# Linear analysis of quadrature domains. II *

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Abstract

The natural correspondence between bounded planar quadrature domains, in the terminology of Aharonov-Shapiro, and certain square matrices with a distinguished cyclic vector is further exploited. Two different cubature formulas on quadrature domains, that is the computation of the integral of a real polynomial, are presented. The minimal defining polynomial of a quadrature domain is decomposed uniquely into a linear combination of moduli squares of complex polynomials. The geometry of a canonical rational embedding of a quadrature domain into the projective complement of a real affine ball is also investigated. Explicit computations on order-two quadrature domains illustrate the main results.

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INTRODUCTION. This note is a continuation of [P1] and it is devoted to some constructive aspects of the relation between quadrature domains and their linear data. We assume that the defining polynomial of a bounded quadrature domain is given and we try to find explicit formulas for the real moments of the domain and other naturally associated objects. Our approach is based on the observation that a quadrature domain is the level set of the norm of the resolvent of a square matrix, localized at a specific cyclic vector.

The first set of formulas we propose starts from the equation of a quadrature domain, it passes through an inversion of a Hankel matrix (formula (8) in text) and requires a logarithm of formal series (formula (9)) in order to compute all the moments of the domain.

The second method of computing the same moments starts again from the equation of the quadrature domain, then it identifies from this equation a square matrix with a cyclic vector, called in the sequel the linear data of the domain, and finally exploits Helton-Howe trace formula for seminormal operators in order to evaluate the moments. In particular, this method gives a non-commutative cubature formula on quadrature domains (formula (12) in text) which is exact on all $n$-polyharmonic polynomials, for $n$ specified. An error formula for this cubature is then obtained.

The rest of the paper deals with some specific properties of the resolvent of the linear data of a quadrature domain. The minimal polynomial which defines a quadrature domain of order $d$ is canonically decomposed into the modulus square of the minimal polynomial of the associated matrix minus exactly $d$ moduli squares of complex polynomials, of exact degrees $d-1, d-2, \ldots, 1, 0$. Thus a natural set of parameters of a quadrature domain is exhibited.

A canonical rational embedding of the quadrature domain $\Omega$ of order $d$ in the exterior of the unit ball of $\mathbf{C}^d$ is obtained. Then we prove that the multivalued Schwarz reflection in the boundary of $\Omega$ maps the exterior of $\Omega$ into $\overline{\Omega}$, and in this transformation the boundary covers the boundary exactly once via the identity map. A uniqueness result for this embedding of a quadrature domain in the exterior of a multidimensional ball constitutes the subject of Section 5. As a consequence certain rational maps from $\mathbf{C}$ into $\mathbf{C}^n$ which commute with the reflections in the unit balls of the two affine spaces are classified.

A few simple examples of the interplay between planar domains and pairs of matrices with a cyclic vector end the paper.

# 1 PRELIMINARIES

We recall from [P1] a few formulas which relate a quadrature domain to a matrix with a cyclic vector. Although these formulas have been motivated by the study of the $L$-problem of moments in the real plane, we do not make any precise reference to this relationship; see for details [P2].

Let $\Omega$ be a bounded planar domain and let $dA$ stand for the area measure in

**C**. The coordinate in the complex plane **C** will be denoted by $z$. The domain $\Omega$ is called, following the terminology of Aharonov and Shapiro [AS], a *quadrature domain* if there exists a distribution $u$ with finite support in $\Omega$ such that:

$$\int_\Omega f dA = u(f),$$

for every integrable analytic function $f$ in $\Omega$. Quadrature domains tend to be very rigid; they are remarkable in many respects as it is amply illustrated by the recent monograph [Sh].

The *order* of the quadrature domain $\Omega$ is the cardinality of the support of $u$, counting multiplicities. To be more specific, there are points $\lambda_j \in \Omega$ and constants $\gamma_{jk}, 0 \leq k \leq m(j)-1, 1 \leq j \leq m$, with the property that:

$$u(f) = \sum_{j=1}^{m} \sum_{k=0}^{m(j)-1} \gamma_{jk} f^{(k)}(\lambda_j),$$

where $u$ and $f$ are as above. To make the above decomposition optimal, we assume that $\gamma_{j,m(j)-1} \neq 0$ for all $j, 1 \leq j \leq m$. The order $d = d(\Omega)$ of $\Omega$ is then by definition:

$$d = \sum_{j=1}^{m} m(j).$$

The quadrature domains of order one are precisely the disks, see [Sh]. In general the equation of the boundary of a quadrature domain $\Omega$ of order $d$ is given, up to a finite set, by a monic self-adjoint irreducible polynomial

$$Q(z, \overline{z}) = \sum_{k,l=0}^{d} \alpha_{kl} z^k \overline{z}^l,$$

where by self-adjoint we mean $\alpha_{kl} = \overline{\alpha_{lk}}$ and by monic we mean $\alpha_{dd} = 1$. For details we refer to [G1].

A quadrature domain $\Omega$ is characterized by the existence of a meromorphic function $S(z)$ in $\Omega$, continuous on $\overline{\Omega} \setminus \{\lambda_1, \ldots, \lambda_m\}$, with the property $S(z) = \overline{z}$ for $z \in \partial\Omega$. The function $S$ is called *the Schwarz function* of $\Omega$, see [D] and [Sh]. The poles of $S(z)$ coincide, including the multiplicities, with the nodes $\lambda_j, 1 \leq j \leq m$, of the quadrature identity. Let us define the polynomial:

$$P(z) = \prod_{j=1}^{m} (z - \lambda_j)^{m(j)},$$

so that $P(z)S(z)$ is a holomorphic function in $\Omega$.

The following facts were established in [G1].

THEOREM 1.1 ([G1], SECTION 6) *Let $\Omega$ be a quadrature domain. Then, with the above notation, we have:*

$$P(z) = z^d + \sum_{j=0}^{d-1} \alpha_{jd} z^j, \tag{1}$$

*and*

$$\frac{1}{\pi} \sum_{k=1}^{m} \sum_{l=0}^{m(k)-1} \frac{l!\gamma_{kl}}{(z-\lambda_k)^{l+1}} = \alpha_{d,d-1} - \frac{\sum_{j=0}^{d} \alpha_{j,d-1} z^j}{P(z)} = S(z) + A(z), \tag{2}$$

*where $A(z)$ is an analytic function in $\Omega$.*

An explanation of these formulas will become available later in this and the next section. Roughly speaking, Theorem 1.1 above asserts that the first two lines in the matrix of coefficients $\alpha_{kl}$ (of the defining polynomial $Q(z,\overline{z})$) determine the quadrature data $\lambda_j, \gamma_{jk}$, as well as the polar part of the Schwarz function $S(z)$.

Actually there is more structure in the defining polynomial $Q$. Namely, there exists a linear transformation $U : \mathbf{C}^d \longrightarrow \mathbf{C}^d$ with a cyclic vector $\xi \in \mathbf{C}^d$ for $U^*$ and with $P(z)$ as characteristic polynomial, such that:

$$\frac{Q(z,\overline{z})}{|P(z)|^2} = 1 - \|(U^* - \overline{z})^{-1}\xi\|^2, \tag{3}$$

where the equality is understood in the sense of rational functions, see for details [P1]. It is clear from the above discussion that both the polynomial $Q(z,\overline{z})$ or the pair $(U,\xi)$ form complete invariants for the quadrature domain $\Omega$.

Similarly to Theorem 1.1 we have the following result.

THEOREM 1.2 ([P1] SECTION 3) *Let $\Omega$ be a quadrature domain. With the above notation we have:*

$$u(f) = \pi\langle f(U)\xi, \xi\rangle, \tag{4}$$

*for every analytic function $f$ in $\Omega$. Moreover,*

$$S(z) = -\langle (U-z)^{-1}\xi, \xi\rangle + B(z), \tag{5}$$

*where $B(z)$ is an analytic function in $\Omega$.*

To relate Theorems 1.1 and 1.2, we remark that $B(z) = -A(z), z \in \Omega$. Indeed, this follows from the fact that both the left member of (2) and $-\langle (U - z)^{-1}\xi, \xi\rangle$ are rational functions which vanish at infinity.

NOTATION. Throughout this paper we keep generically unchanged the notation introduced in this section. That is, $\Omega$ is a bounded quadrature domain of order $d$ and the quadrature data are $\lambda_j, \gamma_{jk}, 1 \leq j \leq m, 0 \leq k \leq m(j)-1$. The defining polynomial of $\Omega$ is $Q(z,\overline{z})$ with the coefficients $\alpha_{jk}, 0 \leq j,k \leq d, \alpha_{dd} = 1$. The Schwarz function is $S(z)$ with denominator $P(z)$, and the linear data of $\Omega$ are $(U,\xi)$. In addition, we will consider the moments of the domain $\Omega$:

$$a_{mn} = \int_\Omega z^m \overline{z}^n dA(z),$$

and the scalar products

$$g_{nm} = \langle U^{*m}\xi, U^{*n}\xi \rangle,$$

where $m,n$ are non-negative integers.

## 2 From the equation of a quadrature domain to its moments

The aim of this section is to find explicit formulas for computing the integral of a polynomial in $z$ and $\overline{z}$ on a quadrature domain $\Omega$ (against the area measure), knowing only the defining polynomial of the boundary of $\Omega$. A first naive approach to this problem would be to use Stokes formula and one of the known expressions for the polar parts of the Schwarz function of $\Omega$. In what follows we go beyond this step and present two other alternative ways of computing such integrals.

Let us recall the basic exponential transformation which relates the finite matrix ${(\alpha_{jk})_{j,k=0}}^d$ to the infinite matrix ${(a_{mn})_{m,n=0}}^\infty$ :

$$\frac{Q(z,\overline{z})}{|P(z)|^2} = exp(\frac{-1}{\pi}\sum_{m,n=0}^{\infty}\frac{a_{mn}}{z^{m+1}\overline{z}^{n+1}}), \tag{6}$$

which is valid for large values of $|z|$, see for details [P2]. Thus, by taking a logarithm at the level of formal series, the moments $a_{mn}$ can be determined from the defining polynomial $Q(z,\overline{z})$. In its turn, the pair of matrices $(U,\xi)$ can be used in simplifying the above computation:

$$\frac{Q(z,\overline{z})}{|P(z)|^2} = 1 - \sum_{m,n=0}^{\infty}\frac{\langle U^{*n}\xi, U^{*m}\xi\rangle}{z^{m+1}\overline{z}^{n+1}} = 1 - \sum_{m,n=0}^{\infty}\frac{g_{mn}}{z^{m+1}\overline{z}^{n+1}}. \tag{7}$$

Therefore, a direct relation between the matrix of coefficients $\alpha_{kl}$ and the Gram matrix $\langle U^{*n}\xi, U^{*m}\xi\rangle$ becomes possible. To simplify the following computation we put $\alpha_k = \alpha_{kd}, 0 \leq k \leq d$, so that $P(z) = \sum_{k=0}^d \alpha_k z^k$. Note that $\alpha_d = 1$ by a convention we have adopted in the previous section.

We begin with a series of elementary computations:

$$\overline{P(z)}(U^* - \overline{z})^{-1}\xi = (\overline{P(z)} - P(U)^*)(U^* - \overline{z})^{-1}\xi = -\sum_{k=1}^{d}\overline{\alpha}_k(\sum_{s=0}^{k-1}U^{*k-s-1}\xi\overline{z}^s).$$

Later we will return to a second possible form of the same polynomial (see formula (14) below).

Accordingly we obtain:

$$|P(z)|^2\|(U^*-\overline{z})^{-1}\xi\|^2 = \sum_{k,l=1}^{d}\sum_{s=0}^{k-1}\sum_{t=0}^{l-1}\overline{\alpha_k}\alpha_l\langle U^{*k-s-1}\xi, U^{*l-t-1}\xi\rangle z^t\overline{z}^s.$$

If we fix $s,t$ in the last formula and perform the other summations we obtain the coefficient of $z^t\overline{z}^s$ in $|P(z)|^2-Q(z,\overline{z})$.

Let us recall that $g_{kj}=\langle U^{*j}\xi, U^{*k}\xi\rangle$ are the elements of the Gram matrix $G=(g_{jk})_{j,k=0}{}^{d-1}$. Let us also introduce the Hankel matrix:

$$H(\alpha)=\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & 1\\ \alpha_2 & \alpha_3 & \alpha_4 & \dots & 0\\ \alpha_3 & \alpha_4 & \alpha_5 & \dots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ \alpha_{d-1} & 1 & 0 & \dots & 0\\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and the matrix $A(\alpha)$ of coefficients of the polynomial $|P(z)|^2-Q(z,\overline{z})$:

$$A(\alpha)_{jk}=\alpha_j\overline{\alpha_k}-\alpha_{jk}, \qquad (0\le j,k\le d-1).$$

Thus we obtain:

$$A(\alpha)_{jk}=\sum_{r,s}\alpha_r\overline{\alpha}_s g_{r-j-1,s-k-1}=\sum_{p,q}\alpha_{j+p+1}g_{pq}\overline{\alpha}_{k+q+1}.$$

Then the previous computation can be summarized in the following result.

PROPOSITION 2.1. *The Gram matrix $G$ of the linear data $(U,\xi)$ of a quadrature domain can be obtained from the coefficients $(\alpha_{jk})_{j,k=0}{}^{d}$ of the defining polynomial by the formula:*

$$H(\alpha)GH(\alpha)^*=A(\alpha). \tag{8}$$

Finally, let us write the announced formula for the moments of a quadrature domain:

$$\sum_{m,n=0}^{\infty}\frac{a_{mn}}{z^{m+1}\overline{z}^{n+1}}=-\pi log(1-\sum_{m,n=0}^{\infty}\frac{g_{mn}}{z^{m+1}\overline{z}^{n+1}}). \tag{9}$$

We remark that the above transformation, from the matrix $(g_{mn})$ to the matrix of moments $(a_{mn})$ is triangular, in the sense that $a_{mn}$ depends only on $g_{kl}$ where $0 \leq k \leq m$ and $0 \leq l \leq n$. A couple of examples of low order quadrature domains which illustrate the preceding formulas are included in the last section of the paper.

Our next aim is to factor the Gram matrix $G$ into the linear data $(U, \xi)$ and then to use them in another formula for the moments of the quadrature domain, this time the computations being carried only at the level of linear algebra (and avoiding non-linear operations such as the above logarithm).

## 3 A non-commutative cubature formula

In this section we exploit Helton-Howe trace formula in the construction of a cubature formula on quadrature domains. Traditionally, cubature formulas in one or several variables arise from the evaluation of functions at the zeroes of some families of orthogonal polynomials, see [ST], [Xu]. Below we approximate the integral of a (real analytic) function on a quadrature domain $\Omega$ by its values on the matrix $U$ and some bigger matrices constructed recurrently from $U$. An error formula is obtained, similar to the errors in the well studied one dimensional theory, see [ST] Chapter IV.

Let $\Omega$ be a quadrature domain of order $d$ with the linear data $(U, \xi)$ on the Hilbert space $K$ of dimension $d$. Let $T$ be the unique irreducible hyponormal operator, acting on the Hilbert space $H, K \subset H$, such that $[T^*, T] = \xi \otimes \xi$ and with principal function equal to the characteristic function of $\Omega$, see for details [P1],[P2]. We recall from [P1] that $K$ is the linear span of the vectors $\{T^{*n}\xi; n \geq 0\}$ and that $U^* = T^*|K$.

For a polynomial $p \in \mathbf{C}[z, \overline{z}]$,

$$p(z, \overline{z}) = \sum_{\alpha+\beta \leq n} c_{\alpha\beta} z^\alpha \overline{z}^\beta,$$

we introduce the symmetrized operator valued functional calculus:

$$p^\sharp(T, T^*) = \sum_{\alpha+\beta \leq n} \frac{c_{\alpha\beta}}{\beta+1} \sum_{\gamma=0}^{\beta} T^{*\gamma} T^\alpha T^{*\beta-\gamma}. \tag{10}$$

Lemma 3.1. *With the above notation we have:*

$$\int_\Omega p dA = \pi \langle p^\sharp(T, T^*)\xi, \xi \rangle. \tag{11}$$

Proof. Indeed, for a monomial $z^n \overline{z}^m$, Helton-Howe trace formula (see for details [P2]) yields:

$$\pi^{-1} \int_\Omega z^n \overline{z}^m dA = \frac{1}{m+1} Tr[T^{*m+1} T^n, T] =$$

$$\frac{1}{m+1}Tr\sum_{k=0}^{m} T^{*k}[T^*,T]T^{*m-k}T^n = \frac{1}{m+1}\sum_{k=0}^{m}\langle T^{*m-k}T^nT^{*k}\xi,\xi\rangle.$$

The previous formula becomes effective as soon as we recall the block structure of the operator $T$. To be more precise, let us define recurrently:

$$U_0 = U, \qquad {A_0}^2 = \xi\otimes\xi - [{U_0}^*, U_0],$$

and for $k \geq 0$:

$$U_{k+1} = {A_k}^{-1}U_kA_k, \qquad {A_{k+1}}^2 = {A_k}^2 - [{U_{k+1}}^*, U_{k+1}].$$

We know from [P1], Theorem 4.2 that, for all $k \geq 0$, $A_k$ are positive matrices on the space $K$. Then the operator $T$ is unitarily equivalent to an infinite block matrix with $U_k$ on the diagonal, $A_k$ under the diagonal and zero elsewhere.

For a fixed positive integer $n$ we denote by $T_n$ the $(n+1)\times(n+1)$- block truncation of $T$. More specifically:

$$T_n = \begin{pmatrix} U_0 & 0 & 0 & \dots & 0 & 0 \\ A_0 & U_1 & 0 & \dots & 0 & 0 \\ 0 & A_1 & U_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & U_{n-1} & 0 \\ 0 & 0 & 0 & \dots & A_{n-1} & U_n \end{pmatrix}.$$

For a polynomial $p(z,\overline{z})$ we denote by $deg_z(p), deg_{\overline{z}}(p)$ the corresponding degrees in $z$ and $\overline{z}$.

THEOREM 3.2. *Let $\Omega$ be a quadrature domain with associated hyponormal operator $T$ and let $p \in \mathbf{C}[z,\overline{z}]$. Then:*

$$\int_\Omega pdA = \pi\langle p^\sharp(T_n, {T_n}^*)\xi,\xi\rangle \tag{12}$$

*whenever $n \geq min(deg_z(p), deg_{\overline{z}}(p))$.*

PROOF. Let $P_n$ denote the orthogonal projection of the Hilbert space $H$ onto the finite dimensional subspace $K_n = K + TK + \ldots + T^nK$. Then we find the identities $T_n = P_nTP_n$ and $T^*P_n = P_nT^*P_n$ from the block structure of the matrix $T$. Moreover, $T^kx = (P_nTP_n)^kx = {T_n}^kx$ for every $x \in K$ and $k \leq n$.

Let $p(z,\overline{z})$ be a polynomial satisfying $deg_z(p) \leq n$. For a typical monomial in $p^\sharp(T,T^*)$ we have:

$$\langle T^{*\gamma}T^\alpha T^{*\beta-\gamma}\xi,\xi\rangle = \langle T^{*\gamma}P_nT^\alpha P_0T^{*\beta-\gamma}\xi,\xi\rangle = \langle {T_n}^{*\gamma}{T_n}^\alpha{T_n}^{*\beta-\gamma}\xi,\xi\rangle,$$

because $\alpha \leq n$.

Similarly, assume instead that $deg_{\overline{z}}(p) \leq n$. Then, in the above notation $\gamma \leq \beta \leq n$, whence:

$$\langle T^{*\gamma} T^{\alpha} T^{*\beta-\gamma} \xi, \xi \rangle = \langle P_0 T^{*\gamma} P_n T^{\alpha} P_0 T^{*\beta-\gamma} \xi, \xi \rangle = \langle {T_n}^{*\gamma} {T_n}^{\alpha} {T_n}^{*\beta-\gamma} \xi, \xi \rangle.$$

This completes the proof of Theorem 3.2.

Let us remark that for analytic polynomials $p(z)$, formula (12) reduces to the quadrature identity (4). In the spirit of some recent advances in multivariable cubature formulas (cf. [Xu]), relation (12) holds in particular for $deg(p) \leq 2n+1$.

For an arbitrary polynomial $p$, the error in formula (12) depends only on the monomials in $p$ of the form $z^{\alpha}\overline{z}^{\beta}$ with both $\alpha$ and $\beta$ strictly larger than $n$, hence only on $\Delta^{n+1}p$, where $\Delta$ is the Laplace operator. Actually we can make this statement more precise.

For a disk $D(0, \rho)$ centered at zero, of radius $\rho$ and a polynomial $p(z) = \sum_{\alpha+\beta \leq N} c_{\alpha\beta} z^{\alpha}\overline{z}^{\beta}$ we introduce the norm:

$$\|p\|_{\rho} = \sum_{\alpha+\beta \leq N} |c_{\alpha\beta}| \rho^{\alpha+\beta}.$$

In virtue of Cauchy inequalities for functions of two variables, for every positive $\epsilon$, the preceding norm can be estimated from above by the uniform norm of $p(z, w)$ for $|z|, |w| \leq \rho + \epsilon$. However, we do not make use of this estimate below.

PROPOSITION 3.3. *Let $\Omega$ be a quadrature domain contained in the disk $D(0, \rho)$ and let $p(z, \overline{z})$ be an arbitrary polynomial.*

*Then for every positive integer $n$ we have:*

$$|\pi^{-1} \int_{\Omega} p dA - \langle p^{\sharp}(T_n, {T_n}^*) \xi, \xi \rangle| \leq \frac{Area(\Omega)}{\pi} \frac{(\rho/2)^{2n+2}}{(n+1)!^2} \|\Delta^{n+1} p\|_{\rho}. \tag{13}$$

PROOF. Since the domain $\Omega$ is contained in the disk $D(0, \rho)$, the spectral radius of the operator $T$ is less or equal than $\rho$. But for hyponormal operators this implies $\|T\| \leq \rho$, see [P2] and the references there.

According to Lemma 3.1 we have to estimate

$$|\langle (p^{\sharp}(T, T^*) - p^{\sharp}(T_n, {T_n}^*)) \xi, \xi \rangle|.$$

For a typical monomial in this expression we obtain:

$$|\langle (T^{*\gamma} T^{\alpha} T^{*\beta-\gamma} - {T_n}^{*\gamma} {T_n}^{\alpha} {T_n}^{*\beta-\gamma}) \xi, \xi \rangle| =$$

$$|\langle T^{*\gamma} (I - P_n) T^{\alpha} T^{*\beta-\gamma} \xi, \xi \rangle| \leq \|T\|^{\alpha+\beta} \|\xi\|^2 \leq \frac{Area(\Omega)}{\pi} \rho^{\alpha+\beta}.$$

Let $p(z,\overline{z})$ be as above, with coefficients $c_{\alpha\beta}$. Then

$$\frac{\Delta^{n+1}p}{4^{n+1}} = \partial^{n+1}\overline{\partial}^{n+1}p =$$

$$\sum_{\alpha,\beta>n} c_{\alpha\beta}\alpha\beta(\alpha-1)(\beta-1)\ldots(\alpha-n)(\beta-n)z^{\alpha-n-1}\overline{z}^{\beta-n-1}.$$

On the other hand,

$$|\langle (p^\sharp(T,T^*) - p^\sharp(T_n,{T_n}^*))\xi,\xi\rangle| =$$

$$|\sum_{\alpha,\beta>n}\frac{c_{\alpha\beta}}{\beta+1}\sum_{\gamma=0}^{\beta}\langle T^{*\gamma}(I-P_n)T^\alpha T^{*\beta-\gamma}\xi,\xi\rangle| \le \|\xi\|^2\sum_{\alpha,\beta>n}|c_{\alpha\beta}|\rho^{\alpha+\beta} \le$$

$$\frac{\|\xi\|^2\rho^{2n+2}}{4^{n+1}}\sum_{\alpha,\beta>n}\frac{4^{n+1}|c_{\alpha\beta}|\alpha\beta(\alpha-1)(\beta-1)\ldots(\alpha-n)(\beta-n)\rho^{\alpha+\beta-2n-2}}{(n+1)!(n+1)!} \le$$

$$\frac{Area(\Omega)}{\pi}\frac{(\rho/2)^{2n+2}}{(n+1)!^2}\|\Delta^{n+1}p\|_\rho.$$

This completes the proof of Proposition 3.3.

Suppose that $f(z,\overline{w})$ is an analytic function of two complex variables which is convergent in the polydisk $D(0,\rho')\times D(0,\rho')$, where $\rho'>\rho$. Then, by restricting $f$ to the diagonal, the norm $\|f\|_\rho$ is finite. This shows in particular that the functional claculus $f^\sharp(T_n,{T^*}_n)$ makes sense; and so does Proposition 3.3. Thus the quadrature formula (12) is exact for such functions which in addition are $(n+1)$-polyharmonic when restricted to the diagonal.

# 4 A RATIONAL EMBEDDING OF QUADRATURE DOMAINS

The resolvent of the matrix associated to a quadrature domain of order $d$ gives a canonical embedding in an affine, or projective complex space of dimension $d$. This embedding will prove to be functorial with respect to the reflections in the boundaries of the original quadrature domain and respectively the unit sphere in $\mathbf{C}^d$.

The compactification of the complex plane with one point at infinity is denoted in the sequel by $\hat{\mathbf{C}}$ or $\mathbf{P}_1(\mathbf{C})$. The projective space of dimension $d$ will be denoted by $\mathbf{P}_d(\mathbf{C})$ or simply $\mathbf{P}_d$.

In this section we treat a slightly more general situation than that encountered in the case of quadrature domains. Namely, let $d$ be a positive integer,

$d > 1$, let $A$ be a linear transformation of $\mathbf{C}^d$, and assume for the beginning that $\xi \in \mathbf{C}^d$ is a cyclic vector for $A$. Let us denote:

$$R(z) = (A - z)^{-1}\xi, \qquad z \in \mathbf{C} \setminus \sigma(A),$$

the resolvent of $A$, localized at the vector $\xi$.

LEMMA 4.1. *The map $R : \hat{\mathbf{C}} \setminus \sigma(A) \longrightarrow \mathbf{C}^d$ is one to one and its range is a smooth complex curve.*

PROOF. Indeed, according to the resolvent equation we find:

$$R(z) - R(w) = (z - w)(A - z)^{-1}(A - w)^{-1}\xi, \qquad z, w \in \mathbf{C} \setminus \sigma(A).$$

Thus $R(z) - R(w) \neq 0$ for $z \neq w$. For the point at infinity we have $R(\infty) = 0 \neq R(z)$, for $z \in \mathbf{C} \setminus \sigma(A)$.

Moreover, the same resolvent equation shows that:

$$R'(z) = (A - z)^{-1}R(z) \neq 0,$$

and similarly for the point at infinity we obtain:

$$\frac{d}{dt}R(1/t) = -lim_{t\to 0}[t^{-2}(A - t^{-1})^{-2}\xi] = -\xi \neq 0.$$

Actually we can pass to projective spaces and complete the above curve. Let us denote by $(z_0 : z_1)$ the homogeneous coordinates in $\mathbf{P}_1$, and by $(u_0 : u_1 : \ldots : u_d)$ the homogeneous coordinates in $\mathbf{P}_d$. Let $z = z_1/z_0$ in the affine chart $z_0 \neq 0$ and $w = (u_1/u_0, \ldots, u_d/u_0)$ in the affine chart $u_0 \neq 0$.

Let $P(z) = det(A - z)$, so that $P(z)$ is a common denominator in the rational entries of the map $R(z)$. Let us define the function:

$$q(z, A)\xi = P(z)R(z) = (P(z) - P(A))R(z),$$

and remark that $q(z, A)$ is a polynomial in $z$ and $A$, of the form:

$$q(z, A) = -A^{d-1} + O(A^{d-2}).$$

Actually we need for later use a more precise form of the polynomial $q(z, A)$. We pause here to derive it by a series of elementary computations.

We have

$$\frac{P(w) - P(z)}{w - z} = \sum_{k=0}^{d} \alpha_k \frac{w^k - z^k}{w - z} =$$

$$\sum_{k=0}^{d} \alpha_k \sum_{j=0}^{k-1} z^{k-j-1} w^j = \sum_{j=0}^{d-1} ( \sum_{k=j+1}^{d} \alpha_k z^{k-j-1}) w^j =$$

$$T_0(z)w^{d-1} + T_1(z)w^{d-2} + \ldots + T_{d-1}(z),$$

where $\alpha_d = 1$ and

$$T_k(z) = \alpha_d z^k + \alpha_{d-1} z^{k-1} + \ldots + \alpha_{d-k+1} z + \alpha_{d-k}.$$

Note that $T_0(z) = 1$.

Therefore we obtain:

$$-q(z, A) = T_0(z)A^{d-1} + T_1(z)A^{d-2} + \ldots + T_{d-1}(z). \tag{14}$$

Since $\xi$ is a cyclic vector for $A$ and $dim \vee_{k=0}^{\infty} A^k \xi = d$, we infer that $q(z, A)\xi \neq 0$ for all $z \in \mathbf{C}$. In addition, for an eigenvalue $\lambda$ of $A$ (multiple or not), we have:

$$(A - \lambda) q(\lambda, A)\xi = P(\lambda)\xi = 0,$$

therefore $q(\lambda, A)\xi$ is a corresponding (non-trivial) eigenvector.

Let us define the completion of the map $R$ as follows:

$$R(z_0 : z_1) = \begin{cases} (P(z_1/z_0) : q(z_1/z_0, A)\xi), & z_0 \neq 0, \\ (1 : 0 : \ldots : 0), & z_0 = 0. \end{cases} \tag{15}$$

LEMMA 4.2. *The map $R : \mathbf{P}_1 \longrightarrow \mathbf{P}_d$ is a smooth embedding, that is, $R$ is one to one and its image is a smooth projective curve.*

PROOF. Indeed, for a point $\lambda \in \sigma(A)$ we obtain $R(1, \lambda) = (0 : q(\lambda, A)\xi)$, while for two distinct points $\lambda, \mu \in \sigma(A)$ we have $R(1 : \lambda) \neq R(1 : \mu)$ because the complex lines $\mathbf{C}\lambda$ and $\mathbf{C}\mu$ are different (they belong to different eigenspaces of the operator $A$).

It remains to prove that the differential of $R$ does not vanish at any point $\lambda \in \sigma(A)$. This fact in its turn follows from the observation that the vectors $(P(z), q(z, A)\xi)$ and $(P'(z), \frac{d}{dz} q(z, A)\xi)$ are not colinear for $z = \lambda$, where $\lambda \in \sigma(A)$. Indeed, if these vectors are colinear, then there exists a scalar $\alpha$ such that:

$$(0, q(\lambda, A)\xi) = \alpha(P'(\lambda), \frac{d}{dz} q(\lambda, A)\xi).$$

In particular this implies the identity:

$$q(\lambda, A)\xi = \alpha \frac{d}{dz} q(\lambda, A)\xi$$

which contradicts the fact that the vector $\xi$ is cyclic for $A$ and $dim \vee_{k=0}^{\infty} A^k \xi = d$.

Note that $R(\mathbf{P}_1)$ is a smooth unirational curve of degree $d$ in $\mathbf{P}_d$ and the rational map $R$ has degree $d$. According to a classical result in algebraic geometry, $R(\mathbf{P}_1)$ is projectively isomorphic to the *rational normal curve* of degree $d$ in $\mathbf{P}_d$ obtained as the range of the Veronese embedding

$$(z_0 : z_1) \longmapsto ({z_0}^d : {z_0}^{d-1} z_1 : \ldots : {z_1}^d).$$

See for details [GH] pg. 178. Later on, Proposition 4.6 will make more precise this projectivity.

Actually the cyclicity condition on $\xi$ can be dropped, because the resolvent $(A-z)^{-1}\xi$ has values in the cyclic subspace generated by $\xi$. Therefore, as a conclusion of these computations we can state the following result.

THEOREM 4.3. *Let $A$ be a linear transformation of $\mathbf{C}^d$ and let $\xi$ be a non-zero vector of $\mathbf{C}^d$. Then the map $R(z) = (A-z)^{-1}\xi$ extends to a rational embedding:*

$$R : \mathbf{P}_1 \longrightarrow \mathbf{P}_d.$$

*The range of $R$ is contained in a linear subspace $E$ of $\mathbf{P}_d$ of dimension equal to dim $\vee_{k=0}^{\infty} A^k \xi$ and the values $R(z)$ span $E$ as a linear space.*

Above, and throughout this note, by embedding we mean a (rational) map which separates the points and the directions at every point. In particular this implies that $R(\mathbf{P}_1)$ is a smooth rational curve.

In analogy with the previous sections we define the open set:

$$\Omega = \{z \in \mathbf{C}; \|(A-z)^{-1}\xi\| > 1\} \cup \sigma(A).$$

The singular points $a$ in the boundary of the bounded domain $\Omega$ are given by the equation $\langle R'(a), R(a)\rangle = 0$. But we know from Lemma 4.1 that $\|R'(a)\| \neq 0$, and on the other hand the Hessian $H(a)$ at $a$ of the defining equation $\|R(z)\|^2 = 1$ is:

$$H(a) = \begin{pmatrix} \langle R'(a), R'(a)\rangle & \langle R''(a), R(a)\rangle \\ \langle R(a), R''(a)\rangle & \langle R'(a), R'(a)\rangle \end{pmatrix}.$$

In particular $rank H(a) \geq 1$, which shows that $a$ is either an isolated point or a singular double point of $\partial\Omega$. In the case of a non-isolated singular point $a$ in the boundary of a quadrature domain $\Omega$ it is known that $a$ is a cusp or a double tangency point, and in this case $rank H(a) = 1$, see [G1].

Our next aim is to study the reflection in the boundary of the domain $\Omega$ defined above. More precisely, for a point $s \in \mathbf{P}_1(\mathbf{C})$ we consider the multivalued reflections in $\partial\Omega$ as the solutions $z = r_1(s), \ldots, r_d(s)$ of the equation:

$$\langle R(s), R(z)\rangle = 1. \tag{16}$$

Proposition 4.4. *The multivalued reflection* $s \mapsto (r_j(s))_{j=1}{}^d$ *satisfies:*

*a). All* $r_j(s) \in \Omega, 1 \leq j \leq d$, *for* $s \in \mathbf{P}_1(\mathbf{C}) \setminus \overline{\Omega}$;

*b). For an appropriate numbering of the* $r_j$*'s,* $r_1(s) = s$ *and* $r_j(s) \in \Omega, 2 \leq j \leq d$, *for* $s \in \partial\Omega$.

Proof. Indeed, $\|R(s)\| < 1$ whenever $s$ does not belong to $\overline{\Omega}$. Therefore $\|R(z)\| > 1$ for every solution $z$ of the equation (16). For $s \in \partial\Omega$ we obtain $\|R(s)\| = 1$, hence one solution of (16), say $r_1$, satisfies $r_1(s) = s$ and all other solutions $z$ satisfy necessarily $\|R(z)\| > 1$.

In particular Theorem 4.3 and Proposition 4.4 apply to quadrature domains $\Omega$. These domains are characterized by the property that there is an antianalytic meromorphic extension of the root $r_1(s)$ in Proposition 4.4.b), from $s \in \partial\Omega$ to $s \in \Omega$. Indeed, this extension is given by the Schwarz function by the fomula $r_1(s) = \overline{S(s)}, s \in \Omega$, and conversely, if the extension of the root $r_1$ exists, then this formula defines the Schwarz function of the domain. For the same domains, a simple rephrasing of Lemma 4.2 reads as follows.

Corollary 4.5. *A quadrature domain of order* $d$ *is rationally isomorphic to the intersection of a smooth rational curve of degree* $d$ *in* $\mathbf{P}_d$ *and the complement of a real affine ball.*

Next we consider the case of a simply connected quadrature domain $\Omega$ of order $d$. With the notation in Section 1, we have:

$$\Omega = \{z; \|(U^* - \overline{z})^{-1}\xi\| > 1\}$$

modulo a finite set. So by denoting $\Omega^* = \{z; \overline{z} \in \Omega\}$, the set $\Omega^*$ is of the form considered above ( with $A = U^*$), and it is still a simply connected quadrature domain. Let $\phi : \mathbf{D} \longrightarrow \Omega^*$ be a rational conformal map of the unit disk onto $\Omega^*$, see [AS] or [G1]; let $S(z)$ be the Schwarz function of the boundary of $\Omega^*$ and let $R(z)$ be the rational embedding of $\Omega^*$ established in Lemma 4.2.

In general the Schwarz function is subject to the identity:

$$\langle R(z), R(\overline{S(z)})\rangle = 1, \qquad z \in \overline{\Omega}. \tag{17}$$

Since $\overline{S(\phi(\zeta))} = \phi(\frac{1}{\overline{\zeta}})$ for $|\zeta| = 1$, the rational map $r(\zeta) = R(\phi(\zeta)), \zeta \in \mathbf{P}_1$, satisfies the duality relation:

$$\langle r(\zeta), r(\frac{1}{\overline{\zeta}})\rangle = 1, \tag{18}$$

identically on $\mathbf{P}_1$, in the sense of rational functions. Thus, in some extended sense, the rational map $r$ is commuting with the reflections in the boundaries of the unit ball of $\mathbf{C}$ and respectively $\mathbf{C}^d$. The next section contains in particular a partial converse to this remark.

The special nature of the defining equation of a quadrature domain is also reflected in the following general observation.

Proposition 4.6. *Let $A$ be a linear transformation of $\mathbf{C}^d$ with cyclic vector $\xi$ and let $P(z)$ be the minimal polynomial of $A$. Then there is a unique representation:*

$$|P(z)|^2(1-\|(A-z)^{-1}\xi\|^2) = |P(z)|^2 - \sum_{k=0}^{d-1}|Q_k(z)|^2, \tag{19}$$

*where $Q_k(z)$ are polynomials with positive leading coefficient and $deg(Q_k) = k, 0 \leq k \leq d-1$.*

Proof. To prove the uniqueness of the decomposition (19) we remark that there exists a simple algorithm of finding the polynomials $Q_k$. Indeed, let us denote:

$$F(z,\overline{z}) = |P(z)|^2(1-\|(A-z)^{-1}\xi\|^2),$$

and assume that a representation like (19) exists. Then the coefficient of $\overline{z}^d$ in $F(z,\overline{z})$ is $P(z)$. (We assume the minimal polynomial $P(z)$ to be monic). Hence the polynomial $F_{d-1}(z,\overline{z}) = -F(z,\overline{z}) + |P(z)|^2$ has degree $d-1$ in each variable. By assumption the coefficient $\gamma_1$ of $z^{d-1}\overline{z}^{d-1}$ in $F_{d-1}$ is positive, so that:

$$F_{d-1}(z,\overline{z}) = {\gamma_1}^{1/2}\overline{z}^{d-1}Q_{d-1}(z) + O(z^{d-1},\overline{z}^{d-2}).$$

Therefore the polynomial $Q_{d-1}(z)$ is determined by $F_{d-1}(z,\overline{z})$.

Proceeding by descending recurrence in $k$, ($k < d-1$) we are led to the polynomial

$$F_k(z,\overline{z}) = F_{k+1}(z,\overline{z}) - |Q_{k+1}(z)|^2$$

which has as leading term a positive constant $\gamma_k$ times $z^k\overline{z}^k$. Then necessarily

$$F_k(z,\overline{z}) = {\gamma_k}^{1/2}\overline{z}^kQ_k(z) + O(z^k,\overline{z}^{k-1}).$$

Thus $Q_k(z)$ is determined by $F_k(z,\overline{z})$. And so on until we end by setting $F_0(z,\overline{z}) = \gamma_0 = |Q_0(z,\overline{z})|^2 > 0$.

To prove the existence of the decomposition (19) we start from the identity (14):

$$-P(z)(A-z)^{-1}\xi = (P(A)-P(z))(A-z)^{-1}\xi =$$
$$T_{d-1}(z)\xi + T_{d-2}(z)A\xi + \ldots + T_0(z)A^{d-1}\xi.$$

Recall that $T_k(z)$ is a polynomial of degree equal to $k, 0 \leq k \leq d-1$.

Next we orthonormalize the system of vectors $\xi, A\xi, \ldots, A^{d-1}\xi$:

$$e_0 = \frac{\xi}{\|\xi\|},$$

$$e_1 = \frac{A\xi - \langle A\xi, e_0\rangle e_0}{\|\ldots\|},$$

$$e_2 = \frac{A^2\xi - \langle A^2\xi, e_1\rangle e_1 - \langle A^2\xi, e_0\rangle e_0}{\| \dots \|},$$

and so on.

Then

$$\xi = \|\xi\| e_0 = c_0 e_0 \qquad (c_0 > 0),$$
$$A\xi = c_1 e_1 + \langle A\xi, e_0\rangle e_0 \qquad (c_1 > 0),$$
$$A^2\xi = c_2 e_2 + \langle A^2\xi, e_1\rangle e_1 + \dots \qquad (c_2 > 0),$$

and so on.

These computations and relation (14) give:

$$-P(z)(A-z)^{-1}\xi = T_0(z)A^{d-1}\xi + \dots + T_{d-1}(z)\xi =$$
$$T_0(z)(c_{d-1}e_{d-1} + \langle A^{d-1}\xi, e_{d-2}\rangle e_{d-2} + \dots) +$$
$$T_1(z)(c_{d-2}e_{d-2} + \langle A^{d-2}\xi, e_{d-3}\rangle e_{d-3} + \dots) + \dots + T_{d-1}(z)c_0 e_0 =$$
$$c_{d-1}T_0(z)e_{d-1} + (c_{d-2}T_1(z) + \langle A^{d-1}\xi, e_{d-2}\rangle T_0(z))e_{d-2} + \dots +$$
$$(c_0 T_{d-1}(z) + \langle A\xi, e_0\rangle T_{d-2}(z) + \dots)e_0 =$$
$$Q_0(z)e_{d-1} + Q_1(z)e_{d-2} + \dots + Q_{d-1}(z)e_0,$$

where

$$Q_k(z) = c_{d-1-k}T_k(z) + O(z^{k-1}).$$

Hence $Q_k(z)$ is a polynomial of degree $k$ with leading coefficient $c_{d-1-k} > 0$. This finishes the proof of Proposition 4.6.

The case of a non-cyclic vector $\xi$ for the matrix $A$ can be treated similarly. If $P(z)$ stands for the characteristic polynomial of $A$, then there is an integer $d'$, $1 \le d' < d = deg(P)$, and there are unique polynomials $M_j(z)$, $d' \le j < d$ with the property that:

$$\|P(z)(A-z)^{-1}\xi\|^2 = \sum_{j=d'}^{d-1} |M_j(z)|^2.$$

We omit the details.

For a quadrature domain $\Omega$, the leading coefficient of $Q_{d-1}(z)$ is:

$$c_0 = \|\xi\| = \left(\frac{Area(\Omega)}{\pi}\right)^{1/2}.$$

The quadrature data of $\Omega$ (that is the nodes and the weights) is determined by the knowledge of the rational function (2). By comparing

$$Q(z,\overline{z}) = |P(z)|^2 - |Q_{d-1}(z)|^2 + O(z^{d-2}, \overline{z}^{d-2})$$

with the middle term in (2), and by considering the behaviour at infinity of these functions, we find that the rational function (2) coincides with

$$\frac{c_0 Q_{d-1}(z)}{P(z)}.$$

Therefore the quadrature data of $\Omega$ is in a natural bijection with the pair of polynomials $P(z), Q_{d-1}(z)$. The other polynomials $Q_{d-2}(z), \ldots, Q_0(z)$ determine the domain $\Omega$ (via its defining function) and they depend on $(d-1)^2$ real parameters, see also [G1] Theorem 10.

In the particular case $d = 2$ we have $deg(P(z)) = 2$ and:

$$|P(z)|^2(1 - \|(A-z)^{-1}\xi\|^2) = |P(z)|^2 - |az+b|^2 - c, \tag{20}$$

where $a, c > 0$ and $b \in \mathbf{C}$. Examples 6.1 and 6.2 below treat such cases.

As an application of Proposition 4.6 we discuss the structure of the exponential transform:

$$E_\Omega(z, \overline{w}) = exp[\frac{-1}{\pi} \int_\Omega \frac{dA(\zeta)}{(\zeta - z)(\overline{\zeta} - \overline{w})}], \quad |z|, |w| \gg 0,$$

of a quadrature domain $\Omega$ which possesses rotational symmetries. In the above notation, for large values of $|z|, |w|$ we have by (6) :

$$E_\Omega(z, \overline{z}) = \frac{|P(z)|^2 - \sum_{k=0}^{d-1} |Q_k(z)|^2}{|P(z)|^2}.$$

PROPOSITION 4.7 *Let $\Omega$ be a quadrature domain and let $\epsilon$ be a primitive root of unity, of order $n$. Assume that $\Omega = \epsilon\Omega$.*

*Then for all $z \in \mathbf{C}$, $|P(\epsilon z)| = |P(z)|$ and $|Q_k(\epsilon z)| = |Q_k(z)|$, $0 \le k \le d-1$.*

PROOF. From the very definition of the exponential transform, it follows by the change of variables $\zeta \mapsto \epsilon^{-1}\zeta$ that $E_\Omega(z, \overline{w}) = E_\Omega(\epsilon z, \epsilon^{-1}\overline{w})$. In virtue of the uniqueness of the decomposition (19) above, the $\epsilon$-rotational symmetry of the functions $|P|$ and $|Q_k|$ follows.

If a monic polynomial $Q$ satisfies $|Q(z)| = |Q(\epsilon z)|$, then $Q(z)$ is a product of $z^m$, with $m$ arbitrary, and factors like $(z^n - a^n)$, $a \neq 0$. This remark leads to the following result, noted originally by the first author in [G2].

COROLLARY 4.8 *Let $\Omega$ be a quadrature domain of order $d$ and let $\epsilon$ be a primitive root of unity, of order $d$, so that $\Omega = \epsilon\Omega$.*

*Then there exists a complex number $a \neq 0$, so that the nodes of $\Omega$ are $\epsilon^k a$, $0 \le k \le d-1$, and the defining equation of $\partial\Omega$ has the form:*

$$|z^d - a^d|^2 = \sum_{k=0}^{d-1} c_k |z^k|^2,$$

*where* $c_k > 0,\ 0 \le k \le d-1$.

PROOF. The case $d = 1$ leads to the defining equation $|z-a|^2 = c$, hence $\Omega$ is a disk.

Assume that $d > 1$. By degree reasons, it is clear that $|Q_k(z)| = c_k|z^k|^2$, for all $0 \le k \le d-1$. By Proposition 4.6, the constants $c_k$ must be positive.

For the leading polynomial $P(z)$, of degree $d$, there are two possibilities. Either $P(z) = z^d - a^d$, with $a \neq 0$, or $P(z) = z^d$. The latter case is excluded, because $\Omega$ would be defined by a polynomial in $|z|^2$, hence it would be a union of annuli, which is not a quadrature domain.

# 5 Universality of the rational embedding of a quadrature domain

The aim of this section is to prove that the rational embedding $R$ of a quadrature domain established by Corollary 4.5 is universal, in the sense that , in a natural degree range, any similar embedding is the composition of a projective transformation, a linear embedding and $R$.

In what follows we identify $\mathbf{C}^k$ with a fixed affine chart of coordinates in $\mathbf{P}_k, k > 0$. Thus, the unit ball centered at zero $\mathbf{B}_k \subset \mathbf{C}^k$ is unambiguously defined. An element $z \in \mathbf{P}_k$ belonging to the privileged affine chart is identified with $(1 : z) \in \mathbf{C}^{k+1}$. A linear transformation $\alpha \in SU(1,k)$ induces a biholomorphic projective transformation $\alpha : \mathbf{P}_k \longrightarrow \mathbf{P}_k$. In addition $\alpha$ is an automorphism of the privileged ball $\mathbf{B}_k$; the group of all these automorphisms acts transitively on $\mathbf{B}_k$, see for instance [H]. We call $\alpha$ a *Möbius transform* of $\mathbf{P}_k$ or respectively of the ball $\mathbf{B}_k$.

Fix a pair of positive integers $k, n$. We begin by analyzing a smooth rational embedding

$$u : \mathbf{P}_1 \longrightarrow \mathbf{P}_n$$

of degree $k$. That is $C = u(\mathbf{P}_1)$ is a smooth curve in $\mathbf{P}_n$ and any generic hyperplane $L \subset \mathbf{P}_n$ intersects $C$ in exactly $k$ points $u(z_1), \dots, u(z_k)$, where $z_1, \dots, z_k \in \mathbf{P}_1$.

Given any point $a \in C$ we can choose the above hyperplane to be:

$$L = L_a = \{z \in \mathbf{P}_n; \langle z, a\rangle = 1\}.$$

(The scalar product was written in the fixed affine chart $z = (1 : z)$). Thus we obtain the multivalued (1 to k) reflection map:

$$a \mapsto L_a \cap C, \qquad a \in C.$$

If $a \in C \cap \partial\mathbf{B}_n$, then $a \in L_a \cap C$. Set by definition:

$$C_+ = C \setminus \overline{\mathbf{B}_n}, \qquad C_- = C \cap \mathbf{B}_n.$$

In the domain of the rational embedding $u$ we distinguish the open set $\Omega = u^{-1}(C_+)$.

Throughout this section we assume that , in the privileged chart of coordinates, $\infty$ does not belong to the closure of the domain $\Omega$. This excludes from discussion the class of unbounded quadrature domains. A full classification and analysis of unbounded quadrature domains appears in [Sa].

LEMMA 5.1. *With the above notation, $\Omega$ is a ( possibly disconnected) quadrature domain if and only if there exists an anti-analytic map $J : C_+ \longrightarrow C$, continuous up to $\overline{C_+}$, such that:*
*a). $J(a) \in L_a \cap C \qquad (a \in C_+)$,*
*b). $J(a) = a, \qquad (a \in C \cap \partial \mathbf{B}_n = \partial C_+)$.*

PROOF. We simply remark that, given a map $J$ as in the statement,

$$S(z) = \overline{u^{-1}(J(u(z)))}, \qquad (z \in \Omega)$$

is a Schwarz function for the set $\Omega$, and conversely.

We will prove later that, whenever the conditions of Lemma 5.1 are fulfilled, the order of the quadrature domain $\Omega$ is greater or equal than the degree $k$ of the curve $C$.

To fix ideas for the forthcoming proof, let us review again the embedding of a bounded quadrature domain $\Omega \subset \mathbf{C}$ of order $d$. Let $(U, \xi)$ be the linear data of the complex conjugate domain $\Omega^*$, so that $R(z) = R_\Omega(z) = (U^* - z)^{-1}\xi$ extends to a rational embedding $R : \mathbf{P}_1 \longrightarrow \mathbf{P}_d$. In projective coordinates we have

$$R(1 : z) = (P(z) : P(z)(U^* - z)^{-1}\xi) = (P(z) : Q(z)),$$

say, where $Q = (Q_1, \ldots, Q_d)$ is a $d$-tuple of polynomials satisfying:

$$deg(P) = d > deg(Q) = max_{j=1}^{d} deg(Q_j).$$

Thus $R(\infty) = 0 = (1 : 0 : \ldots : 0)$. Also $R(\Omega) = R(\mathbf{P}_1) \setminus \overline{\mathbf{B}_d}$ and $R(z_k) = (0 : *) \in H$, where $H$ is the hyperplane at infinity in $\mathbf{P}_d$ and $z_1, \ldots, z_d$ are the quadrature nodes, that is the zeroes of $P(z)$.

THEOREM 5.2. *Let $u : \mathbf{P}_1 \longrightarrow \mathbf{P}_n$ be a rational embedding of degree $k$ such that $\Omega = u^{-1}(u(\mathbf{P}_1) \setminus \overline{\mathbf{B}_n})$ is a bounded quadrature domain of order $d \geq k$. Then $d = k$ and there exists a Möbius transform $\alpha$ of $\mathbf{P}_n$ and a linear embedding $i : \mathbf{P}_k \longrightarrow \mathbf{P}_n$ with the property that $u = \alpha \circ i \circ R_\Omega$.*

Note that the assumption that $\Omega$ is a bounded domain means simply $u(\infty) \in \mathbf{B}_n$, a requirement which can be satisfied by a Möbius transform of $\mathbf{P}_1$.

PROOF. Let us denote as above by $R(z) = (P(z) : Q(z))$ the rational embedding of the quadrature domain $\Omega$. We denote by $S(z)$ its Schwarz function.

Via the action of a Möbius transform $\alpha^{-1}$ on $\mathbf{P}_n$ we can assume that $u(\infty) = 0$. This means that, in the fixed affine charts we have:

$$u(z) = (p(z) : q(z)), \qquad deg(p) = k > deg(q),$$

where $p(z), q(z) = (q_1(z), \ldots, q_n(z))$ are polynomial maps. We choose the polynomial $p(z)$ to be monic.

Since $u(\partial\Omega) \subset \partial\mathbf{B}_n$, we obtain:

$$p(z)\overline{p(z)} = \langle q(z), q(z) \rangle, \qquad (z \in \partial\Omega),$$

therefore,

$$p(z)\overline{p(\overline{S(z)})} = \langle q(z), q(\overline{S(z)}) \rangle, \qquad (z \in \mathbf{P}_1). \tag{21}$$

Similarly we have:

$$P(z)\overline{P(\overline{S(z)})} = \langle Q(z), Q(\overline{S(z)}) \rangle, \qquad (z \in \mathbf{P}_1), \tag{22}$$

and we know that this is the minimal polynomial of the algebraic function $S(z)$. This means that we can find $g \in \mathbf{C}(z)[w]$ with the property:

$$p(z)\overline{p(\overline{w})} - \langle q(z), q(\overline{w}) \rangle = g(z, w)(P(z)\overline{P(\overline{w})} - \langle Q(z), Q(\overline{w}) \rangle).$$

Indeed, by dividing $p(z)\overline{p(\overline{w})} - \langle q(z), q(\overline{w}) \rangle$ by $P(z)\overline{P(\overline{w})} - \langle Q(z), Q(\overline{w}) \rangle$ in the ring $\mathbf{C}(z)[w]$, there will be no remainder since this would be of lower degree than the degree of the minimal polynomial of $S(z)$.

By changing the roles of $z$ and $w$ we find actually that $g(z, w)$ is a polynomial in both variables. Due to the degree assumption and the equality $2k = deg(g) + 2d$ we obtain $d = k$ and $deg(g) = 0$. Thus $g$ is a constant. Since the leading coefficients of both $p(z)$ and $P(z)$ are equal to one, we find that $g(z, w) = 1$. Moreover, $p(z) = P(z)$ as the coefficients of the higher power of $w$ in the previous identity.

In conclusion, we have proved the relation:

$$\langle u(z), u(w) \rangle = \langle R(z), R(w) \rangle, \qquad (z, w \in \mathbf{C}).$$

Since the range of the map $R(z)$ spans the vector space $\mathbf{C}^d$, $d$ being the order of the quadrature domain $\Omega$, there is a uniquely defined isometry $V : \mathbf{C}^d \longrightarrow \mathbf{C}^n$, such that $V(R(z)) = u(z), z \in \mathbf{C}$. At the level of projective spaces $V$ induces a linear embedding $i : \mathbf{P}_d \longrightarrow \mathbf{P}_n$ and $u = i \circ R$, as desired. This finishes the proof of Theorem 5.2.

COROLLARY 5.3. *Let $r : \mathbf{P}_1 \longrightarrow \mathbf{P}_n$ be a rational map such that $r(\mathbf{P}_1)$ is a smooth unirational curve $C$ and $r|\mathbf{B}_1$ is an isomorphism of $\mathbf{B}_1$ onto $C_+$.*

*Then $C_+$ is rationally isomorphic to a simply connected quadrature domain $\Omega \subset \mathbf{C}$. Assume that $deg(r) \leq d^2$, where $d$ denotes the order of $\Omega$. Then*

$$r = \alpha \circ i \circ R_\Omega \circ s,$$

*where $\alpha, i, R_\Omega$ are as in Theorem 5.2 and $s : \mathbf{P}_1 \longrightarrow \mathbf{P}_1$ is a rational map of degree $d$, which is one to one on $\mathbf{B}_1$.*

*In particular the degree of $r$ is $d^2$.*

PROOF. Let $u : \mathbf{P}_1 \longrightarrow \mathbf{P}_n$ be a rational embedding which parametrizes (by assumption) the curve $C$. Let $d'$ be the degree of $u$, or which is the same, the degree of $C$.

We can define the antianalytic map $J : C_+ \longrightarrow C$ by the formula:

$$J(r(\zeta)) = r(\frac{1}{\overline{\zeta}}), \qquad (\zeta \in \mathbf{B}_1).$$

Since necessarily $r(\partial \mathbf{B}_1) \subset \partial C_+ \subset \partial \mathbf{B}_n$ we obtain:

$$\langle r(\frac{1}{\overline{\zeta}}), r(\zeta) \rangle = 1, \qquad (\zeta \in \partial \mathbf{B}_1).$$

Hence, by analytic continuation, the same identity holds for $\zeta \in \mathbf{B}_1$. Therefore

$$J(r(\zeta)) \in L_{r(\zeta)} \cap C, \qquad (\zeta \in \mathbf{B}_1).$$

We also have $J(r(\zeta)) = r(\zeta)$ for all $\zeta \in \partial \mathbf{B}_1$ and by assumption $C_+$ is isomorphic to the unit disk, hence it is simply connected.

Let us denote, as in Theorem 5.2, $\Omega = u^{-1}(C_+)$. Thus $\Omega$ is a simply connected quadrature domain of order $d$, and necessarily $d \leq d'$ because the defining equation $\|u(z)\| > 1$ of $\Omega$ has degree $2d'$ and may not be irreducible. We remark next that the map $s = u^{-1} \circ r$ is well defined, hence analytic, hence rational. By construction $s$ is a conformal transformation of the unit disk $\mathbf{B}_1$ onto $\Omega$. Then necessarily the rational map $s$ has degree equal to the order of $\Omega$, see for instance [AS]. Thus, by the degree assumption we have:

$$d^2 \geq deg(r) = deg(u)deg(s) = d.d' \geq d^2.$$

Therefore $d = d'$ and Theorem 5.2 can be applied to the rational embedding $u$. This proves the conclusion of Corollary 5.3.

It would be interesting to relax the degree condition in Theorem 5.2 to the range $d < k$. However, additional assumptions are needed to insure that the conclusion of Theorem 5.2 holds. A simple counterexample is the embedding $u : \mathbf{P}_1 \longrightarrow \mathbf{P}_2$ given in the standard charts of coordinates by the formula:

$$u(z) = (\frac{1}{2^{1/2}z}, \frac{1}{2^{1/2}z^2}).$$

Indeed the degree of $u$ is two and $u^{-1}(\mathbf{P}_2 \setminus \mathbf{B}_2)$ is the unit disk, a quadrature domain of order one.

# 6 Examples

The complexity of computations of the basic objects attached to a quadrature domain increases very fast with the order. At least for order two quadrature domains such computations are possible, and they have appeared, from different perspectives in [AS], [D], [G1], [S]. Below we show how the matrix $U$ and the vector $\xi$ enter into the picture of order two quadrature domains.

6.1. The Limaçon. Let $z = w^2 + bw$, where $|w| < 1$ and $b \geq 2$. Then $z$ describes a quadrature domain $\Omega$ of order 2, whose boundary has the equation:

$$Q(z, \overline{z}) = |z|^4 - (2 + b^2)|z|^2 - b^2 z - b^2 \overline{z} + 1 - b^2 = 0,$$

see for instance [DL], Section 5.1.

The Schwarz function of $\Omega$ has a double pole at $z = 0$, whence the $2 \times 2$-matrix $U$ is nilpotent. Moreover, we know that:

$$|z|^4 \|(U^* - \overline{z})^{-1}\xi\|^2 = |z|^4 - P(z, \overline{z}).$$

Therefore

$$\|(U^* + \overline{z})\xi\|^2 = (2 + b^2)|z|^2 + b^2 z + b^2 \overline{z} + b^2 - 1,$$

or equivalently: $\|\xi\|^2 = 2 + b^2, \langle U^*\xi, \xi\rangle = b^2$ and $\|U^*\xi\|^2 = b^2 - 1$.

Consequently the linear data of the quadrature domain $\Omega$ are:

$$U^* = \begin{pmatrix} 0 & \frac{b^2-1}{(b^2-2)^{1/2}} \\ 0 & 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} \frac{b^2}{(b^2-1)^{1/2}} \\ (\frac{b^2-2}{b^2-1})^{1/2} \end{pmatrix}.$$

This shows in particular that the pair $(U, \xi)$ is subject to some other restrictions than $U^2 = 0$ and $\xi$ being a cyclic vector for $U^*$. For an abstract version of these restrictions, see [P1].

The rational embedding of the conjugated domain $\Omega^*$ can easily be computed from the definition (15):

$$R(1 : z) = (-z^2 : \frac{b^2}{(b^2-1)^{1/2}} z + \frac{b^2 - 1}{(b^2-1)^{1/2}} : (\frac{b^2-2}{b^2-1})^{1/2} z).$$

Notice that in this situation $\Omega^* = \Omega$, therefore the rational conformal map $\phi : \mathbf{D} \longrightarrow \Omega^*$ is $z = \phi(w) = w^2 + bw$. According to the previous computations, the rational map $r(w) = R(1 : w^2 + bw)$ satisfies the symmetry condition (18).

6.2. Two distinct nodes.

**a).** Suppose that $\Omega$ is a quadrature domain with the quadrature distribution:

$$u(f) = af(0) + bf(1),$$

where we choose the constants $a, b$ to be positive numbers. Then $P(z) = z(z-1)$ and

$$\overline{z}(\overline{z}-1)(U^* - \overline{z})^{-1}\xi = -U^*\xi + \xi - \overline{z}\xi.$$

Therefore the equation of the boundary of $\Omega$ is:

$$Q(z, \overline{z}) = |z(z-1)|^2 - \|U^*\xi - \xi + \overline{z}\xi\|^2.$$

According to the quadrature relations (4) we obtain:

$$\|\xi\|^2 = \frac{a+b}{\pi}, \qquad \langle U\xi, \xi\rangle = \frac{b}{\pi}.$$

Let us denote $\|U^*\xi\|^2 = c$. Then the defining polynomial becomes:

$$Q(z, \overline{z}) = |z(z-1)|^2 - \pi^{-1}(a|z-1|^2 + b(|z|^2 - 1)) - c.$$

The constant $c$ actually depends on $a, b$, via, for instance, the relation $Area(\Omega) = a + b$, or, whenever $a = b$, the fact that $Q(1/2, 1/2) = 0$, see [G1], Corollary 10.1.

We can choose an orthonormal basis with respect to which we have:

$$U^* = \begin{pmatrix} 0 & \alpha \\ 0 & 1 \end{pmatrix}, \qquad \xi = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}.$$

The matricial elements $\alpha, \beta, \gamma$ are then subject to the relations:

$$|\beta|^2 + |\gamma|^2 = \pi^{-1}(a+b), \qquad \overline{\alpha}\beta\overline{\gamma} + |\gamma|^2 = \pi^{-1}b, \qquad |\alpha|^2|\gamma|^2 + |\gamma|^2 = c.$$

An inspection of the arguments shows that the above system of equations has real solutions $\alpha, \beta, \gamma$ given by the formulas:

$$\alpha^2 = \frac{(\pi c - b)^2}{\pi(a+b)c - b^2},$$

$$\beta^2 = \frac{a^{-2}}{\pi(a-b) + \pi^2 c},$$

$$\gamma^2 = \frac{\pi(a+b)c - b^2}{\pi(a-b) + \pi^2 c}.$$

Let us remark that, if $a = b > \pi/4$, the constant $c$ is effectively computable, as mentioned earlier, and becomes:

$$c = \frac{1}{16} + \frac{a}{2\pi}.$$

This again illustrates the special nature of the pair $(U, \xi)$. A simple computation shows that the corresponding canonical embedding of the domain $\Omega = \Omega^*$ is:

$$R(1 : z) = (z(z-1) : \beta(1-z) - \alpha\gamma : \gamma z).$$

We remark that in both Examples 1 and 2, the matrix $U$ and the vector $\xi$ are uniquely determined, as soon as we require that $U$ is upper triangular.

**b).** In complete analogy, we can treat the case of two nodes with equal weights as follows.

Assume that the nodes are fixed at $\pm 1$. Hence $P(z) = z^2 - 1$. The defining equation of the quadrature domain $\Omega$ of order two with these nodes is:

$$Q(z, \overline{z}) = (|z+1|^2 - r^2)(|z-1|^2 - r^2) - c,$$

where $r$ is a positive constant and $c \geq 0$ is chosen so that either $\Omega$ is a union of two disjoint open disks (in which case $c = 0$), or $Q(0,0) = 0$. For details see [G2]. A short computation yields:

$$Q(z, \overline{z}) = z^2\overline{z}^2 - 2rz\overline{z} - z^2 - \overline{z}^2 + \alpha(r),$$

where

$$\alpha(r) = \begin{cases} (1-r^2)^2, & r < 1 \\ 0, & r \geq 1 \end{cases}$$

Equivalently, for the derivation of the formula of $Q$ we can invoke Proposition 4.7, which gives for $\partial\Omega$ the equation:

$$|z^2 - 1|^2 = c_2|z^2|^2 + c_1|z|^2 + c_0,$$

with positive constants $c_k,\ k = 0, 1, 2$. Then we proceed as above.

Exactly as in the preceding two situations, the identification

$$|P(z)|^2(1 - \|(U^* - \overline{z})^{-1}\xi\|^2) = Q(z, \overline{z}) \tag{23}$$

leads to the following simple linear data:

$$\xi = \begin{pmatrix} \sqrt{2}r \\ 0 \end{pmatrix}, \qquad U^* = \begin{pmatrix} 0 & \frac{\sqrt{2}r}{\sqrt{1-\alpha(r)}} \\ \frac{\sqrt{1-\alpha(r)}}{\sqrt{2}r} & 0 \end{pmatrix}.$$

We leave the reader the verification of formula (23).

6.3. Domains corresponding to a nilpotent matrix. To give a basic example for the class of domains discussed in Section 4, we consider the nilpotent matrix $A$ and the cyclic vector $\xi$ :

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

where $a, b, c$ are complex numbers, $c \neq 0$. A simple computation shows that:

$$\|(A-z)^{-1}\xi\|^2 = |\frac{a}{z} + \frac{b}{z^2} + \frac{c}{z^3}|^2 + |\frac{b}{z} + \frac{c}{z^2}|^2 + |\frac{c}{z}|^2.$$

Therefore the equation of the associated domain is:

$$|z|^6 < |az^2 + bz + c|^2 + |bz^2 + cz|^2 + |cz^2|^2.$$

According to Proposition 4.4, the reflection in the boundary of this domain maps the exterior completely into its interior.

The rational embedding associated to this example is:

$$R(1:z) = (-z^3 : az^2 + bz + c : bz^2 + cz : cz^2).$$

Similarly one can compute without difficulty the corresponding objects associated to a nilpotent Jordan block and an arbitrary cyclic vector of it. For instance the nilpotent $n \times n$-Jordan block and the vector $\xi = (0, 0, \ldots, 0, -1)$ give precisely the Veronese embedding:

$$R(1:z) = (z^n : 1 : z : \ldots : z^{n-2} : z^{n-1}).$$

See the remarks preceding Theorem 4.3.

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