Determining Radii of Meromorphy via Orthogonal Polynomials on the Unit Circle

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Abstract

Using a convergence theorem for Fourier-Padé approximants constructed from orthogonal polynomials on the unit circle, we prove an analogue of Hadamard's theorem for determining the radius of m-meromorphy of a function analytic on the unit disk and apply this to the location of poles of the reciprocal of Szegő functions.

1 Introduction

Let σ be a finite positive Borel measure whose support $\text{supp}(\sigma)$ is contained in $\Gamma = \{z : |z| = 1\}$ and $\varphi_n(z) = \kappa_n z^n + \cdots \in \mathcal{P}_n$, $\kappa_n > 0$, the orthonormal polynomial of degree n with respect to σ. It is said that σ satisfies Szegő's condition, and we write $\sigma \in \mathbf{S}$, if

$$\int_\Gamma \log \sigma'(\zeta)|d\zeta| > -\infty\,,$$

where σ' denotes the Radon-Nikodym derivative of σ with respect to the arc length on Γ. The associated (interior) Szegő function is given by

$$S_\sigma(z) = \exp\left\{\frac{1}{4\pi}\int_\Gamma \frac{\zeta+z}{\zeta-z}\log\sigma'(\zeta)|d\zeta|\right\}\,, \qquad |z| < 1\,.$$

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We denote by $\widehat{\mathbf{S}}$ the class of all finite positive Borel measures on Γ such that

$$\limsup_n |\varphi_n(0)|^{1/n} = 1/\rho(\sigma) < 1\,. \tag{1}$$

It is well-known (see (2.1), (2.5), Theorems 6.2 and 7.4 in [5]) that this class is made up of all measures satisfying Szegő's condition such that the largest disk with center at $z=0$ to which S_σ^{-1} can be extended analytically has radius $\rho(\sigma) > 1$. Moreover, σ is absolutely continuous with respect to the Lebesgue measure and $d\sigma(z) = |S_\sigma(z)|^2\, d\theta\,,\ z = e^{i\theta}$. For these and other characterizations of this class of measures see also Theorem 1 and its Corollary in [10].

Let $D = \{z : |z| < 1\}$ and $f \in H(\overline{D})$; that is, f is analytic in a neighborhood of the closed unit disk. We define the following determinants

$$\Delta_{n,m} = \begin{vmatrix} \langle z^{m-1}f, \varphi_n\rangle & \langle z^{m-2}f, \varphi_n\rangle & \cdots & \langle f, \varphi_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle z^{m-1}f, \varphi_{n+m-1}\rangle & \langle z^{m-2}f, \varphi_{n+m-1}\rangle & \cdots & \langle f, \varphi_{n+m-1}\rangle \end{vmatrix}, \tag{2}$$

where $\langle \cdot, \cdot\rangle$ denotes the usual inner product in the Hilbert space $L_2(\sigma)$. Set

$$l_m = l_m(f) = \limsup_n |\Delta_{n,m}|^{1/n}\ ,\quad l_0 = 1\,. \tag{3}$$

It is not difficult to verify that $l_m \le 1$ for all $m \in \mathbb{Z}_+$ (see Lemma 1 below). For $f \in H(\overline{D})$, by $D_m(f) = \{z : |z| < R_m(f)\}$ we denote the largest disk centered at $z = 0$ to which f can be extended to a meromorphic function with at most m poles. We write D_m or R_m when it is clear to which function the notation refers. Here and in the following, poles are counted according to their multiplicities.

Our main result is

Theorem 1 *Let $\sigma \in \widehat{\mathbf{S}}$ and $f \in H(\overline{D})$. Then for all $m \in \mathbb{Z}_+$, we have*

$$R_m = \frac{l_m}{l_{m+1}}\,, \tag{4}$$

where by convention $0/0 = \infty$.

Theorem 1 is an analogue of J. Hadamard's celebrated result for determining the radius of m-meromorphy of an analytic function in terms of its Taylor coefficients. For a proof of Hadamard's Theorem see [6] or [3].

Theorem 2 (J. Hadamard) *Let $g(z) = \sum_{n=0}^{\infty} g_n z^n$ be an analytic function on some neighborhood of $z = 0$. Then, for each $m \ge 0$ we have*

$$R_m(g) = \frac{\widehat{l}_m}{\widehat{l}_{m+1}}\,,$$

where $\widehat{l}_0 = 1$ and $\widehat{l}_m = \limsup_n |H_{n,m}|^{1/n}$,

$$H_{n,m} = \begin{vmatrix} g_{n-m+1} & g_{n-m+2} & \cdots & g_n \\ g_{n-m+2} & g_{n-m+3} & \cdots & g_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ g_n & g_{n+1} & \cdots & g_{n+m-1} \end{vmatrix}, \quad m \in \mathbb{N}\,, \quad n \ge m-1\,, \tag{5}$$

(here, as in (4), by convention $0/0 = \infty$).

The proof of Theorem 2 relies on the behavior of row sequences of Padé approximants. Let us consider the analogous construction for general Fourier expansions in terms of the orthogonal polynomials with respect to σ.

Let n, m be two fixed non-negative integers. Then, there exist polynomials $Q_{n,m}$ and $P_{n,m}$ such that

i) $\deg P_{n,m} \leq n$, $\deg Q_{n,m} \leq m$, $Q_{n,m} \not\equiv 0$,

ii) $(Q_{n,m} f - P_{n,m})(z) = A_{n,1}\varphi_{n+m+1} + A_{n,2}\varphi_{n+m+2} + \cdots$.

The quotient $R_{n,m} = P_{n,m}/Q_{n,m}$ of any solution of the system above is called an (n, m) Fourier-Padé approximant of f (relative to the measure σ). Given (n, m), more than one rational function may be defined (even after cancelling out common factors). If all solutions of the system above satisfy that $\deg Q_{n,m} = m$, then $R_{n,m}$ is uniquely determined (see [14]). Since by construction $Q_{n,m} \not\equiv 0$, we will normalize it with leading coefficient equal to 1.

Let σ be a measure on Γ such that

$$\lim_n \Phi_n(0) = 0 \,, \tag{6}$$

where $\Phi_n = \varphi_n/\kappa_n$ denotes the monic polynomial of degree n orthogonal with respect to σ. In this case, we write $\sigma \in \mathcal{N}_0$. It is well-known (see [11] and the references therein) that $\sigma' > 0$ a. e. on Γ is sufficient for (6) to take place. The proof of Theorem 1 is based on the following result.

Theorem 3 *Let $\sigma \in \mathcal{N}_0$ and $f \in H(\overline{D})$. Then the following statements are equivalent:*

a) f has exactly m poles $\{z_1, z_2, \ldots, z_m\}$ in D_m,

b) there exists a polynomial $Q_m(z) = \prod_{k=1}^m (z - z_k)$ such that

$$\limsup_n \|Q_{n,m} - Q_m\|^{1/n} = q < 1 \,. \tag{7}$$

where $\|\cdot\|$ denotes the norm in the space of polynomial coefficients.

If either a) or b) takes place, then

$$R_m = \max_{1\leq k\leq m} |z_k|/q \tag{8}$$

and

$$\limsup_n \|f - R_{n,m}\|_{\mathcal{K}}^{1/n} \leq \max_{z\in\mathcal{K}} |z|/R_m \,, \tag{9}$$

where $\mathcal{K}$ denotes an arbitrary compact subset of $D_m \setminus \{z_1, \ldots, z_m\}$ and $\|\cdot\|_{\mathcal{K}}$ denotes the sup-norm on $\mathcal{K}$.

This theorem is basically due to S. P. Suetin. For Fourier-Padé approximants with respect to measures supported on the real line, he proves in [13] that a) implies b), (9), and that $R_m \leq \max_{1\leq k\leq m} |z_k|/q$. He also states without proof the corresponding result for Fourier-Padé approximants relative to a measure supported on an arc of the complex plane whose orthogonal polynomials have Szegő type strong asymptotic behavior. In the theory of Padé approximants such results are called *of direct type* and follow the structure of R. de Montessus de Ballore's Theorem [9]. In [14], for measures supported on the real line, S. P. Suetin proves that b) implies a) and $R_m \geq \max_{1\leq k\leq m} |z_k|/q$. These are *inverse type* results. In [14], other types of Fourier expansions are not mentioned. The proof for measures on the unit circle (of the direct and inverse statements) is essentially the same as the one given by Suetin for the case of the real line, so we omit it. In Section 2, some auxiliary lemmas are given. Theorem 1 is proved in Section 3.

Because of the analytic properties of S_σ^{-1} when $\sigma \in \widehat{\mathbf{S}}$, we can apply Theorem 1 to $f = S_\sigma^{-1}$ in order to obtain the distribution of poles of this function along circles centered at the origin. We get

$$R_m(S_\sigma^{-1}) = l_m(S_\sigma^{-1})/l_{m+1}(S_\sigma^{-1}), \qquad m \in \mathbb{Z}_+ . \tag{10}$$

Since (see (2.1), (2.5), and (2.10) in [5])

$$S_\sigma^{-1}(z) = \frac{1}{\kappa}\sum_{n=0}^{\infty} \overline{\varphi_n(0)}\varphi_n(z)\,,$$

where

$$\kappa = \exp\left\{\frac{-1}{4\pi}\int_\Gamma \log\sigma'(\zeta)|d\zeta|\right\},$$

by analogy with Hadamard's Theorem one is tempted to replace $l_m(S_\sigma^{-1})$ by $\limsup_n |\widetilde{H}_{n,m}|^{1/n}$ in formula (10) with

$$\widetilde{H}_{n,m} = \begin{vmatrix} \varphi_{n-m+1}(0) & \varphi_{n-m+2}(0) & \cdots & \varphi_n(0) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_n(0) & \varphi_{n+1}(0) & \cdots & \varphi_{n+m-1}(0) \end{vmatrix},$$

as was conjectured by one of the authors in [8]. In this form the formula is not correct as the following example shows.

Let σ be the measure whose reflection coefficients are given by $\Phi_n(0) = a^n$ where $0 < |a| < 1$. In page 180 of [2] it was shown that the corresponding Szegő function has a simple pole at $1/\overline{a}$ (thus $R_0 = 1/|a|$) and $\lim_n \|Q_{n,1}(z) - (z - 1/\overline{a})\| = |a|^2$. According to Theorem 3, we have that $R_1(S_\sigma^{-1}) = 1/|a|^3$. On the other hand, it is obvious that $\widetilde{H}_{n,2} = 0$ and we would get the wrong formula for $R_1(S_\sigma^{-1})$.

Nevertheless, using (10) it is possible to prove the following formula

Corollary 1 *Let $\sigma \in \widehat{\mathbf{S}}$. Set*

$$\widetilde{\Delta}_{n,m} = \begin{vmatrix} \varphi_n^{(m-1)}(0) & \varphi_n^{(m-2)}(0) & \cdots & \varphi_n(0) \\ \varphi_{n+1}^{(m-1)}(0) & \varphi_{n+1}^{(m-2)}(0) & \cdots & \varphi_{n+1}(0) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{n+m-1}^{(m-1)}(0) & \varphi_{n+m-1}^{(m-2)}(0) & \cdots & \varphi_{n+m-1}(0) \end{vmatrix}$$

and

$$\widetilde{l}_m = \limsup_n \left|\widetilde{\Delta}_{n,m}\right|^{1/n}, \quad \widetilde{l}_0 = 1\,.$$

Then, $\widetilde{l}_m = l_m$ *and*

$$R_m = \frac{\widetilde{l}_m}{\widetilde{l}_{m+1}}\,, \quad m \in \mathbb{Z}_+\,.$$

Corollary 2 *Let* $\Phi_n(0) = a^n(c+\epsilon_n), n \in \mathbb{N}$, *be a sequence of reflection coefficients, where* $c \neq 0$ *and* $\limsup_n \left(\max_{j\geq n} |\epsilon_j|\right)^{1/n} = \delta < |a|^{m^2}, m \geq 1$. *Let* σ *denote the associated measure on the unit circle. Then,*

$$R_{m-1}(S_\sigma^{-1}) = \frac{1}{|a|^{2m-1}}$$

and S_σ^{-1} *has exactly* $m-1$ *simple poles in* D_{m-1} *located at the points* $1/\overline{a}|a|^{2(k-1)}, k = 1, 2, \ldots, m-1$.

A direct consequence of Corollary 2 is

Corollary 3 *Under the same assumptions as in the previous corollary, if* $\delta = 0$, *then* S_σ^{-1} *is a meromorphic function in the complex plane with simple poles located at the points* $1/\overline{a}|a|^{2(k-1)}, k \in \mathbb{N}$.

When $\Phi_n(0) = a^n, n \in \mathbb{Z}_+$, an explicit expressions in the form of an infinite product for the Szegő function was obtained by G. Szegő (see [15]). In particular, it was known that the Szegő function is analytic in the whole complex plane with simple zeros precisely at each of the points indicated in Corollary 3.

Some special cases to which Corollaries 2 and 3 respectively may be applied are:

1. $\Phi_n(0) = ca^n + p(b^n)$, where $c \neq 0$, p is a polynomial, and $|b| < |a|^{m^2}$ for a given $m \in \mathbb{Z}_+$.

2. $\Phi_n(0) = ca^n + p((1/n)^n)$, where $c \neq 0$ and p is a polynomial,

(It is assumed in these examples that c and p are such that the necessary condition for the existence of an orthogonality measure $|\Phi_n(0)| < 1, n \in \mathbb{N}$, is satisfied.)

The proof of the corollaries is carried out in Section 4.

2 Lemmas and auxiliary results

Let $\varphi_n(z) = \kappa_n \Phi_n(z), \kappa_n > 0$, be the nth orthonormal polynomial with respect to $\sigma \in \mathcal{N}_0$ and

$$g_n(z) = \int \frac{\overline{\varphi_n(\zeta)}}{z-\zeta} d\sigma(\zeta)$$

the associated function of second type. Using that

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z)\,, \tag{11}$$

where $\Phi_n^*(z) = z^n\overline{\Phi_n(1/\overline{z})}$ is the so-called nth *reversed polynomial*, and

$$1 - (\kappa_n/\kappa_{n+1})^2 = |\Phi_{n+1}(0)|^2$$

(see e.g. formulas (1.2) and (1.5) in [5]), it is easy to verify that (6) is equivalent to each one of the following relations

$$\lim_n \frac{\kappa_{n+1}}{\kappa_n} = 1\,, \tag{12}$$

and

$$\lim_n \frac{\varphi_{n+1}(z)}{\varphi_n(z)} = z\,. \tag{13}$$

uniformly in $\{z : |z| \geq 1\}$. On the other hand (see Theorems 4 and 7.4 in [7]), condition (6) implies that $|\varphi_n|^2 d\sigma$ converges in the weak-star topology of measures to the unit Lebesgue measure on Γ. Since from orthogonality

$$(g_n \varphi_n)(z) = \int \frac{|\varphi_n(\zeta)|^2}{z-\zeta} d\sigma(\zeta)\,,$$

using (13) and the weak-star convergence, it follows that

$$\lim_n \frac{g_{n+1}}{g_n} = \frac{1}{z}\,, \tag{14}$$

uniformly on compact subsets of $\{z : |z| > 1\}$.

Properties (12), (13), and (14) are all that it is needed from the measure in order to prove Theorem 3 following the scheme employed by S. P. Suetin in [13] and [14] in proving the analogous result for measures supported on the real line.

For most parts of the proof of Theorem 1, it is only required that

$$\lim_n \kappa_n^{1/n} = 1 \tag{15}$$

and

$$\lim_n |\varphi_n(z)|^{1/n} = |z|\ ,\ \ |z| \geq 1\ , \tag{16}$$

where convergence is uniform on compact subsets of $\{z : |z| \geq 1\}$. These properties immediately follow from (6) on account of (12) and (13) respectively. In proving Theorem 1 we are forced to restrict the class of measures to $\sigma \in \widehat{\mathbf{S}}$ because at one point we need (1) to derive the estimate (25) below. Either (15) or (16) characterizes the set of all measures with $\mathrm{supp}(\sigma) = \Gamma$ belonging to the class **Reg** of regular measures (for details about this class of measures see Theorem 3.1.1 in [12]).

The following lemma is obtained using (16) in the same way as similar statements are proved for Taylor series. The only delicate point is to ensure that the sum of the series is f, but this is guaranteed because

$$\|f - S_n\|_2^2 \leq \|f - T_n\|_2^2 \leq \sigma(\Gamma)\|f - T_n\|_\infty^2\,,$$

where S_n and T_n denote the nth Fourier and Taylor partial sums whereas $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the $L_2(\sigma)$ and the uniform norms, respectively. Since $f \in H(\overline{D})$, it follows that

$$\lim_n \|f - S_n\|_2 = 0\,,$$

and by the principle of analytic continuation f and the sum of the Fourier series coincide. For details see Theorems 6.2 and 7.4 in [5]. This lemma will serve as the induction basis for the proof of Theorem 1.

Lemma 1 *Let supp*$(\sigma) = \Gamma$, $\sigma \in \mathbf{Reg}$, *and* $f \in H(\overline{D})$. *Then,*

$$f(z) = \sum_{n \geq 0} \langle f, \varphi_n \rangle \varphi_n(z) \,, \tag{17}$$

uniformly on each compact subset of D_0, *where*

$$\langle f, \varphi_n \rangle = \int f(z) \overline{\varphi_n(z)} d\sigma(z) \,.$$

For each z *such that* $|z| > R_0$ *the series in* (17) *diverges. Moreover,*

$$l_1 = \limsup_n |\langle f, \varphi_n \rangle|^{1/n} = \limsup_n |\langle f, \Phi_n \rangle|^{1/n} = \frac{1}{R_0} < 1 \,.$$

The following Lemma will also be quite useful in proving Theorem 1.

Lemma 2 *Let supp*$(\sigma) = \Gamma$, $\sigma \in \mathbf{Reg}$, *and* $f \in H(\overline{D})$. *Then,*

$$l_m \leq (R_0 \cdots R_{m-1})^{-1} < 1 \,, \qquad m \in \mathbb{N} = \{1, 2, \ldots\} \,.$$

Proof.- For each $i \in \mathbb{Z}_+$, let i_0 denote the number of poles which f has in $D_i(f)$ (counting their order). Take q_i as the monic polynomial of degree i which has a zero at each pole of f in $D_i(f)$ and $i - i_0$ zeros at $z = 0$. Therefore, $\deg q_i = i$ and

$$R_0(q_i f) = R_i(f) \,.$$

Because of Lemma 1, it follows that

$$\limsup_n |\langle q_i f, \varphi_n \rangle|^{1/n} = 1/R_0(q_i f) = 1/R_i(f) \,. \tag{18}$$

Fix $m \in \mathbb{N}$. Consider the determinant $\Delta_{n,m}$ defined in (2). Adding to the first column an appropriate linear combination of the rest of the columns, from the properties of the determinants, we obtain

$$\Delta_{n,m} = \begin{vmatrix} \langle q_{m-1} f, \varphi_n \rangle & \langle z^{m-2} f, \varphi_n \rangle & \cdots & \langle f, \varphi_n \rangle \\ \langle q_{m-1} f, \varphi_{n+1} \rangle & \langle z^{m-2} f, \varphi_{n+1} \rangle & \cdots & \langle f, \varphi_{n+1} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle q_{m-1} f, \varphi_{n+m-1} \rangle & \langle z^{m-2} f, \varphi_{n+m-1} \rangle & \cdots & \langle f, \varphi_{n+m-1} \rangle \end{vmatrix} .$$

Proceeding analogously form the second column on, it follows that

$$\Delta_{n,m} = \begin{vmatrix} \langle q_{m-1} f, \varphi_n \rangle & \langle q_{m-2} f, \varphi_n \rangle & \cdots & \langle f, \varphi_n \rangle \\ \langle q_{m-1} f, \varphi_{n+1} \rangle & \langle q_{m-2} f, \varphi_{n+1} \rangle & \cdots & \langle f, \varphi_{n+1} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle q_{m-1} f, \varphi_{n+m-1} \rangle & \langle q_{m-2} f, \varphi_{n+m-1} \rangle & \cdots & \langle f, \varphi_{n+m-1} \rangle \end{vmatrix} .$$

Expanding this determinant, we obtain a sum of $m!$ terms each one of which has exactly one factor representing each column. According to (18), it follows that the nth root of each one of these terms has lim sup not greater than $(R_0(f) \cdots R_{m-1}(f))^{-1}$. Since the number of terms in the expansion of the determinants remains fixed with n, the statement of the lemma follows. $\square$

3 Proof of Theorem 1

The proof of this theorem is carried out by induction on $m \in \mathbb{Z}_+$. By definition $l_0 = 1$, therefore, for $m = 0$ formula (4) follows from Lemma 1. Fix $m \geq 1$ and suppose that (4) holds for all indices up to $m-1$. Let us prove that it is also true for m.

From Lemma 2, we have that $l_i \leq 1$ for all $i \in \mathbb{Z}_+$. If $R_m = \infty$, according to Lemma 2 we have that $l_{m+1} = 0$. Hence, $R_m = l_m/l_{m+1}$ as needed (recall that by convention $0/0 = \infty$). Therefore, we can assume that $R_m < \infty$. Consequently, $R_i < \infty$ for $i = 0, 1, \ldots, m$. By the hypothesis of induction, we have that $R_i = l_i/l_{i+1}, i = 0, 1, \ldots, m-1$; therefore, $l_i > 0, i = 0, \ldots, m$. Multiplying these equalities, we obtain that $R_0 \cdots R_{m-1} = 1/l_m < \infty$. Using again Lemma 2 (for the index $m+1$), we get

$$l_m/l_{m+1} \geq R_0 \cdots R_m/R_0 \cdots R_{m-1} = R_m \,.$$

Now, it rests to show that $R_m \geq l_m/l_{m+1}$.

Notice that

$$l_{m-1}/l_m = R_{m-1} \leq R_m \leq l_m/l_{m+1} \,.$$

If $l_{m-1}/l_m = l_m/l_{m+1}$, we would have equality throughout and, in particular, $R_m = l_m/l_{m+1}$ as needed. Hence, it is sufficient to consider the case when $R_m < \infty, R_i = l_i/l_{i+1}, i = 0, \ldots, m-1$, and $l_{m-1}/l_m < l_m/l_{m+1}$, or what is the same

$$l_{m-1}l_{m+1}/l_m^2 < 1 \,. \tag{19}$$

Our next goal is to prove that under these conditions there exists a polynomial Q_m of exact degree m such that

$$\limsup_n \|Q_{n,m} - Q_m\|^{1/n} \leq l_{m-1}l_{m+1}/l_m^2 < 1 \,. \tag{20}$$

Suppose this has been proved. Then, according to (7) and (8) in Theorem 3, we have that

$$\max_{1 \leq k \leq m} |z_k|/R_m \leq l_{m-1}l_{m+1}/l_m^2$$

and f has exactly m poles in D_m at the zeros $z_1, \ldots, z_m$ of the polynomial Q_m. This implies that $R_{m-1} = \max_{1 \leq k \leq m} |z_k|$. Consequently,

$$R_{m-1}/R_m \leq l_{m-1}l_{m+1}/l_m^2 = R_{m-1}l_{m+1}/l_m \,.$$

Cancelling out R_{m-1} on both sides of this inequality, we get

$$R_m \geq l_m/l_{m+1} \,,$$

and we are done. Therefore, to conclude the proof of Theorem 1, we must show that under the induction hypothesis (20) holds if (19) takes place and $R_m < \infty$.

First, let us prove that $Q_{n,m}$ is of degree m for all sufficiently large n. Set $Q_{n,m}(z) = c_{n,0}z^m + c_{n,1}z^{m-1} + \cdots + c_{n,m}$. By definition, $\langle Q_{n,m}f, \varphi_{n+k}\rangle = 0$ for $k = 1, \ldots, m$. This is equivalent to

$$-c_{n,0}\langle z^m f, \varphi_{n+k}\rangle = c_{n,1}\langle z^{m-1}f, \varphi_{n+k}\rangle + \cdots + c_{n,m}\langle f, \varphi_{n+k}\rangle \,, \qquad k = 1, \ldots, m \,. \tag{21}$$

The determinant of this system is $\Delta_{n+1,m}$. If we can show that $\Delta_{n+1,m} \neq 0$ for all sufficiently large n, then we can guarantee the existence of a unique solution on the coefficients $c_{n,1}, \ldots, c_{n,m}$ of the non-homogeneous system which is obtained above taking $c_{n,0} = 1$.

In the sequel, we write $\Delta_{n,m}(g)$ and $l_m(g)$ to specify that the notation is relative to some function g. In particular,

$$\Delta_{n,m}(zf) = \begin{vmatrix} \langle z^m f, \varphi_n \rangle & \cdots & \langle zf, \varphi_n \rangle \\ \vdots & \ddots & \vdots \\ \langle z^m f, \varphi_{n+m-1} \rangle & \cdots & \langle zf, \varphi_{n+m-1} \rangle \end{vmatrix} .$$

By Sylvester's determinant identity (see (30) page 33 in [4]), we have

$$\Delta_{n+1,m+1}(f)\Delta_{n+2,m-1}(zf) = \Delta_{n+1,m}(zf)\Delta_{n+2,m}(f) - \Delta_{n+2,m}(zf)\Delta_{n+1,m}(f) . \qquad (22)$$

From Lemma 2 applied to zf and the induction hypothesis, we have that

$$\limsup_n |\Delta_{n+1,m+1}(f)\Delta_{n+2,m-1}(zf)|^{1/n} \leq l_{m+1}(f) l_{m-1}(zf) \leq \frac{l_{m+1}(f)}{R_0(zf)\cdots R_{m-2}(zf)} =$$

$$\frac{l_{m+1}(f)}{R_0(f)\cdots R_{m-2}(f)} = l_{m+1}(f) l_{m-1}(f) < l_m^2(f) . \qquad (23)$$

Now, let us show that

$$\Delta_{n+1,m}(zf)\Delta_{n+2,m}(f) - \Delta_{n+2,m}(zf)\Delta_{n+1,m}(f) = \Delta_{n,m}(f)\Delta_{n+2,m}(f) - \Delta^2_{n+1,m}(f) + \varepsilon_n , \qquad (24)$$

where $\limsup_n |\varepsilon_n|^{1/n} < l_m^2(f)$. Having proved this, using (22), (23), and (24), we obtain

$$\limsup_n \left|\Delta_{n,m}(f)\Delta_{n+2,m}(f) - \Delta^2_{n+1,m}(f)\right|^{1/n} < l_m^2(f) . \qquad (25)$$

Let us compare the determinants $\Delta_{n+1,m}(zf)$ and $\Delta_{n,m}(f)$. Let q_i be the polynomials introduced in the proof of Lemma 2. Notice that $R_0(zq_i f) = R_0(q_i f) = R_i(f)$. Proceeding as in the proof of Lemma 2, using (11) and the distributive law for determinants, we have

$$\frac{\Delta_{n+1,m}(zf)}{\kappa_{n+1}\cdots\kappa_{n+m}} =$$

$$\begin{vmatrix} \langle zq_{m-1}f, z\Phi_n + \Phi_{n+1}(0)\Phi_n^* \rangle & \cdots & \langle zf, z\Phi_n + \Phi_{n+1}(0)\Phi_n^* \rangle \\ \vdots & \ddots & \vdots \\ \langle zq_{m-1}f, z\Phi_{n+m-1} + \Phi_{n+m}(0)\Phi_{n+m-1}^* \rangle & \cdots & \langle zf, z\Phi_{n+m-1} + \Phi_{n+m}(0)\Phi_{n+m-1}^* \rangle \end{vmatrix} =$$

$$\begin{vmatrix} \langle q_{m-1}f, \Phi_n \rangle & \cdots & \langle f, \Phi_n \rangle \\ \vdots & \ddots & \vdots \\ \langle q_{m-1}f, \Phi_{n+m-1} \rangle & \cdots & \langle f, \Phi_{n+m-1} \rangle \end{vmatrix} + \delta_n = \frac{\Delta_{n,m}(f)}{\kappa_n\cdots\kappa_{n+m-1}} + \delta_n ,$$

where δ_n denotes the sum of the remaining $2^m - 1$ determinants. Each one of them has at least one column of the form

$$\begin{pmatrix} \overline{\Phi_{n+1}(0)}\langle zq_k f, \Phi_n^* \rangle \\ \vdots \\ \overline{\Phi_{n+m}(0)}\langle zq_k f, \Phi_{n+m-1}^* \rangle \end{pmatrix} , \qquad k = 0, \ldots, m-1 ,$$

$(q_0 \equiv 1)$. Those columns not of this form are as

$$\begin{pmatrix} \langle q_k f, \Phi_n \rangle \\ \vdots \\ \langle q_k f, \Phi_{n+m-1} \rangle \end{pmatrix}, \qquad k = 0, \ldots, m-1 \,.$$

For $i = 1, \ldots, m$, and $k = 0, \ldots, m-1$, we have that

$$\left| \overline{\Phi_{n+i}(0)} \langle z q_k f, \Phi^*_{n+i-1} \rangle \right| = \left| \overline{\Phi_{n+i}(0)} \langle z (q_k f - S_{k,n+i-2}), \Phi^*_{n+i-1} \rangle \right|$$

where $S_{k,n+i-2}$ denotes the $n+i-2$ Fourier sum of $q_k f$. Notice that

$$\langle z S_{n+i-2}, \Phi^*_{n+i-1} \rangle = 0$$

because zS_{n+i-2} is a polynomial of degree at most $n+i-1$ with a zero of multiplicity ≥ 1 at $z = 0$ and Φ^*_{n+i-1} is orthogonal to all such polynomials. Therefore, using Lemma 1, (1), and the Holder inequality, it follows that

$$\limsup_n \left| \overline{\Phi_{n+i}(0)} \langle z q_k f, \Phi^*_{n+i-1} \rangle \right|^{1/n} < \limsup_n \| q_k f - S_{n+i-2} \|_2^{1/n} \leq \frac{1}{R_k} \,.$$

Hence, the expansion of any one of the $2^m - 1$ determinants included in the sum δ_n is made of $m!$ terms each one of which has one factor with nth root of order smaller than $1/R_k, 0 \leq k \leq m-1$, and for each $i \neq k, 0 \leq i \leq m-1$, a factor with nth root of order at most $1/R_i$. Therefore,

$$\limsup_n \delta_n^{1/n} < 1/R_0 \cdots R_{m-1} = l_m \,.$$

Taking account of (15), we have proved that

$$\limsup_n |\Delta_{n+1,m}(zf) - \Delta_{n,m}(f)|^{1/n} < l_m \,, \tag{26}$$

from which (24) and (25) follow.

According to a result of Hadamard in [6], any sequence of complex numbers $\{d_n\}$ such that

$$\limsup_n |d_n|^{1/n} = 1 \,, \qquad \limsup_n |d_{n+1} d_{n-1} - d_n^2|^{1/n} < 1$$

has the regular limit

$$\lim_n |d_n|^{1/n} = 1 \,.$$

For a proof due to Ostrowski see Lemma 2, page 330 in [3]. In fact the stronger statement

$$\lim_n \frac{d_{n+1}}{d_n} = \delta \,, \qquad |\delta| = 1 \,,$$

is deduced.

By definition we have that $\limsup_n |\Delta_{n+1,m}(f)|^{1/n} = l_m$ and by assumption $l_m \neq 0$. On the other hand, we have (25). According to what was said above applied to the sequence $\{d_n = \Delta_{n,m}/l_m\}$, these conditions imply the regular limit

$$\lim_n |\Delta_{n,m}(f)|^{1/n} = l_m \,. \tag{27}$$

Therefore, we have that $\Delta_{n,m}(f) \neq 0$ for all sufficiently large n, and $Q_{n,m}$ can be taken as a monic polynomial of exact degree m for all large n as we set out to prove.

In the sequel, all the determinants $\Delta_{n,m}$ refer to the function f and we drop the explicit reference to it. Using the system of equations (21) with $a_{n,0} = 1$, by Cramer's rule $c_{n,i} = \Delta^i_{n+1,m}/\Delta_{n+1,m}, i = 1, \ldots, m$, where $\Delta^i_{n+1,m}$ is the determinant obtained substituting the i-th column of $\Delta_{n+1,m}$ by the column vector

$$-\left(\begin{array}{c} \langle z^m f, \varphi_{n+1} \rangle \\ \vdots \\ \langle z^m f, \varphi_{n+m} \rangle \end{array} \right).$$

Therefore,

$$c_{n+1,i} - c_{n,i} \;=\; \frac{\Delta^i_{n+2,m}}{\Delta_{n+2,m}} - \frac{\Delta^i_{n+1,m}}{\Delta_{n+1,m}} = \frac{\Delta^i_{n+2,m}\Delta_{n+1,m} - \Delta_{n+2,m}\Delta^i_{n+1,m}}{\Delta_{n+2,m}\Delta_{n+1,m}}.$$

Let $H_{n+1,m+1}$ be the matrix defining the determinant $\Delta_{n+1,m+1}$. Fix $i \in \{1, \ldots, m\}$. Let $\Delta^{(i)}_{n+1}$ be the determinant of the matrix of order $m-1$ obtained from $H_{n+1,m+1}$ eliminating its first and last rows and its first and $i+1$ columns. Applying Sylvester's Theorem to $H_{n+1,m+1}$, it is easy to check that

$$\Delta_{n+1,m+1}\Delta^{(i)}_{n+1} = (-1)^{m+1}\left(\Delta^i_{n+2,m}\Delta_{n+1,m} - \Delta_{n+2,m}\Delta^i_{n+1,m}\right).$$

Consequently,

$$c_{n+1,i} - c_{n,i} = (-1)^{m+1}\frac{\Delta_{n+1,m+1}\Delta^{(i)}_{n+1}}{\Delta_{n+2,m}\Delta_{n+1,m}}. \tag{28}$$

Reasoning as before with the polynomials q_i, it is not difficult to show that

$$\limsup_n \left|\Delta^{(i)}_{n+1}\right|^{1/n} \leq 1/R_0 \cdots R_{m-2} = l_{m-1}. \tag{29}$$

On account of (27), (28), and (29), we find that

$$\begin{aligned}\limsup_n |c_{n+1,i} - c_{n,i}|^{1/n} &\leq \frac{\limsup_n |\Delta_{n+1,m+1}|^{1/n} \limsup_n \left|\Delta^{(i)}_{n+1}\right|^{1/n}}{\lim_n |\Delta_{n+2,m}|^{1/n} \lim_n |\Delta_{n+1,m}|^{1/n}} \\ &\leq l_{m+1}l_{m-1}/l_m^2 < 1.\end{aligned}$$

Therefore, $\sum_n |c_{n+1,i} - c_{n,i}|$ is convergent. Let $\lim_n c_{n,i} = c_i$, $i = 1, \ldots, m$, and

$$Q(z) = z^m + c_1 z^{m-1} + \cdots + c_m.$$

Then

$$\limsup_n |c_{n,i} - c_i|^{1/n} \leq l_{m+1}l_{m-1}/l_m^2 < 1$$

and, consequently,

$$\limsup_n \|Q_{n,m}(z) - Q(z)\|^{1/n} \leq l_{m+1}l_{m-1}/l_m^2 < 1.$$

With this we conclude the proof of Theorem 1. □

4 Proof of the Corollaries

Proof of Corollary 1:

Since $\sigma \in \widehat{\mathbf{S}}$ we have that S_σ^{-1} is analytic in $\overline{D}$ and

$$\langle z^k S_\sigma^{-1}, \varphi_n\rangle = \frac{1}{2\pi}\int_\Gamma z^k S_\sigma^{-1}(z)\overline{\varphi_n(z)}\,|S_\sigma(z)|^2\,d\theta = \frac{1}{2\pi}\int_\Gamma z^k \overline{\varphi_n(z)S_\sigma(z)}d\theta\,, \quad z = e^{i\theta}\,.$$

Thus,

$$\overline{\langle z^k S_\sigma^{-1}, \varphi_n\rangle} = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi_n(z)S_\sigma(z)}{z^{k+1}}dz = \frac{(\varphi_n S_\sigma)^{(k)}}{k!}(0) = \frac{1}{k!}\sum_{s=0}^{k}\binom{k}{s}\varphi_n^{(s)}(0)S_\sigma^{(k-s)}(0)$$

and

$$\langle z^k S_\sigma^{-1}, \varphi_n\rangle = \frac{1}{k!}\sum_{s=0}^{k}\binom{k}{s}\overline{\varphi_n^{(s)}(0)S_\sigma^{(k-s)}(0)}\,.$$

Notice that the coefficients on the right hand side do not depend on n. From this it is easy to reduce $\Delta_{n,m}$ to the following expression

$$\overline{\Delta_{n,m}} = \frac{S_\sigma^m(0)}{(m-1)!(m-2)!\cdots 1!}\widetilde{\Delta}_{n,m}\,, \tag{30}$$

where $\overline{\Delta_{n,m}}$ denotes the complex conjugate of the determinant given in (2). Since $S_\sigma(0) \neq 0$ and m is fixed, it is obvious that $l_m = \widetilde{l}_m$ and the statement follows immediately. □

Proof of Corollary 2:

The assumptions on the reflection coefficients imply that $\Phi_n(0) \neq 0$ for all sufficiently large n. For simplicity in the deduction of some formulas, we will assume that $\Phi_n(0) \neq 0$ for all n. It is easy to see that this causes no restriction in the validity of the general result.

Let us begin showing by induction on k that, for each $n \in \mathbb{N}$,

$$\frac{\Phi_n^{(k)}(0)}{\Phi_{n+1}(0)} = A_0^{(k)} + A_1^{(k)}|a|^{2n} + A_2^{(k)}|a|^{4n} + \cdots + A_k^{(k)}|a|^{2kn} + \varepsilon_n^{(k)}\,, \tag{31}$$

where $A_0^{(k)},\ldots,A_k^{(k)}$ are constants independent of n, $A_k^{(k)} \neq 0$, and

$$\limsup_n \left(\max_{j\geq n}|\varepsilon_j^{(k)}|\right)^{1/n} \leq \delta.$$

To this end, we make use of the three-term recurrence relation satisfied by the monic orthogonal polynomials on the unit circle

$$\Phi_{n+1}(z) = \left(z + \frac{\Phi_{n+1}(0)}{\Phi_n(0)}\right)\Phi_n(z) - (1-|\Phi_n(0)|^2)\frac{\Phi_{n+1}(0)}{\Phi_n(0)}z\Phi_{n-1}(z)\,, \quad n \geq 0 \tag{32}$$

($\Phi_{-1}(z) \equiv 0$). For a proof of this formula see Lemma 2 in [1] or 8.3 in [5]. Taking derivatives in (32) it is easy to deduce (by induction) that

$$\begin{aligned}\Phi_{n+1}^{(k)}(z) \quad &= \quad \left(z + \frac{\Phi_{n+1}(0)}{\Phi_n(0)}\right)\Phi_n^{(k)}(z) - z(1-|\Phi_n(0)|^{2n})\frac{\Phi_{n+1}(0)}{\Phi_n(0)}\Phi_{n-1}^{(k)}(z) + \\ &+ \quad k\left[\Phi_n^{(k-1)}(z) - (1-|\Phi_n(0)|^{2n})\frac{\Phi_{n+1}(0)}{\Phi_n(0)}\Phi_{n-1}^{(k-1)}(z)\right]\,, \quad n \geq k\,, \quad k \geq 1\,.\end{aligned} \tag{33}$$

Set $z = 0$ in (32) and divide by $\Phi_{n+1}(0)$. It follows that

$$\frac{\Phi_{n+1}^{(k)}(0)}{\Phi_{n+1}(0)} - \frac{\Phi_n^{(k)}(0)}{\Phi_n(0)} = k\left[\frac{\Phi_n^{(k-1)}(0)}{\Phi_{n+1}(0)} - \frac{\Phi_{n-1}^{(k-1)}(0)}{\Phi_n(0)} + \frac{\Phi_{n-1}^{(k-1)}(0)}{\Phi_n(0)}|\Phi_n(0)|^2\right].$$

Substituting in this expression n by j and adding the corresponding formulas for $j = k$ up to n, we obtain

$$\frac{\Phi_{n+1}^{(k)}(0)}{\Phi_{n+1}(0)} - \frac{\Phi_k^{(k)}(0)}{\Phi_k(0)} = k\left(\frac{\Phi_n^{(k-1)}(0)}{\Phi_{n+1}(0)} - \frac{\Phi_{k-1}^{(k-1)}(0)}{\Phi_k(0)}\right) + k\sum_{j=k}^{n}\frac{\Phi_{j-1}^{(k-1)}(0)}{\Phi_j(0)}|\Phi_j(0)|^2.$$

Since $\Phi_k^{(k)}(0)/\Phi_k(0) = k\Phi_{k-1}^{(k-1)}(0)/\Phi_k(0)$ it follows that

$$\frac{\Phi_{n+1}^{(k)}(0)}{\Phi_{n+1}(0)} = k\frac{\Phi_n^{(k-1)}(0)}{\Phi_{n+1}(0)} + k\sum_{j=k}^{n}\frac{\Phi_{j-1}^{(k-1)}(0)}{\Phi_j(0)}|\Phi_j(0)|^2. \tag{34}$$

(In the general case, when $\Phi_n(0) \neq 0$ for $n \geq n_0$, one obtains a formula equal to (34) except for an extra constant term on the right hand of the form $\left(\frac{\Phi_{n_0}^{(k)}(0)}{\Phi_{n_0}(0)} - k\frac{\Phi_{n_0-1}^{(k-1)}(0)}{\Phi_{n_0}(0)}\right)$ which causes no problem in the rest of the proof.)

Let us verify (31) for $k = 0$. In fact, using the assumptions of the Corollary, we have that

$$\frac{\Phi_n(0)}{\Phi_{n+1}(0)} = \frac{1}{a} + \frac{1}{a}\frac{\varepsilon_n - \varepsilon_{n+1}}{c + \varepsilon_n} \tag{35}$$

and the formula holds with $A_0^{(0)} = \dfrac{1}{a}$ and $\varepsilon_n^{(1)} = \dfrac{\varepsilon_n - \varepsilon_{n+1}}{c + \varepsilon_n}$. Assume that (31) holds for the index $k-1$, $k \geq 1$, and let us prove that it is also verified for the index k.

Using the induction hypothesis, we substitute (31), for the index $k-1$, into (34). We have

$$\begin{aligned}\frac{\Phi_{n+1}^{(k)}(0)}{\Phi_{n+1}(0)} = k\sum_{i=0}^{k-1}A_i^{(k-1)}|a|^{2in} + k\varepsilon_n^{(k-1)} \quad &+ \quad k\sum_{i=0}^{k-1}A_i^{(k-1)}\sum_{j=k-1}^{n-1}|a|^{2ij}\,|\Phi_{j+1}(0)|^2 \\ &+ \quad k\sum_{j=k-1}^{n-1}\varepsilon_j^{(k-1)}\,|\Phi_{j+1}(0)|^2\,.\end{aligned} \tag{36}$$

Set $S_0 = k\displaystyle\sum_{j=k-1}^{\infty}\varepsilon_j^{(k-1)}\,|\Phi_{j+1}(0)|^2 < \infty$. Then

$$k\sum_{j=k-1}^{n-1}\varepsilon_j^{(k-1)}\,|\Phi_{j+1}(0)|^2 = S_0 - k\sum_{j\geq n}\varepsilon_j^{(k-1)}\,|\Phi_{j+1}(0)|^2 = S_0 + \varepsilon_{n,0}^{(k)}, \tag{37}$$

where

$$\begin{aligned}\max_{m\geq n}|\varepsilon_{m,0}^{(k)}| \quad &\leq \quad \max_{m\geq n} k\sum_{j\geq m}|\varepsilon_j^{(k-1)}|\,|\Phi_{j+1}(0)|^2 \\ &\leq \quad k\max_{m\geq n}\max_{j\geq m}|\varepsilon_j^{(k-1)}|\sum_{j\geq m}|\Phi_{j+1}(0)|^2 \\ &\leq \quad kc_0\max_{m\geq n}|\varepsilon_m^{(k-1)}|\end{aligned}$$

and $c_0 = \sum_{j\geq 0} |\Phi_{j+1}(0)|^2 < \infty$. Therefore,

$$\limsup_n \left(\max_{m\geq n} |\varepsilon^{(k)}_{m,0}| \right)^{1/n} \leq \delta\,. \tag{38}$$

On the other hand, $|\Phi_{j+1}(0)|^2 = |a|^{2(j+1)}(|c|^2 + \varepsilon_{j+1,1})$ and $\varepsilon_{j+1,1} = 2\Re\,(\overline{c}\varepsilon_{j+1}) + |\varepsilon_{j+1}|^2$ also satisfies

$$\limsup_n \left(\max_{m\geq n} |\varepsilon_{m,1}| \right)^{1/n} \leq \delta\,.$$

For each $i \in \{0, \dots, k-1\}$ fixed

$$\begin{aligned} \sum_{j=k-1}^{n-1} |a|^{2ij}\,|\Phi_{j+1}(0)|^2 &= |ca|^2 \sum_{j=k-1}^{n-1} |a|^{2(i+1)j} + |a|^2 \sum_{j=k-1}^{n-1} |a|^{2(i+1)j}\varepsilon_{j+1,1} \qquad (39)\\ &= |ca|^2 \frac{|a|^{2(i+1)(k-1)} - |a|^{2(i+1)n}}{1-|a|^{2(i+1)}} + |a|^2 \sum_{j=k-1}^{n-1} |a|^{2(i+1)j}\varepsilon_{j+1,1}\,. \end{aligned}$$

Set $S_{i+1} = k|a|^2 A_i^{(k-1)} \sum_{j\geq k-1} |a|^{2(i+1)j}\varepsilon_{j+1,1} < \infty$. Then

$$\begin{aligned} k|a|^2 A_i^{(k-1)} \sum_{j=k-1}^{n-1} |a|^{2(i+1)j}\varepsilon_{j+1,1} &= S_{i+1} - k|a|^2 A_i^{(k-1)} \sum_{j\geq n} |a|^{2(i+1)j}\varepsilon_{j+1,1}\\ &= S_{i+1} + \varepsilon^{(k)}_{n,i+1}\,, \qquad (40) \end{aligned}$$

where

$$\begin{aligned} \max_{m\geq n} \left|\varepsilon^{(k)}_{m,i+1}\right| &\leq k|a|^2 |A_i^{(k-1)}| \max_{m\geq n} \sum_{j\geq m} |a|^{2(i+1)j} |\varepsilon_{j+1,1}|\\ &\leq k|a|^2 |A_i^{(k-1)}| \max_{m\geq n} \max_{j\geq m} |\varepsilon_{j+1,1}| \sum_{j\geq m} |a|^{2(i+1)j} \leq\\ &\leq k|a|^2 |A_i^{(k-1)}| c_{i+1} \max_{m\geq n} |\varepsilon_{m+1,1}| \end{aligned}$$

and $c_{i+1} = \sum_{j\geq 0} |a|^{2(i+1)j} < \infty$. Therefore

$$\limsup_n \left(\max_{j\geq n} |\varepsilon_{j,i+1}| \right)^{1/n} \leq \delta\,. \tag{41}$$

Putting together (36)-(41) it follows that

$$\frac{\Phi_n^{(k)}(0)}{\Phi_n(0)} = \sum_{i=0}^{k} \widetilde{A}_i^{(k)} |a|^{2in} + \widehat{\varepsilon}_n^{(k)}\,,$$

where

$$\limsup_n \left(\max_{j\geq n} \widehat{\varepsilon}_j^{(k)} \right)^{1/n} \leq \delta\,,$$

and $\widetilde{A}_k^{(k)} = -kA_{k-1}^{(k-1)}|c|^2|a|^{-2(k-1)}(1-|a|^{2k})^{-1} \neq 0$. This is (31) for the index k except that we need $\Phi_{n+1}(0)$ in the denominator in place of $\Phi_n(0)$. This is easy to arrange on account of (35). With this we conclude the proof of (31). Now, it readily follows that

$$\frac{\Phi_n^{(k)}(0)}{a^n} = \sum_{i=0}^{k} B_i^{(k)}|a|^{2in} + \delta_n^{(k)}, \tag{42}$$

where $B_0^{(k)}, \ldots, B_k^{(k)}$ are constants independent of n, $B_k^{(k)} = cA_k^{(k)} \neq 0$, and

$$\limsup_n \left|\delta_n^{(k)}\right|^{1/n} \leq \delta\,. \tag{43}$$

Fix $k \in \{1, \ldots, m\}$. Substituting (42) in the determinant $\widetilde{\Delta}_{n,k}$ defined in Corollary 1 and using elementary properties of the determinant, we have

$$\begin{aligned} \frac{\widetilde{\Delta}_{n,k}}{\kappa_n \cdots \kappa_{n+k-1}} &= \begin{vmatrix} \frac{\Phi_n^{(k-1)}(0)}{a^n} & \frac{\Phi_n^{(k-2)}(0)}{a^n} & \cdots & \frac{\Phi_n(0)}{a^n} \\ \frac{\Phi_{n+1}^{(k-1)}(0)}{a^{n+1}} & \frac{\Phi_{n+1}^{(k-2)}(0)}{a^{n+1}} & \cdots & \frac{\Phi_{n+1}(0)}{a^{n+1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Phi_{n+k-1}^{(k-1)}(0)}{a^{n+k-1}} & \frac{\Phi_{n+k-1}^{(k-2)}(0)}{a^{n+k-1}} & \cdots & \frac{\Phi_{n+k-1}(0)}{a^{n+k-1}} \end{vmatrix} a^n a^{n+1} \cdots a^{n+k-1} \\ &= V \prod_{i=0}^{k-1} B_i^{(i)} |a|^{2n}|a|^{4n} \ldots |a|^{2(k-1)n} a^{(n+\frac{k-1}{2})k} + \delta_n \\ &= V \left(\prod_{i=0}^{k-1} A_i^{(i)}\right) |a|^{nk(k-1)} a^{(n+\frac{k-1}{2})k} + \delta_n\,, \end{aligned} \tag{44}$$

where V denotes the Vandermonde determinant relative to the points $1, |a|^2, \ldots, |a|^{2(k-1)}$ and δ_n denotes the sum of $2^m - 1$ determinants each one of which has at least one column of the form $(\delta_n^{(j)}, \ldots, \delta_{n+k-1}^{(j)})^t, j = 0, \ldots, k-1$. Therefore, on account of (43),

$$\limsup_n |\delta_n|^{1/n} \leq \delta < |a|^{m^2} \leq |a|^{k^2}\,. \tag{45}$$

Hence

$$\widetilde{l}_k = \lim_n \left|\widetilde{\Delta}_{n,k}\right|^{1/n} = |a|^{k^2}\,.$$

According to Corollary 1, for each $k \in \{0, \ldots, m-1\}$ we have that

$$R_k = \widetilde{l}_k / \widetilde{l}_{k+1} = |a|^{-(2k+1)}\,.$$

From this it follows that S_σ^{-1} has exactly $m-1$ simple poles which are located on the circles of radii $|a|^{-(2k+1)}, k = 0, \ldots, m-2$, respectively and has a singularity on the circle of radius equal to $|a|^{-(2m-1)}$. In order to obtain their exact value we use Theorem 3.

We proceed as follows. By Theorem 3, for each $k \in \{1, \ldots, m-1\}$ the sequence $\{Q_{n,k}\}, n \in \mathbb{N}$, of the denominator polynomials of the Fourier-Padé approximants relative to S_σ^{-1} converges to the polynomial Q_k whose zeros are the poles of S_σ^{-1} inside D_k. This is so because in each of the disks D_k this function has exactly k poles. It follows that $Q_k(0)/Q_{k-1}(0), (Q_0 \equiv 1,)$ is equal to the pole which S_σ^{-1} has on the circle of radius $|a|^{-(2k-1)}$. Let us calculate $Q_k(0)$.

From the definition of $R_{n,k}$ it follows immediately that

$$Q_{n,k}(z) = \frac{1}{\Delta_{n+1,k}(S_\sigma^{-1})} \begin{vmatrix} z^k & z^{k-1} & \cdots & 1 \\ \langle z^k S_\sigma^{-1}, \varphi_{n+1}\rangle & \langle z^{k-1} S_\sigma^{-1}, \varphi_{n+1}\rangle & \cdots & \langle S_\sigma^{-1}, \varphi_{n+1}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle z^k S_\sigma^{-1}, \varphi_{n+k}\rangle & \langle z^{k-2} S_\sigma^{-1}, \varphi_{n+k}\rangle & \cdots & \langle S_\sigma^{-1}, \varphi_{n+k}\rangle \end{vmatrix} .$$

Therefore,

$$Q_{n,k}(0) = (-1)^k \frac{\Delta_{n+1,k}(zS_\sigma^{-1})}{\Delta_{n+1,k}(S_\sigma^{-1})} .$$

Using (26), we obtain

$$Q_{n,k}(0) = (-1)^k \frac{\Delta_{n,k}(S_\sigma^{-1}) + \widetilde{\delta}_n}{\Delta_{n+1,k}(S_\sigma^{-1})} , \tag{46}$$

where

$$\limsup_n |\widetilde{\delta}_n|^{1/n} < l_k = \widetilde{l}_k = |a|^{k^2} . \tag{47}$$

From (30) and (44)-(47), it follows that

$$Q_{n,k}(0) = (-1)^k \frac{\kappa_n \cdots \kappa_{n+k-1}}{\kappa_{n+1} \cdots \kappa_{n+k}} \frac{|a|^{nk(k-1)} \overline{a}^{(n+\frac{k-1}{2})k}}{|a|^{(n+1)k(k-1)} \overline{a}^{(n+\frac{k+1}{2})k}} \frac{1+\delta_{n,1}}{1+\delta_{n,2}} ,$$

where $\lim_n \delta_{n,1} = \lim_n \delta_{n,2} = 0$. Cancelling out equal powers of $|a|$ and $\overline{a}$ and taking limit using (12), we obtain that

$$Q_k(0) = (-1)^k \frac{1}{|a|^{k(k-1)} \overline{a}^k} .$$

Therefore,

$$Q_k(0)/Q_{k-1}(0) = \frac{1}{|a|^{2(k-1)} \overline{a}}, \quad k = 1, \ldots, m-1 ,$$

as we needed to prove. □

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