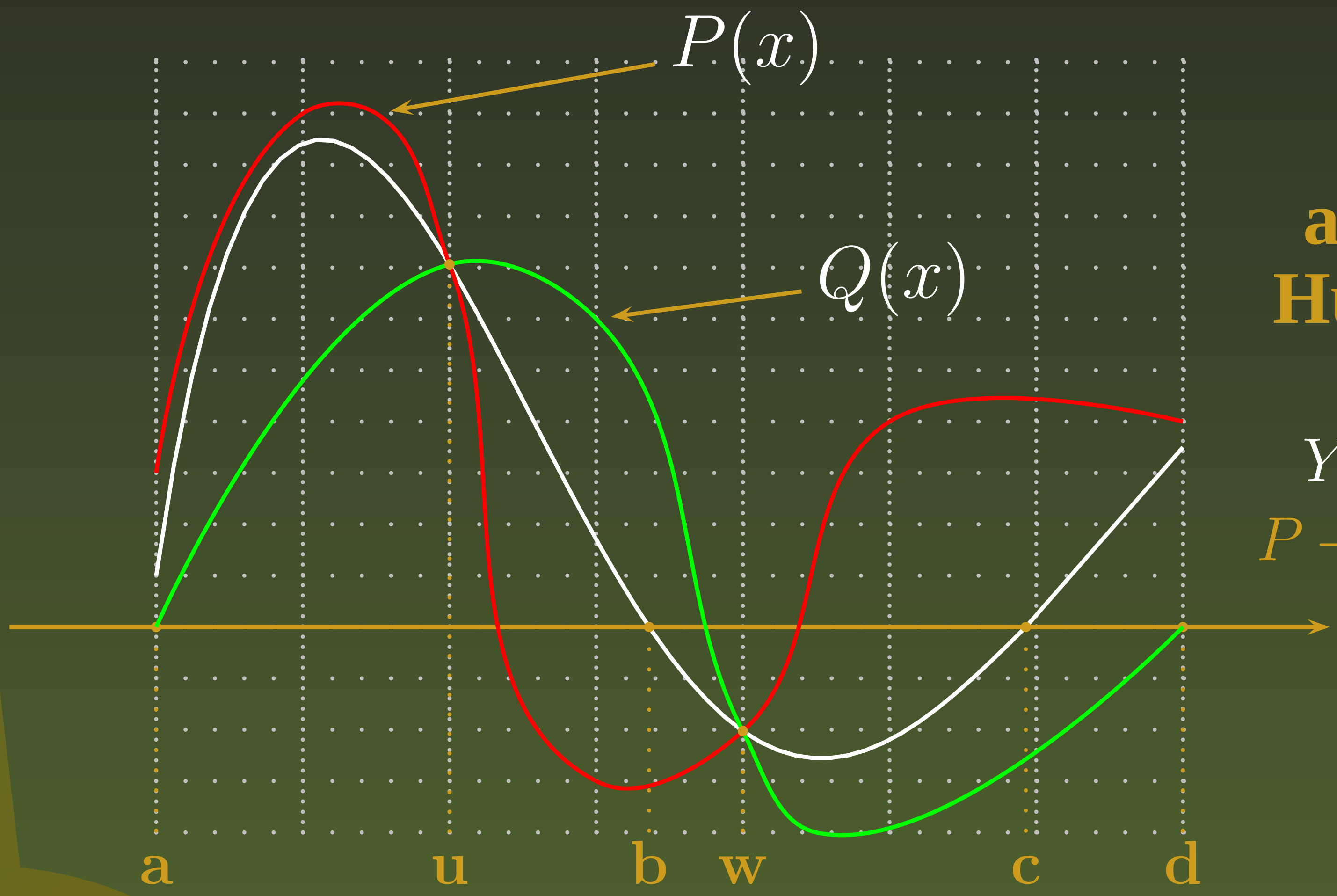


Onesided approximation

$P - f, f - Q \in \Delta^0$

$$\widetilde{E}_n(f)_p := \inf \{\|P - Q\|_p \mid Q \le f \le P\}$$

Shape Preserving Approximation (SPA): examples



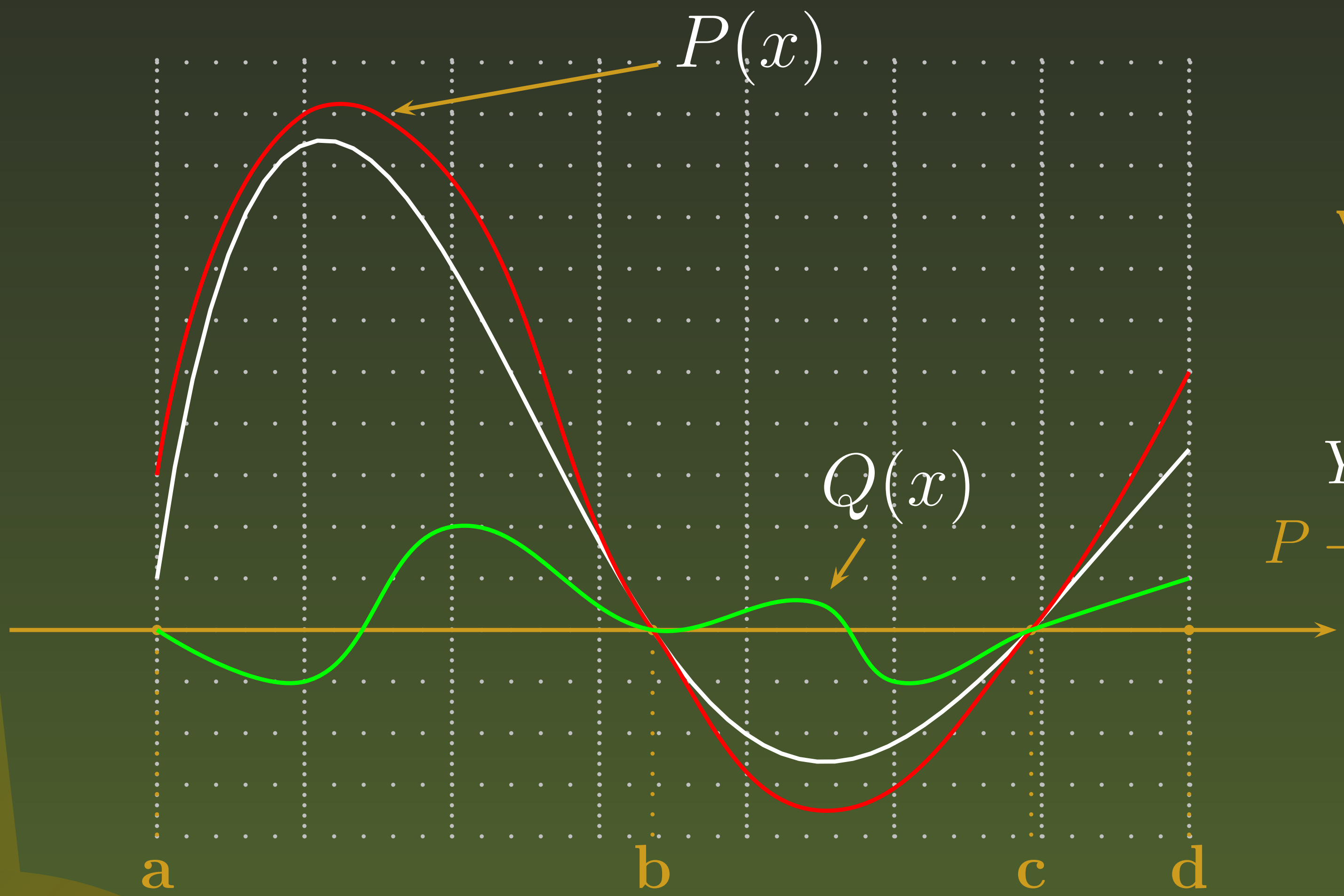
Intertwining approximation: Hu, K., Yu [1997]

$Y_s = \{u, w\}$, $s = 2$

$P - f, f - Q \in \Delta^0(Y_s)$

$$\widetilde{E}_n(f, Y_s)_p := \inf\left\{\|P - Q\|_p \;\middle|\; P - f, f - Q \in \Delta^0(Y_s)\right\}$$

Shape Preserving Approximation (SPA): examples



Intertwining vs. Copositive

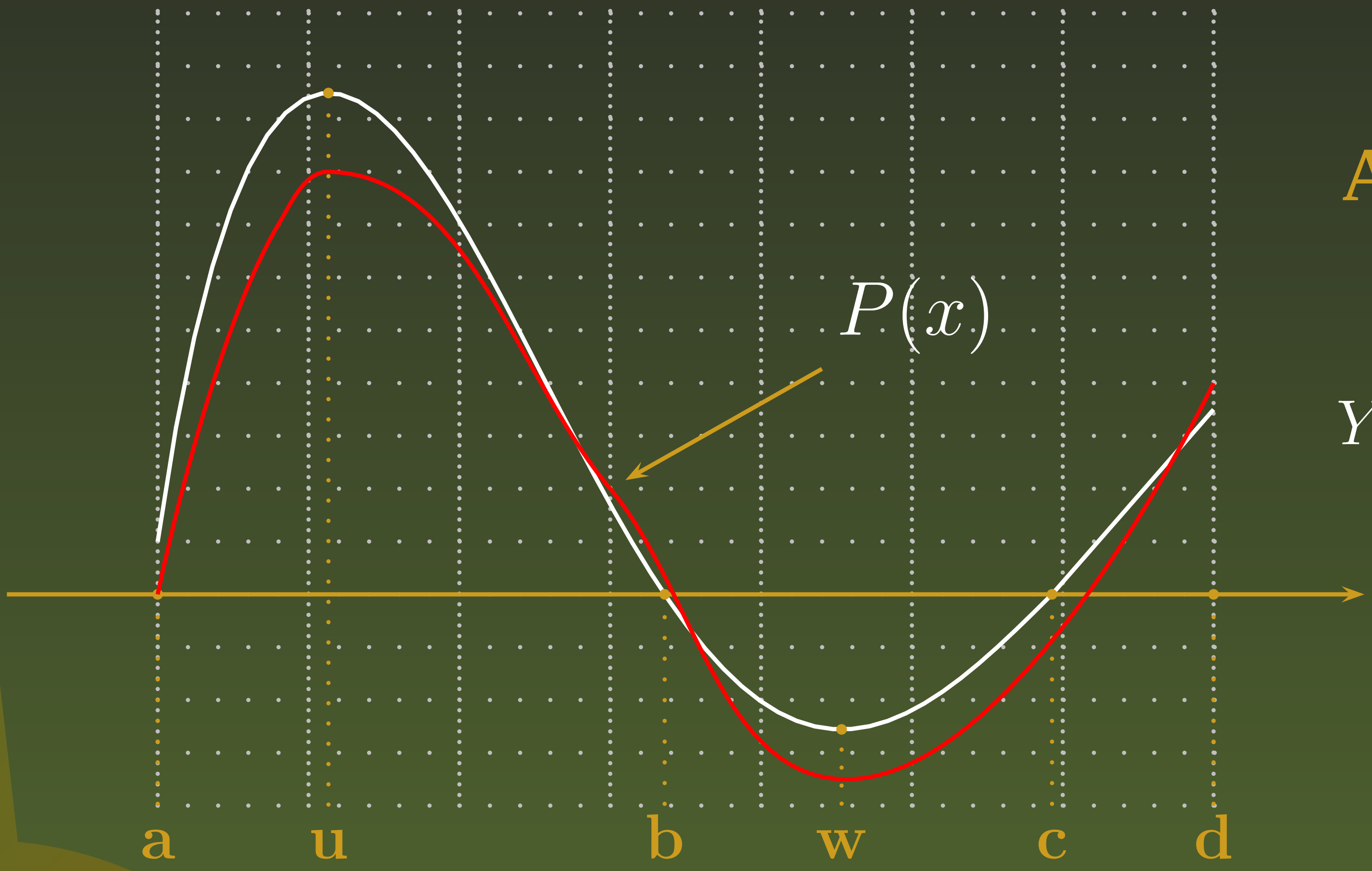
$Y_s = \{b, c\}$, $s = 2$

$P - f,\ f - Q \in \Delta^0(Y_s)$

$P \in \Delta^0(Y_s)$

$$E_n^{(0)}(f, Y_s) \le \widetilde{E}_n(f, Y_s)_p$$

Shape Preserving Approximation (SPA): examples



Comonotone Approximation

$Y_s = \{u, w\}$, $s = 2$

$P' \in \Delta^0(Y_s)$

$$E_n^{(1)}(f, Y_s)_p := \inf\left\{\|f - P\|_p \mid P'(x) \cdot f'(x) \geq 0\right\}$$

q-monotone approximation: definition

Definition: A function $f : [a, b] \mapsto \mathbb{R}$ is q-monotone on $[a, b]$ if its qth divided differences $[x_0, \ldots, x_q]f$ are ≥ 0 for all choices of $(q+1)$ distinct points $x_0, \ldots, x_q$ in $[a, b]$. Notation: $f \in \Delta^q[a, b]$ or $f \in \Delta^q$

Let $Y_s := \{y_i\}_{i=1}^s$ be such that $y_0 := a < y_1 < \cdots < y_s < b =: y_{s+1}$ if $s \geq 1$, and $Y_0 := \emptyset$.

Definition: A function $f : [a, b] \mapsto \mathbb{R}$ is said to be in $\Delta^q(Y_s)$ iff f is q-monotone on $[a, y_1]$ and changes its q-monotonicity at the points in Y_s, *i.e.*, $(-1)^i f \in \Delta^q[y_i, y_{i+1}]$, $0 \leq i \leq s$. Notation: $f \in \Delta^q(Y_s)$.

Ex: Δ^2 is the set of all convex functions;

$\{(g, h) \mid g - f, f - h \in \Delta^0(Y_s)\}$ is the set of all intertwining pairs with respect to Y_s

Errors of q-monotone approximation: notation

- $E(f,Y)_X := \inf_{p\in Y} \|f-p\|_X$: unconstrained
- $E^{(q)}(f,Y)_X := E(f,Y\cap\Delta^q)_X = \inf_{p\in Y\cap\Delta^q} \|f-p\|_X$: q-monotone
- $E_n(f)_p := E(f,\Pi_n)_{\mathbb{L}_p}$ and $E_n^{(q)}(f)_p := E(f,\Pi_n\cap\Delta^q)_{\mathbb{L}_p}$: approximation by polynomials of degree $\le n$ in $\mathbb{L}_p$ (quasi) norm
- $\sigma_{N,r}(f)_p := E(f,\mathcal{S}_{N,r})_{\mathbb{L}_p}$ and $\sigma_{N,r}^{(q)}(f)_p := E(f,\mathcal{S}_{N,r}\cap\Delta^q)_{\mathbb{L}_p}$: approximation by splines of order r with $N-1$ free knots

SPA: orders of approximation

Question: How does $E(f,Y)_X$ compare to $E^{(q)}(f,Y)_X$?

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Obvious: $E(f,Y)_X \leq E^{(q)}(f,Y)_X$

Would be nice: $E^{(q)}(f,Y)_X \leq CE(f,Y)_X$, $f \in \Delta^q$.

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Onesided and positive approximation in $\mathbb{C}[a,b]$:

$f \in \mathbb{C}[a,b]$: $E_n(f)_\infty \le \widetilde{E}_n(f)_\infty \le 2E_n(f)_\infty$

$f \in \mathbb{C}[a,b] \cap \Delta^0$: $E_n(f)_\infty \le E_n^{(0)}(f)_\infty \le 2E_n(f)_\infty$

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$f \in \mathbb{C}[a,b] \cap \Delta^0$: $E_n(f)_\infty \leq E_n^{(0)}(f)_\infty \leq 2E_n(f)_\infty$

Theorem (Lorentz and Zeller [1969]) *For $q \in \mathbb{N}$, there exists a function $f \in \Delta^q$ such that* $\limsup_{n\to\infty} \frac{E_n^{(q)}(f)_p}{E_n(f)_p} = \infty$.

SPA: orders of approximation

Question: Since, for $f \in \Delta^q$, $E_n^{(q)}(f)_p \le C E_n(f)_p$ is not possible in general, and since the "next best thing" is

$$E_n(f)_p \le \frac{C}{n^r} E_{n-r}(f^{(r)})_p$$

what about

$$E_n^{(q)}(f)_p \le \frac{C}{n^r} E_{n-r}(f^{(r)})_p \quad ???$$

SPA: orders of approximation

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what about

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Onesided and positive approximation:

$f \in \mathbb{W}_p^1[a,b]$: $\widetilde{E}_n(f)_p \le \dfrac{C}{n} E_{n-1}(f')_p$ (Stojanova [1988])

$f \in \mathbb{W}_p^1[a,b] \cap \Delta^0$: $E_n^{(0)}(f)_p \le \dfrac{C}{n} E_{n-1}(f')_p$

$f \in \Delta^q$: orders of q-monotone approximation

$$E_n^{(q)}(f)_p \le (b-a)E_{n-1}^{(q-1)}(f')_p, \quad 1 \le p \le \infty$$

Proof: Let $f \in \mathbb{C}^1[a,b] \cap \Delta^q$ (note that $f \in \Delta^q$ is automatically in $\mathbb{C}^1(a,b)$ if $q \ge 3$), and let $P_n(x) := \int_a^x q_{n-1}(t)\,dt + f(a)$, where $E_{n-1}^{(q-1)}(f')_p = \|f' - q_{n-1}\|_p$. Then $P_n'(x) = q_{n-1}(x) \in \Delta^{q-1}$ (*i.e.*, $P_n \in \Delta^q$), and by Hölder's inequality

$$\begin{aligned} E_n^{(q)}(f)_p \le \|f - P_n\|_p &= \left\| \int_a^x \left(f'(t) - P_n'(t)\right)\, dt \right\|_p \\ &\le \quad (b-a)^{1/p} \, \|f' - P_n'\|_1 \le (b-a) \, \|f' - q_{n-1}\|_p \\ &= \quad (b-a)E_{n-1}^{(q-1)}(f')_\infty \end{aligned}$$

■

$f \in \Delta^q$: orders of q-monotone approximation

$$E_n^{(q)}(f)_p \le (b-a) E_{n-1}^{(q-1)}(f')_p, \quad 1 \le p \le \infty$$

Corollaries:

$$0 \le \mu \le q: \quad E_n^{(q)}(f)_p \le (b-a)^\mu E_{n-\mu}^{(q-\mu)}(f^{(\mu)})_p$$

$$\mu = q-2: \quad E_n^{(q)}(f)_p \le C E_{n-q+2}^{(2)}(f^{(q-2)})_p$$

$$\mu = q: \quad E_n^{(q)}(f)_p \le C E_{n-q}^{(0)}(f^{(q)})_p$$

$$p = \infty: \quad E_n^{(q)}(f)_\infty \le C E_{n-q}(f^{(q)})_\infty$$

$f \in \Delta^q$: orders of q-monotone approximation

$f \in \Delta^q$, $p = \infty$: $E_n^{(q)}(f)_\infty \le C E_{n-q}(f^{(q)})_\infty$

For each $q \in \mathbb{N}$, there is an absolute constant $C_0 > 0$ such that, for any $n \ge q$, a function $f \in \mathbb{C}^q[a,b] \cap \Delta^q$ exists such that

$$E_n^{(q)}(f)_\infty \ge C_0 E_{n-q}(f^{(q)})_\infty > 0 \quad \text{(Leviatan and Shevchuk [1995])}$$

$f \in \Delta^q$, $1 \le p < \infty$: $E_n^{(q)}(f)_p \le C E_{n-q}^{(0)}(f^{(q)})_p$

For any $q \in \mathbb{N}$, $n \ge q$, $0 < p < \infty$, and $A > 0$ there exists $f \in \mathbb{C}^\infty[a,b] \cap \Delta^q$ such that

$$E_n^{(q)}(f)_p \ge A E_{n-q}(f^{(q)})_p \quad \text{(K. [1995])}$$

and so even $E_n^{(q)}(f)_p \le C E_{n-q}(f^{(q)})_p$ is NOT true.

Intertwining and Copositive approximation: open problems

Recall the definition of intertwining approximation:

$$\widetilde{E}_n(f, Y_s)_p := \inf \left\{ \|P - Q\|_p \;\middle|\; P - f, f - Q \in \Delta^0(Y_s) \right\}$$

Open Problem (intertwining approx.): Let $0 < p \le \infty$. Does there exist an $r \in \mathbb{N}$ such that $\widetilde{E}_n(f, Y_s)_p \le \dfrac{C}{n^r} E_{n-r}(f^{(r)})_p \quad ???$

Open Problem (copositive approx.): Let $0 < p \le \infty$. Does there exist an $r \in \mathbb{N}$ such that $E_n^{(0)}(f, Y_s)_p \le \dfrac{C}{n^r} E_{n-r}(f^{(r)})_p \quad ???$

Intertwining and Copositive approximation: open problems

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- $E_n^{(0)}(f, Y_s)_p \le \widetilde{E}_n(f, Y_s)_p$ and so intertwining $\longrightarrow$ copositive (positive answer) and copositive $\longrightarrow$ intertwining (negative answer)
- $1 \le p < \infty$: $E_n^{(0)}(f, Y_s)_p \not\le C\omega_3(f', 1)_p$ and so r cannot be 1

Some conclusions

In general, Shape Preserving Approximation by polynomials and fixed knot splines requires special treatment since estimates do not follow from unconstrained results.

Some conclusions

In general, Shape Preserving Approximation by polynomials and fixed knot splines requires special treatment since estimates do not follow from unconstrained results.

Question 1: Can we say the same about approximation from other spaces? e.g. approximation by free knot splines?

Question 2: $E_n^{(q)}(f, Y_s)_p \leq C E_n(f)_p$, $f \in \Delta^q(Y_s) \cap V$???
For example, $V =$ piecewise polynomial functions.

SPA: splines with fixed knots

Usual proof of direct (positive) results:

Function f of certain shape $\longrightarrow$ shape preserving (SP) splines with fixed knots (e.g. Chebyshev knots) $\longrightarrow$ SP polynomials

Open Problem: Let $0 < p \le \infty$, and suppose a spline (or PP) $g \in \Delta^q(Y_s)$. Prove or disprove that

$$E_n^{(q)}(g, Y_s)_p \le C E_n(g)_p$$

or

$$E_n^{(q)}(g, Y_s)_p \le C \omega_m^{\varphi}(g, n^{-1})_p \, .$$

SPA by free knot splines and a different kind of q-monotone approximation

q-monotone approximation: free knot splines

Recall: $\mathcal{S}_{N,r}$ is the set of all PP of order r with N pieces;
$\sigma_{N,r}(f)_p := E(f, \mathcal{S}_{N,r})_{\mathbb{L}_p}$; $\sigma^{(q)}_{N,r}(f)_p := E(f, \mathcal{S}_{N,r} \cap \Delta^q)_{\mathbb{L}_p}$; and

$$\widetilde{\sigma}^{(q)}_{N,r}(f)_p := E(f, \mathcal{S}_{N,r} \cap \mathbb{C}^{r-2} \cap \Delta^q)_{\mathbb{L}_p}$$

Theorem (K. and Shadrin [2003]) *Let* $q, r, N \in \mathbb{N}$, $r \geq q$, *and* $0 < p \leq \infty$. *Then, there exist constants* $c_0 = c_0(r)$ *and* $c_1 = c_1(r, p)$ *such that, for all* $f \in \Delta^q \cap \mathbb{L}_p$,

$$\widetilde{\sigma}^{(q)}_{c_0 N,r}(f)_p \leq c_1 \sigma_{N,r}(f)_p \,.$$

$q = 1, 2$: Leviatan and Shadrin [1997], Petrov [1996]

$q = 3$, $p = \infty$: Petrov [1998]

q-monotone approximation: free knot splines

Recall: $\mathcal{S}_{N,r}$ is the set of all PP of order r with N pieces;

$\sigma_{N,r}(f)_p := E(f, \mathcal{S}_{N,r})_{\mathbb{L}_p}$; $\sigma^{(q)}_{N,r}(f)_p := E(f, \mathcal{S}_{N,r} \cap \Delta^q)_{\mathbb{L}_p}$; and

$$\widetilde{\sigma}^{(q)}_{N,r}(f)_p := E(f, \mathcal{S}_{N,r} \cap \mathbb{C}^{r-2} \cap \Delta^q)_{\mathbb{L}_p}$$

Theorem (K. and Shadrin [2003]) *Let $q, r, N \in \mathbb{N}$, $r \geq q$, and $0 < p \leq \infty$. Then, there exist constants $c_0 = c_0(r)$ and $c_1 = c_1(r,p)$ such that, for all $f \in \Delta^q \cap \mathbb{L}_p$,*

$$\widetilde{\sigma}^{(q)}_{c_0 N,r}(f)_p \leq c_1 \sigma_{N,r}(f)_p \, .$$

$q = 1, 2$: Leviatan and Shadrin [1997], Petrov [1996]

$q = 3$, $p = \infty$: Petrov [1998]

A different kind of q-monotone approximation

Approximation by q-monotone functions: functions which are not in Δ^q are approximated by elements of the entire convex cone Δ^q (Damas, Marano, Ubhaya, Zwick).

Applications in SPA. Main idea: given $f \in \Delta^q$, take $s \in \mathcal{S}_{N,r}$, a best (unconstrained) free knot spline approximant to f (*i.e.*, $\|f - s\|_p = \sigma_{N,r}(f)_p$), and then correct s to $s^* \in \Delta^q$, a best approximant to s from Δ^q. Hence,

$$\|s - s^*\|_p = \inf_{g \in \Delta^q} \|s - g\|_p \leq \|s - f\|_p ,$$

and so

$$C(p)\|f - s^*\|_p \leq \|f - s\|_p + \|s - s^*\|_p \leq 2\|f - s\|_p = 2\sigma_{N,r}(f)_p .$$

Approximation of (non-q-monotone) splines by q-monotone f-s

$$\|s - s^*\|_{\mathbb{L}_p[a,b]} = \inf_{g \in \Delta^q} \|s - g\|_{\mathbb{L}_p[a,b]}$$

Properties of s^*:

$\mathfrak{Z} := \{z | s(z) = s^*(z)\}$ is closed

If the difference $s - s^*$ has no zeros inside $(c, d) \subset (a, b) \setminus \mathfrak{Z}$, then $s^* \in \mathcal{S}_{\lfloor q/2 \rfloor + 1, q}[c, d]$. Hence, if $s - s^*$ has $m - 1 < \infty$ distinct zeros in (c, d), then $s^* \in \mathcal{S}_{m(\lfloor q/2 \rfloor + 1), q}[c, d]$.

Conclusion: For $r \geq q \geq 2$, $0 < p \leq \infty$, let $s \in \mathcal{S}_{N,r} \cap \mathbb{C}$. Then there is $s^* \in \Delta^q$, a best approximant to s from Δ^q, which is a piecewise polynomial of order r.

Approximation by q-monotone functions: open problem

Open Problem

Let $s \in \mathcal{S}_{N,r}$ (or, more generally, there is a partition of $[a, b]$ into $O(N)$ subintervals $[a_i, b_i]$ such that $\pm s$ is in Δ^q on each of $[a_i, b_i]$ – this means that s is a piecewise q-monotone function on $[a, b]$ with at most $O(N)$ pieces), and let $\mathcal{P}_{\Delta^q}(s)_p$ denote the set of all best q-monotone approximants to s from Δ^q on $[a, b]$ in the $\mathbb{L}_p$ (quasi) norm.

Prove or disprove: There exists a $s^* \in \mathcal{P}_{\Delta^q}(s)_p$ such that $s - s^*$ has at most $O(N)$ sign changes.

Rates of Unconstrained and Shape Preserving Approximation: pointwise and uniform estimates

Monotone and Convex Approximation: pointwise estimates

Theorem *Let* $\alpha > 0$. *If for a nondecreasing (convex) function* $f \in \mathbb{C}[-1,1]$ *and* $\forall\, n > \alpha$ *there is a polynomial* p_{n-1} *such that*

$$|f(x) - p_{n-1}(x)| \le \Delta_n(x)^\alpha\,, \quad x \in [-1,1]\,,$$

then $\forall\, n > \alpha$ *there is a nondecreasing (convex) polynomial* p^*_{n-1} *satisfying*

$$|f(x) - p^*_{n-1}(x)| \le C(\alpha)\Delta_n(x)^\alpha\,, \quad x \in [-1,1]\,.$$

Monotone and Convex Approximation: pointwise estimates

Theorem *Let $\alpha > 0$. If for a nondecreasing (convex) function $f \in \mathbb{C}[-1,1]$ and $\forall\, n > \alpha$ there is a polynomial p_{n-1} such that*

$$|f(x) - p_{n-1}(x)| \le \Delta_n(x)^\alpha\,, \quad x \in [-1,1]\,,$$

then $\forall\, n > \alpha$ there is a nondecreasing (convex) polynomial p^*_{n-1} satisfying

$$|f(x) - p^*_{n-1}(x)| \le C(\alpha)\Delta_n(x)^\alpha\,, \quad x \in [-1,1]\,.$$

Monotone Case:

DeVore and Yu [1985]: $0 < \alpha < 2$

Shevchuk [1989]: $\alpha \ge 2$

Monotone and Convex Approximation: pointwise estimates

Theorem *Let* $\alpha > 0$. *If for a nondecreasing (convex) function* $f \in \mathbb{C}[-1,1]$ *and* $\forall\, n > \alpha$ *there is a polynomial* p_{n-1} *such that*

$$|f(x) - p_{n-1}(x)| \le \Delta_n(x)^\alpha\,, \quad x \in [-1,1]\,,$$

then $\forall\, n > \alpha$ *there is a nondecreasing (convex) polynomial* p^*_{n-1} *satisfying*

$$|f(x) - p^*_{n-1}(x)| \le C(\alpha)\Delta_n(x)^\alpha\,, \quad x \in [-1,1]\,.$$

Convex Case:

Yu [1985], Leviatan [1986]: $0 < \alpha < 2$

Mania [$\le$ 1992]: $\alpha > 2$

K. [1994]: $\alpha = 2$

Monotone Approximation by polynomials: uniform estimates

Theorem (K. and Listopad [1994])
Let $\alpha > 0$, $\alpha \neq 2$, *and let* $f \in \Delta^1$ *be such that for each* $n > \alpha$

$$E_n(f) \leq n^{-\alpha}\,.$$

Then

$$E_n^{(1)}(f) \leq C(\alpha) n^{-\alpha}\,.$$

For $\alpha = 2$ *this conclusion is not correct.*

Convex Approximation by polynomials: uniform estimates

Theorem (K. and Listopad [1994])
Let $\alpha \in (0,3) \cup (4,\infty)$, *and let* $f \in \Delta^2$ *be such that for each* $n > \alpha$

$$E_n(f) \le n^{-\alpha}\,.$$

Then

$$E_n^{(2)}(f) \le C(\alpha) n^{-\alpha}\,.$$

For $\alpha \in [3,4]$ *this conclusion is not correct.*

Relative n-widths with constraints: applications in q-monotone approximation

Relative n-widths with the constraints

- Let $W \subset X$, $V \subset X$, and let $\mathbb{M}^n(X, V)$ be the set of all linear manifolds M^n, $\dim M^n \leq n$ such that $M^n \cap V \neq \emptyset$.
  The quantity

  $$d_n(W, V)_X := \inf_{M^n \in \mathbb{M}^n} \sup_{f \in W} E(f, M^n \cap V)_X$$

  is the *relative n-width of W with the constraint V in X* (Konovalov)
- Remark: if $V = X$, then $d_n(W, V)_X = d_n(W)_X$ – Kolmogorov n-width

Shape preserving widths of Sobolev-type classes of q-monotone f-s

- $V := \Delta_+^q \mathbb{L}_{p'} := \mathbb{L}_{p'} \cap \Delta^q$
- $W := \Delta_+^q \mathbb{W}_p^r := \mathbb{W}_p^r \cap \Delta^q$ where $\mathbb{W}_p^r := \{f \,|\, \|f^{(r)}\|_{\mathbb{L}_p} \le 1\}$
- $d_n(\Delta_+^q \mathbb{W}_p^r, \Delta_+^q \mathbb{L}_{p'})_{\mathbb{L}_{p'}} := \inf\limits_{M^n \in \mathbb{M}^n} \sup\limits_{f \in \Delta_+^q \mathbb{W}_p^r} E(f, M^n \cap \Delta_+^q \mathbb{L}_{p'})_{\mathbb{L}_{p'}}$

Theorem (**Konovalov and Leviatan [2003]**) *Let* $r \in \mathbb{N}$, $q \in \mathbb{N}$ *and* $1 \le p, p' \le \infty$. *For* $3 \le q \le r$,

$$d_n(\Delta_+^q \mathbb{W}_p^r, \Delta_+^q \mathbb{L}_{p'})_{\mathbb{L}_{p'}} \asymp n^{-r+q-3+1/p}, \quad n \ge r,$$

and, if $q = r+1$, $r \ge 2$, *then*

$$d_n(\Delta_+^{r+1} \mathbb{W}_p^r, \Delta_+^{r+1} \mathbb{L}_{p'})_{\mathbb{L}_{p'}} \asymp n^{-2}, \quad n \ge r.$$

Corollaries: applications in q-monotone polynomial approximation

- If $3 \le q \le r$, $1 \le p \le \infty$ and $f \in \mathbb{W}_p^r$, then the rate of approximation $E_n^{(q)}(f)_p$ is asymptotically not faster than $n^{-r+q-3+1/p}$.
- If $q \ge 3$ and $r = q - 1$, then this rate is not faster than n^{-2}.

Corollary (Jackson type estimates): The estimates

$$E_n^{(q)}(f)_p \le Cn^{-3}\omega(f^{(3)}, 1/n)_p$$

and, hence,

$$E_n^{(q)}(f)_p \le C\omega_4(f, 1/n)_p$$

are not true for $q \ge 3$ $(0 < p < \infty)$ and $q \ge 4$ $(0 < p \le \infty)$.

Corollaries: applications in q-monotone polynomial approximation

- If $3 \le q \le r$, $1 \le p \le \infty$ and $f \in \mathbb{W}_p^r$, then the rate of approximation $E_n^{(q)}(f)_p$ is asymptotically not faster than $n^{-r+q-3+1/p}$.
- If $q \ge 3$ and $r = q - 1$, then this rate is not faster than n^{-2}.

Corollary ($q \ge 4$, $1 \le p \le \infty$): Let $q \ge 4$, $1 \le p \le \infty$ and $\alpha > 2$. Then there exists a function $f \in \Delta^q \cap \mathbb{W}_p^{\lceil \alpha \rceil}$ such that

$$E_n(f)_p \le C n^{-\alpha} ,$$

and, at the same time,

$$E_n^{(q)}(f)_p \not\le C n^{-\alpha} .$$

Corollaries: applications in q-monotone polynomial approximation

- If $3 \le q \le r$, $1 \le p \le \infty$ and $f \in \mathbb{W}_p^r$, then the rate of approximation $E_n^{(q)}(f)_p$ is asymptotically not faster than $n^{-r+q-3+1/p}$.
- If $q \ge 3$ and $r = q - 1$, then this rate is not faster than n^{-2}.

Corollary ($q = 3$, $p = 1$): Let $q = 3$, $p = 1$ and $\alpha > 2$. Then there exists a function $f \in \Delta^3 \cap \mathbb{W}_1^{[\alpha]}$ such that

$$E_n(f)_1 \le Cn^{-\alpha},$$

and, at the same time,

$$E_n^{(3)}(f)_1 \not\le Cn^{-\alpha}.$$

Jackson type estimates for q-monotone approximation: $p = \infty$

Bondarenko and Prymak: If $p = \infty$, $q \geq 4$, and $r \leq q - 2$, then the rate of $E_n^{(q)}(f)_\infty$ asymptotically cannot be faster than n^{-2} for all $f \in \mathbb{C}^r$: $E_n^{(q)}(x_+^{q-1}) \geq c_q n^{-2}$.

Corollary: The estimate $E_n^{(q)}(f)_\infty \leq C\omega_3(f, 1/n)_\infty$ is not true for $q \geq 4$.

Shvedov [1981]: The estimate $E_n^{(3)}(f)_\infty \leq C\omega_5(f, 1/n)_\infty$ is not true in general.

Bondarenko: $E_n^{(3)}(f)_\infty \leq C\omega_3^\varphi(f, 1/n)_\infty$

Open Problem: Prove or disprove the estimate

$$E_n^{(3)}(f)_\infty \leq C\omega_4(f, 1/n)_\infty \,.$$

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Bondarenko: $E_n^{(3)}(f)_\infty \le C\omega_3^{\varphi}(f, 1/n)_\infty$

Open Problem: Prove or disprove the estimate

$$E_n^{(3)}(f)_\infty \le C\omega_4(f, 1/n)_\infty .$$

Jackson type estimates for q-monotone approximation: $p = \infty$

Corollary: The estimate $E_n^{(q)}(f)_\infty \le C\omega_3(f, 1/n)_\infty$ is not true for $q \ge 4$.

Ma and Yu [1989]: $E_n^{(q)}(f)_\infty \le C\omega_2(f, 1/n)_\infty$
In fact, they proved: $E_n^{(q)}(f)_p \le C\omega_2(f, 1/n)_p$ for $1 \le p \le \infty$.

Cao and Gonska [1994]: For $f \in \mathbb{C}[-1,1] \cap \Delta^q$, $q \in \mathbb{N}$, there exists an algebraic polynomial $p_n \in \Delta^q$ of degree $O(n)$ such that

$$|f(x) - p_n(x)| \le C\omega_2(f, \sqrt{1-x^2}/n), \quad -1 \le x \le 1\,.$$

Jackson type estimates for q-monotone approximation: $p = \infty$

Corollary: The estimate $E_n^{(q)}(f)_\infty \le C\omega_3(f, 1/n)_\infty$ is not true for $q \ge 4$.

Ma and Yu [1989]: $E_n^{(q)}(f)_\infty \le C\omega_2(f, 1/n)_\infty$
In fact, they proved: $E_n^{(q)}(f)_p \le C\omega_2(f, 1/n)_p$ for $1 \le p \le \infty$.

Cao and Gonska [1994]: For $f \in \mathbb{C}[-1,1] \cap \Delta^q$, $q \in \mathbb{N}$, there exists an algebraic polynomial $p_n \in \Delta^q$ of degree $O(n)$ such that

$$|f(x) - p_n(x)| \le C\omega_2(f, \sqrt{1-x^2}/n), \quad -1 \le x \le 1\,.$$

Open Problem: Prove/disprove the above estimates for $p < \infty$, especially positive results – not much seems to be known if $q > 2$, and investigation of cases $q = 1$ and $q = 2$ is far from complete.

Appendix: SPA in terms of $\omega_{k,r}^{\varphi}(f^{(r)}, \delta)_\infty$ moduli

Skip Appendix

Monotone approximation: $E_n^{(1)}(f)_\infty \le C n^{-r} \omega_{k,r}^{\varphi}(f^{(r)}, n^{-1})_\infty$

r \ k	0	1	2	3	4	5	6	7	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
7	+	+	+	+	+	+	+	+	…
6	+	+	+	+	+	+	+	+	…
5	+	+	+	+	+	+	+	+	…
4	+	+	+	+	+	+	+	+	…
3	+	+	+	+	+	+	+	+	…
2	+	−	−	−	−	−	−	−	…
1	+	+	−	−	−	−	−	−	…
0		+	+	−	−	−	−	−	…

Monotone approximation: $E_n^{(1)}(f)_\infty \le C n^{-r} \omega_{k,r}^{\varphi}(f^{(r)}, n^{-1})_\infty$

r \ k	0	1	2	3	4	5	6	7	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
7	+	+	+	+	+	+	+	+	…
6	+	+	+	+	+	+	+	+	…
5	+	+	+	+	+	+	+	+	…
4	+	+	+	+	+	+	+	+	…
3	+	+	+	+	+	+	+	+	…
2	+	−	−	−	−	−	−	−	…
1	+	+	−	−	−	−	−	−	…
0		+	+	−	−	−	−	−	…

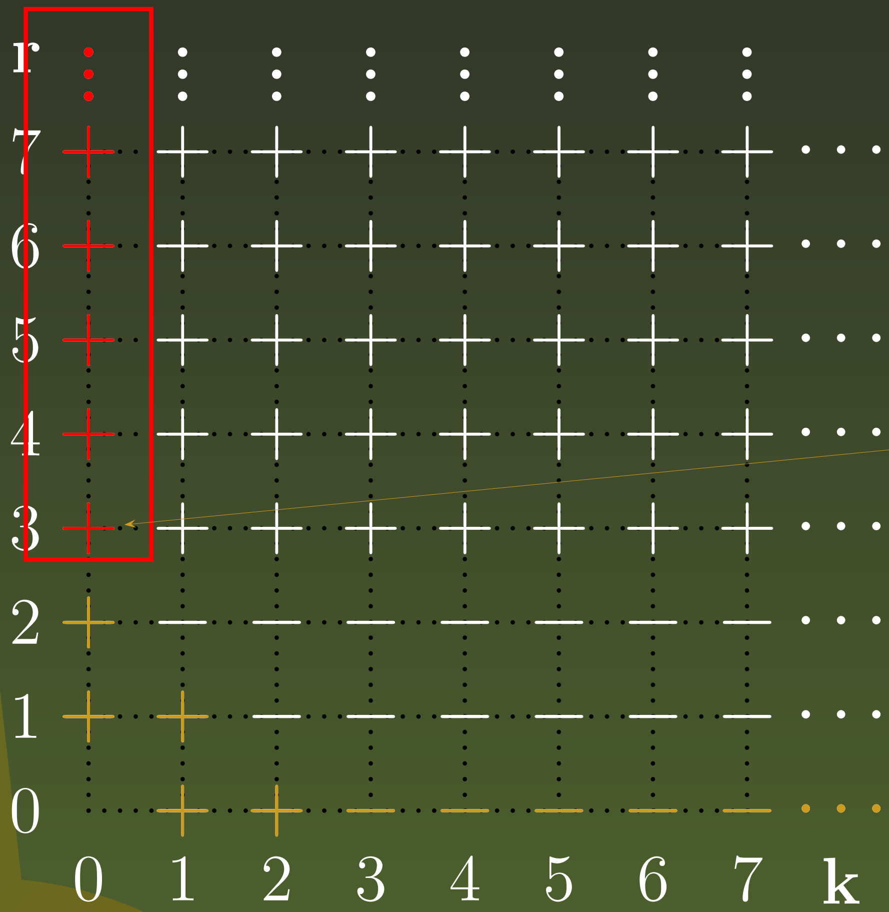
Shvedov [1981]

Monotone approximation: $E_n^{(1)}(f)_\infty \le Cn^{-r}\omega_{k,r}^{\varphi}(f^{(r)}, n^{-1})_\infty$

r									
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
7	+	+	+	+	+	+	+	+	…
6	+	+	+	+	+	+	+	+	…
5	+	+	+	+	+	+	+	+	…
4	+	+	+	+	+	+	+	+	…
3	+	+	+	+	+	+	+	+	…
2	+	−	−	−	−	−	−	−	…
1	+	+	−	−	−	−	−	−	…
0		+	+	−	−	−	−	−	…
	0	1	2	3	4	5	6	7	k

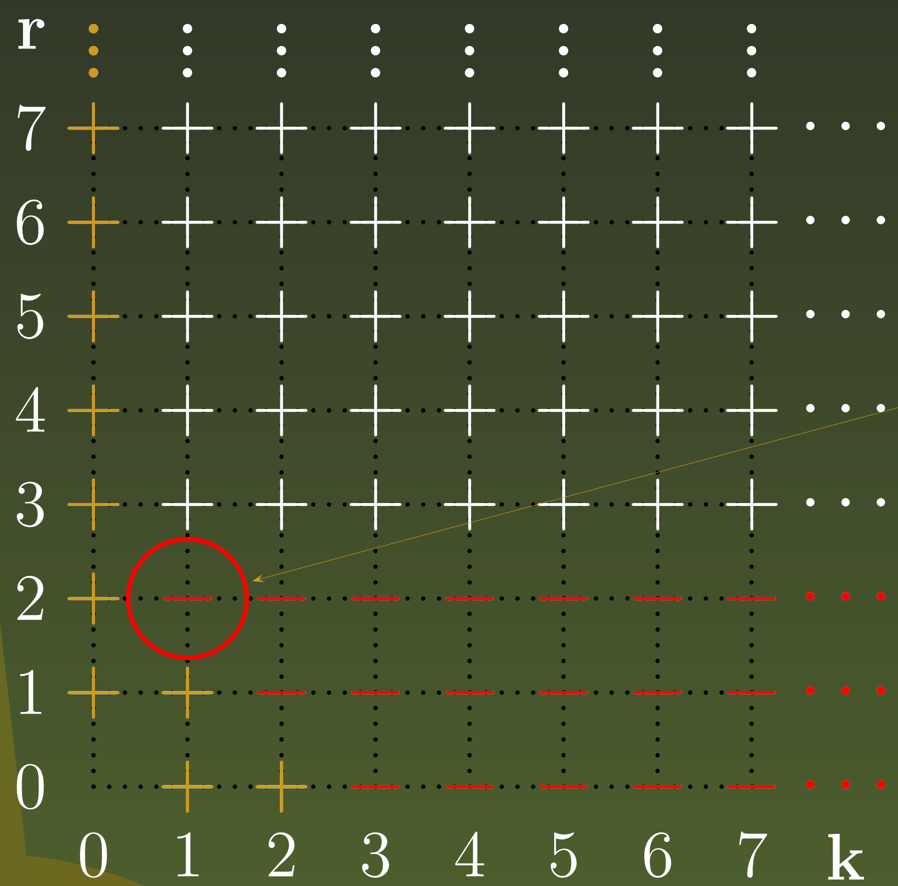
Yu [1987], Leviatan [1988]

Monotone approximation: $E_n^{(1)}(f)_\infty \le Cn^{-r}\omega_{k,r}^{\varphi}(f^{(r)}, n^{-1})_\infty$



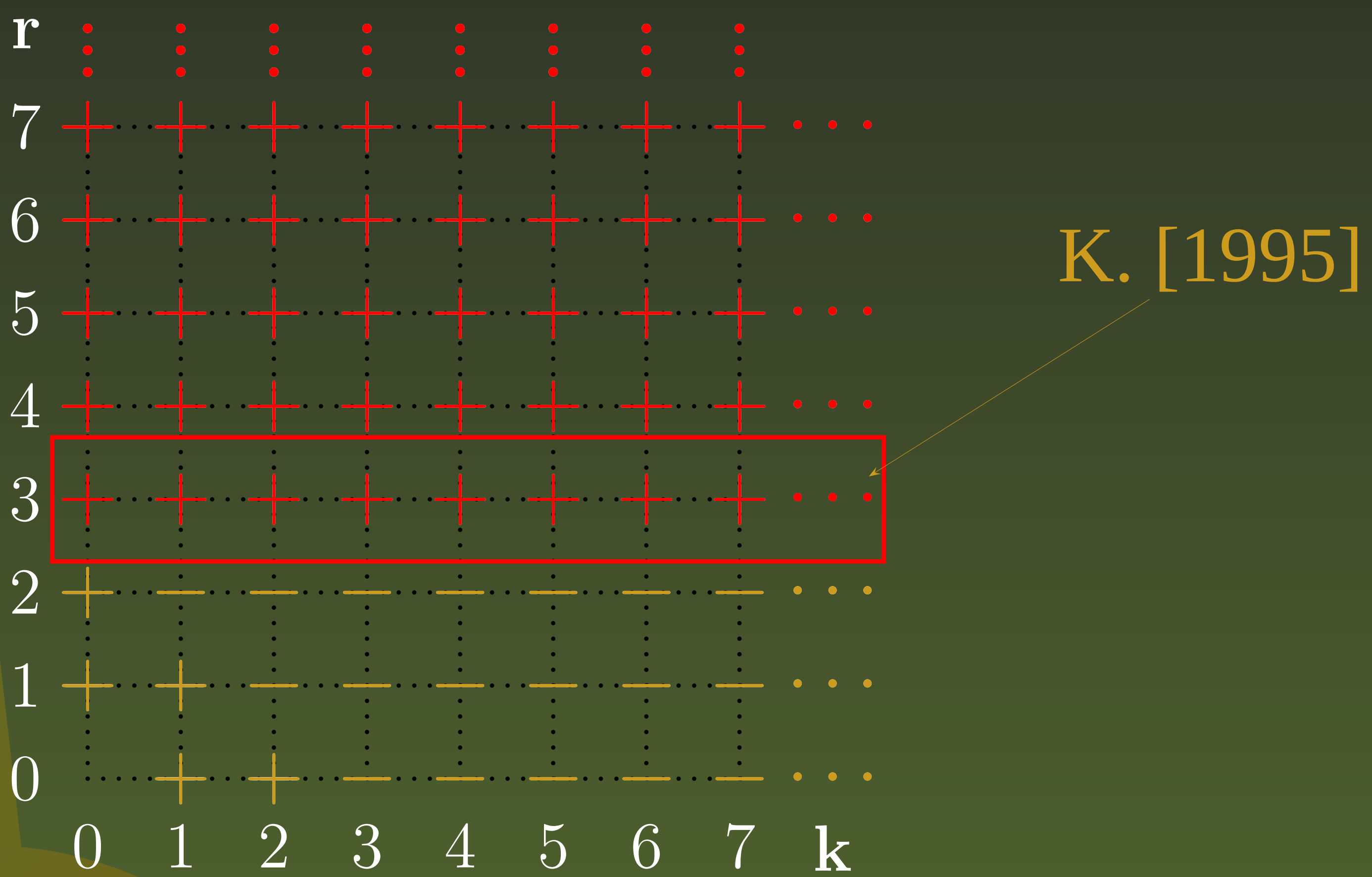
Dzubenko, Listopad, Shevchuk [1993]

Monotone approximation: $E_n^{(1)}(f)_\infty \le C n^{-r} \omega_{k,r}^{\varphi}(f^{(r)}, n^{-1})_\infty$



K. and Listopad [1994]

Monotone approximation: $E_n^{(1)}(f)_\infty \leq Cn^{-r}\omega_{k,r}^{\varphi}(f^{(r)}, n^{-1})_\infty$



Convex approximation: $E_n^{(2)}(f)_\infty \le C n^{-r} \omega_{k,r}^{\varphi}(f^{(r)}, n^{-1})_\infty$

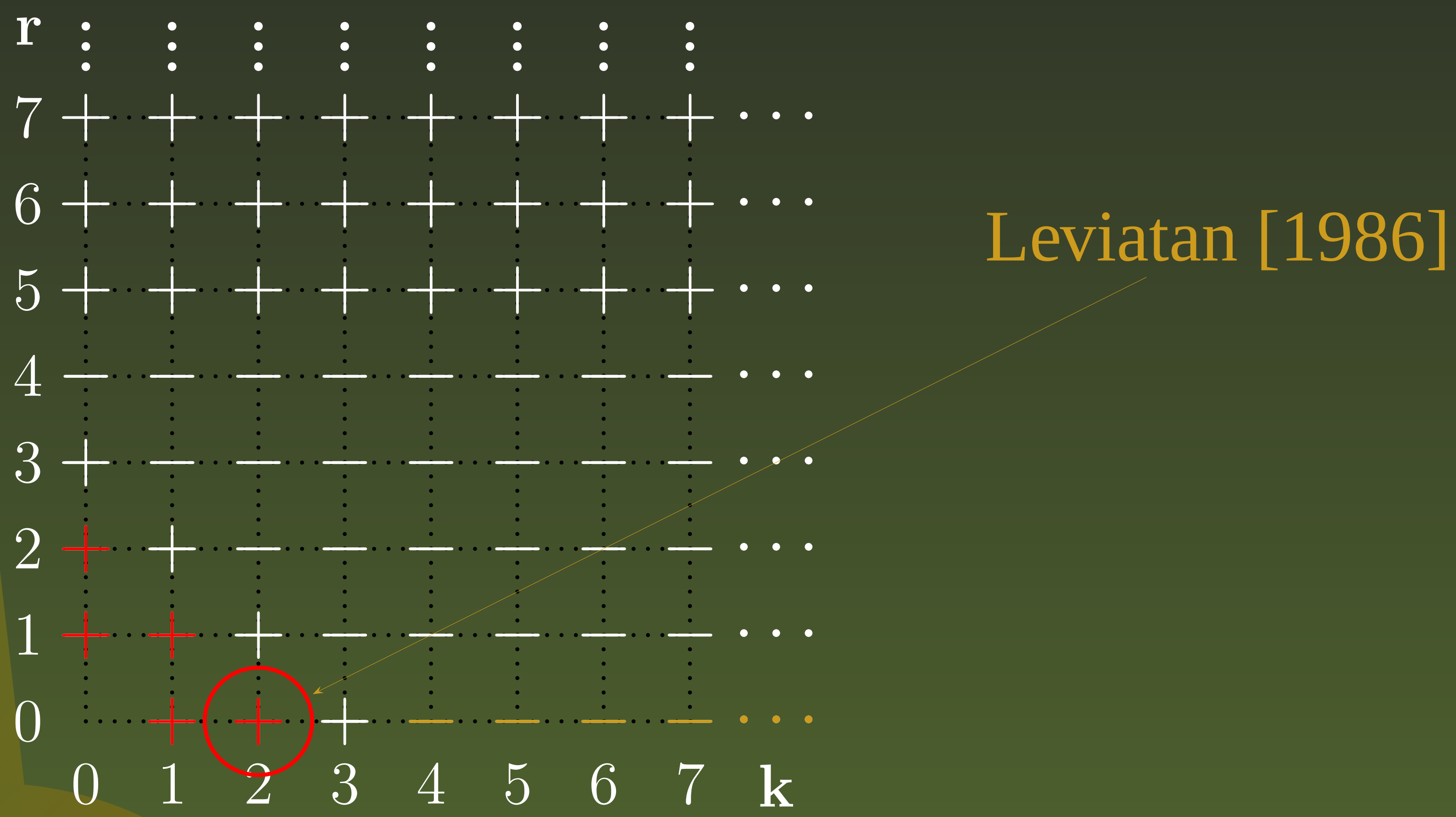
r									
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
7	+	+	+	+	+	+	+	+	…
6	+	+	+	+	+	+	+	+	…
5	+	+	+	+	+	+	+	+	…
4	−	−	−	−	−	−	−	−	…
3	+	−	−	−	−	−	−	−	…
2	+	+	−	−	−	−	−	−	…
1	+	+	+	−	−	−	−	−	…
0		+	+	+	−	−	−	−	…
	0	1	2	3	4	5	6	7	**k**

Convex approximation: $E_n^{(2)}(f)_\infty \leq Cn^{-r}\omega_{k,r}^{\varphi}(f^{(r)}, n^{-1})_\infty$

r \ k	0	1	2	3	4	5	6	7	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
7	+	+	+	+	+	+	+	+	…
6	+	+	+	+	+	+	+	+	…
5	+	+	+	+	+	+	+	+	…
4	−	−	−	−	−	−	−	−	…
3	+	−	−	−	−	−	−	−	…
2	+	+	−	−	−	−	−	−	…
1	+	+	+	−	−	−	−	−	…
0		+	+	+	−	−	−	−	…

Shvedov [1981]

Convex approximation: $E_n^{(2)}(f)_\infty \le Cn^{-r}\omega_{k,r}^{\varphi}(f^{(r)}, n^{-1})_\infty$

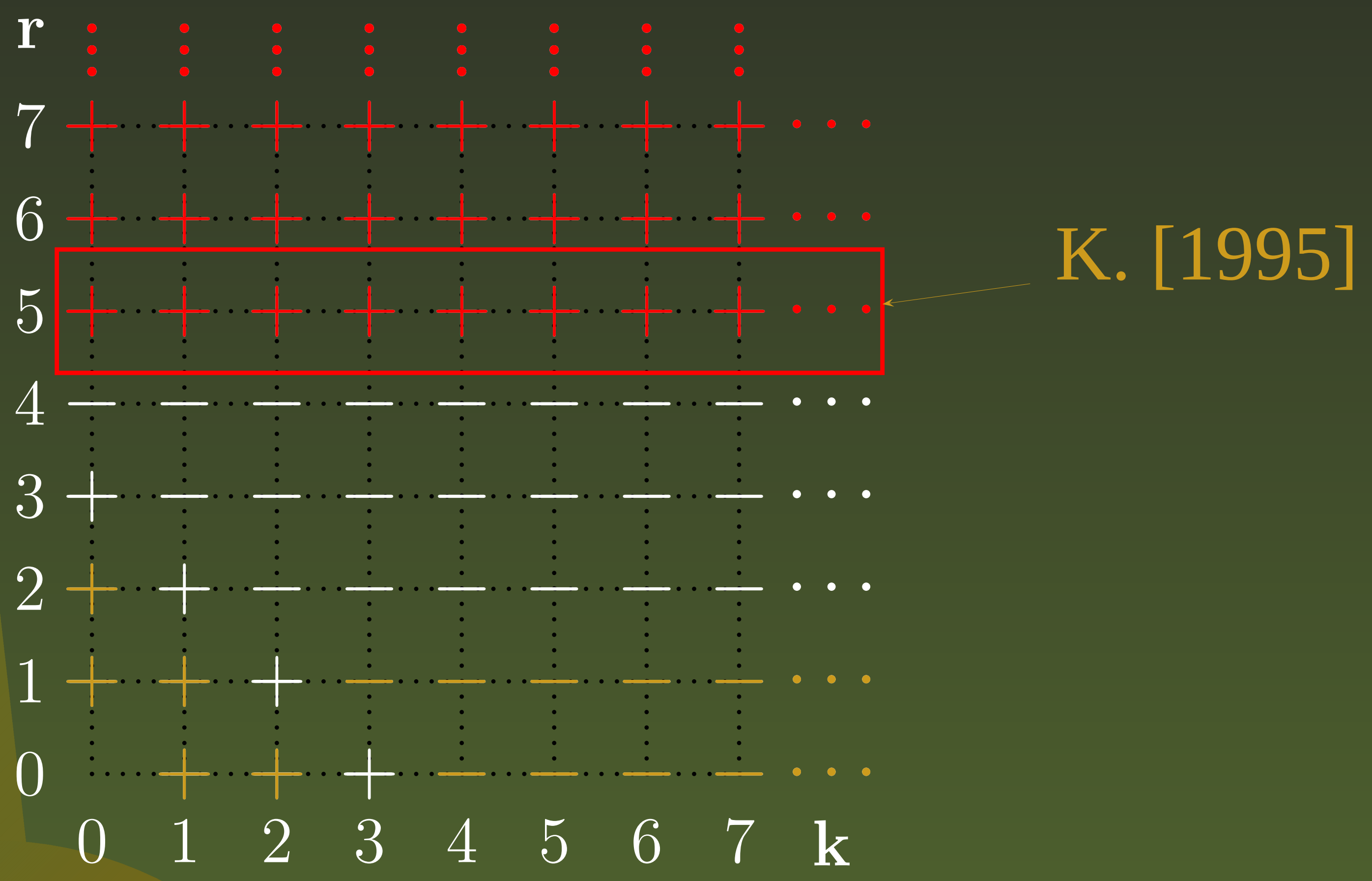


Convex approximation: $E_n^{(2)}(f)_\infty \le C n^{-r} \omega_{k,r}^{\varphi}(f^{(r)}, n^{-1})_\infty$

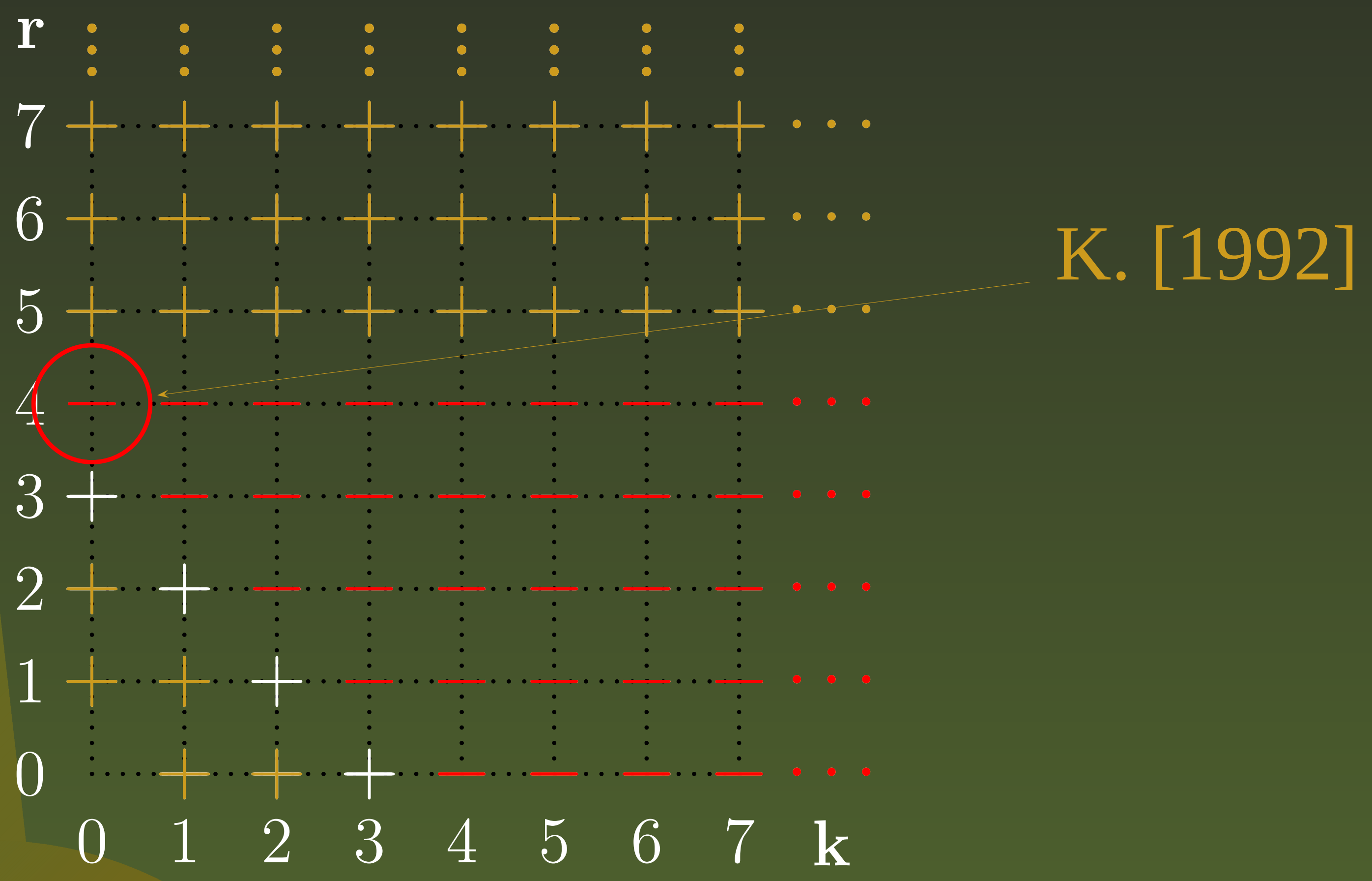
r									
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
7	+	+	+	+	+	+	+	+	…
6	+	+	+	+	+	+	+	+	…
5	+	+	+	+	+	+	+	+	…
4	−	−	−	−	−	−	−	−	…
3	+	−	−	−	−	−	−	−	…
2	+	+	−	−	−	−	−	−	…
1	+	+	+	−	−	−	−	−	…
0		+	+	+	−	−	−	−	…
	0	1	2	3	4	5	6	7	k

Mania [1991]

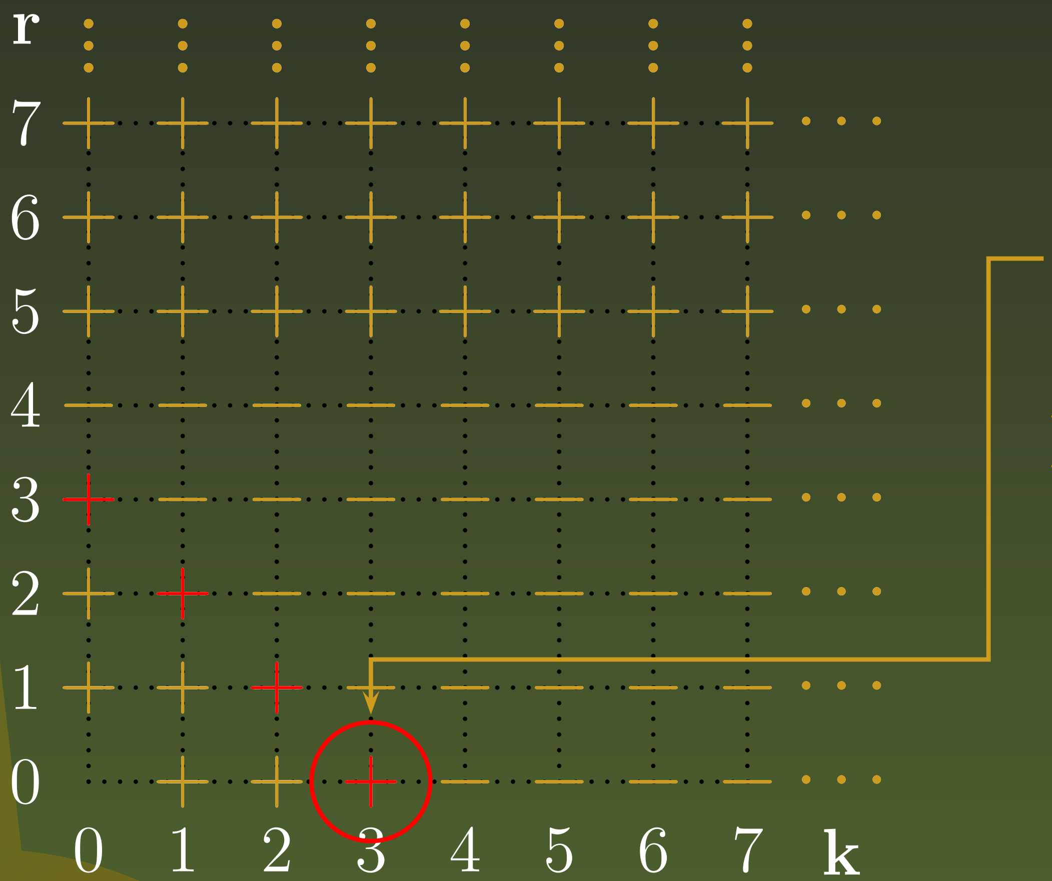
Convex approximation: $E_n^{(2)}(f)_\infty \le C n^{-r} \omega_{k,r}^{\varphi}(f^{(r)}, n^{-1})_\infty$



Convex approximation: $E_n^{(2)}(f)_\infty \le C n^{-r} \omega_{k,r}^{\varphi}(f^{(r)}, n^{-1})_\infty$



Convex approximation: $E_n^{(2)}(f)_\infty \le Cn^{-r}\omega_{k,r}^{\varphi}(f^{(r)}, n^{-1})_\infty$



K. [1994]

Hu, Leviatan, Yu [1994]:

$$E_n^{(2)}(f)_\infty \le C\omega_3(f, n^{-1})_\infty$$

Remarks

- Direct results for SPA in the $\mathbb{L}_p$ (quasi) norm

  Remark 1: $1 \le p < \infty$ is essentially different from $p = \infty$

  Example. Ma and Yu [1988], Shevchuk [1989]: $E_n^{(1)}(f)_\infty \le C n^{-1} \omega_m(f', n^{-1})_\infty$

  However, $E_n^{(1)}(f)_p \le C\omega_2(f', 1)_p$ is NOT true in general for $0 < p < \infty$.

  Remark 2: $0 < p < 1$ is essentially different from $p \ge 1$

  Example. Unconstrained: $\forall A > 0\ \forall B \in \mathbb{R}\ \forall 0 < p < 1\ \forall\ n \in \mathbb{N}\ \exists f \in \mathbb{A}C$ s.t. $E_n(f)_p \ge A n^B \|f'\|_p$.
  Convex ($0 < p < 1$): $E_n^{(2)}(f)_p \le C n^{-1} \omega(f', n^{-1})_p$

  However, $\forall m \ge 2\ \exists 0 < p < 1$ s.t. $E_n^{(2)}(f)_p \le C\omega_2(f^{(m)}, 1)_p$ is not true in general.

- "Co"-approximation (*i.e.*, copositive, intertwining, comonotone, coconvex, co-q-monotone).

  Remark: nothing (*i.e.*, no direct results) is known if $q \ge 3$.

Remarks

- Simultaneous SPA: SPA of a function together with its derivatives
- Weak SPA
- Interpolatory SPA
- SPA approximation by rational functions (convex: B. Gao, D. J. Newman, V. A. Popov)
- Constants depending on the function f

  Example. Shvedov [1981]: $\forall n \in \mathbb{N}\ \forall A > 0\ \exists f \in \Delta^1$ s.t. $E_n^{(1)}(f)_\infty > A\omega_3(f,1)_\infty$

  Leviatan and Shevchuk [1998]: $\forall f \in \Delta^1$: $E_n^{(1)}(f)_\infty \le C\omega_3(f,1/n)_\infty, n \ge N(f)$
- Estimates involving Ivanov moduli of smoothness $\tau_k(f,\psi,\delta)_{q,p}$ as well as generalized Ditzian-Totik moduli $\omega_k^{\varphi^\lambda}(f,\delta)_p$, $0 \le \lambda \le 1$
- Multivariate SPA

"A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street." (David Hilbert)

Thank You!