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CONSTRUCTIVE APPROXIMATION

Asymptotic Distribution of Nodes for Near-Optimal Polynomial Interpolation on Certain Curves in $\mathbb{R}^2$

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Abstract. Let $E \subset \mathbb{R}^s$ be compact and let d_n^E denote the dimension of the space of polynomials of degree at most n in s variables restricted to E. We introduce the notion of an *asymptotic interpolation measure* (AIM). Such a measure, *if it exists*, describes the asymptotic behavior of any scheme $\tau_n = \{\mathbf{x}_{k,n}\}_{k=1}^{d_n^E}$, $n = 1, 2, \ldots$, of nodes for multivariate polynomial interpolation for which the norms of the corresponding interpolation operators do not grow geometrically large with n. We demonstrate the existence of AIMs for the finite union of compact subsets of certain algebraic curves in $\mathbb{R}^2$. It turns out that the theory of logarithmic potentials with external fields plays a useful role in the investigation. Furthermore, for the sets mentioned above, we give a computationally simple construction for "good" interpolation schemes.

1. Introduction

With $\boldsymbol{\pi}_n(\mathbb{R}^s)$ denoting the set of all real polynomials of degree at most n in s variables, i.e.,

$$\boldsymbol{\pi}_n(\mathbb{R}^s) := \left\{ p(\mathbf{x}) : p(\mathbf{x}) = \sum_{|\alpha| \le n} c_\alpha \mathbf{x}^\alpha,\ \boldsymbol{\alpha} \in \mathbb{Z}_+^s,\ c_\alpha \in \mathbb{R},\ \mathbf{x} \in \mathbb{R}^s \right\},$$

($\mathbb{Z}_+^s$ denotes the set of multi-indices with s components) and $E \subset \mathbb{R}^s$ a compact set, the problem of determining the asymptotic behavior of "good points" for polynomial interpolation to functions $f \in C(E)$ is a fundamental question which has been resolved in generality only for the case $s = 1$. If $\boldsymbol{\pi}_n(E)$ denotes the linear space obtained by restricting $\boldsymbol{\pi}_n(\mathbb{R}^s)$ to E, i.e.,

$$\boldsymbol{\pi}_n(E) := \boldsymbol{\pi}_n(\mathbb{R}^s)\big|_E,$$

and d_n^E denotes the dimension of $\boldsymbol{\pi}_n(E)$, then by "good points" $\{\mathbf{x}_{k,n}\}_{k=1}^{d_n^E}$ we mean points for which the interpolation problem is solvable in $\boldsymbol{\pi}_n(E)$ with arbitrary data in

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these points and for which the norm of the interpolation operator

$$L_n : C(E) \to \Pi_n(E)$$

does not grow geometrically large with n. More precisely, if $L_{i,n}^E \in \Pi_n(E)$, $i = \overline{1, d_n^E}$, are the fundamental Lagrange polynomials satisfying

$$L_{i,n}^E(\mathbf{x}_{k,n}) = \delta_{ik} \qquad (i, k = \overline{1, d_n^E}), \tag{1.1}$$

then

$$L_n(f) = \sum_{k=1}^{d_n^E} f(\mathbf{x}_{k,n}) L_{k,n}^E,$$

and (see [5]) the norm $\|L_n\|$ is given by the *Lebesgue constant*

$$\lambda_n^E := \max_{\mathbf{x}\in E} \sum_{i=1}^{d_n^E} |L_{i,n}^E(\mathbf{x})|. \tag{1.2}$$

Thus the basic problem we consider is the determination of sets E that have the following property.

Definition. An infinite compact set $E \subset \mathbb{R}^s$ is said to have an *asymptotic interpolation measure* (more briefly, an AIM) if there exists a measure μ^E on E such that for any interpolation scheme of nodes $\tau_n = \{\mathbf{x}_{k,n}\}_{k=1}^{d_n^E} \subset E$, $n = 0, 1, \ldots$, for which the corresponding interpolation problems are solvable and the sequence of Lebesgue constants λ_n^E satisfies

$$\limsup_{n\to\infty} (\lambda_n^E)^{1/n} \le 1, \tag{1.3}$$

the sequence of normalized counting measures satisfies

$$\nu(\tau_n) := \frac{1}{d_n^E} \sum_{k=1}^{d_n^E} \boldsymbol{\delta}_{\mathbf{x}_{k,n}} \to \mu^E \qquad \text{as} \quad n \to \infty \tag{1.4}$$

in the weak-star sense.

Here, and in what follows, $\boldsymbol{\delta}_{\mathbf{x}}$ denotes the unit point mass at $\mathbf{x}$.

Remark 1.1. The AIM property is clearly invariant under the affine transformations of a set E.

Remark 1.2. Assuming E to contain infinitely many points, the *Auerbach–Fekete points* (often referred to simply as Fekete points) satisfy (1.3). These points are defined as follows. If $\{p_i\}_{i=1}^{d_n^E}$ form a basis for $\Pi_n(E)$, then the Auerbach–Fekete points are points $\{\mathbf{x}_{k,n}^*\}_{k=1}^{d_n^E} \subset E$ that maximize the determinant

$$|(p_i(\mathbf{x}_k))_{1\le i,k\le d_n^E}|$$

over all $\mathbf{x}_1, \ldots, \mathbf{x}_{d_n^E}$ in E. For the Lebesgue constants corresponding to such points we have

$$\lambda_n^E \le d_n^E,$$

and since

$$d_n^E \le \dim \mathbf{\Pi}_n(\mathbb{R}^s) = \binom{n+s}{s} = O(n^s) \qquad \text{as} \quad n \to \infty,$$

it is clear that (1.3) holds.

Remark 1.3. For $s = 1$, it is easy to show that every compact set $E \subset \mathbb{R}^1$ with positive logarithmic capacity has the AIM property[1] and, moreover, that μ^E is just the Robin (equilibrium) measure for E. Indeed, this fact can be proved, for example, from the well-known inequality

$$\|f - L_n(f)\| \le (1 + \|L_n\|) \min_{p \in \mathbf{\Pi}_n(E)} \|f - p\|$$

together with a result on asymptotically minimal polynomials which follows from the theory of logarithmic potentials (see [4], [10]). For $s \ge 2$ the problem is far more difficult; there exist nontrivial compact sets $E \subset \mathbb{R}^2$ that do *not* have the AIM property (see Example 3.4).

In the present paper we shall restrict ourselves to subsets E of algebraic curves in the plane. Using the theory of logarithmic potentials with external fields in the complex plane (see [10]), we shall show that the union of finitely many compact subsets of positive capacity of algebraic curves of genus 0 in $\mathbb{R}^2$ has an AIM. The essential feature of such curves is that they admit a rational parametrization. In particular, our result applies in the case when E is a compact subset of the image of the unit circle under a rational mapping $w = p(z)/q(z)$ in the complex variable z or a compact subset of a curve consisting of piecewise conics.

The outline of the paper is as follows. Section 2 contains some simple consequences of Auerbach's theorem that are essential for the proofs of the main results of the paper. In Section 3 we prove that the union of finitely many subsets of algebraic curves having AIMs again has an AIM. In Section 4 we show that compact subsets of algebraic curves of genus 0 have the AIM property and we determine their asymptotic interpolation measure μ^E. In Section 5 we consider the inverse problem of constructing good interpolation points when the AIM μ^E is known. In the final section, we present several examples that illustrate our results.

2. Some Simple Consequences of Auerbach's Theorem

In this section we prove two auxiliary results in the general Banach space settings which are basic ingredients for establishing our main results.

[1] This assertion need not hold if E has capacity zero.

Proposition 2.1. *Let $\mathbb{X}$ be an r-dimensional subspace of a Banach space $\mathbb{V}$, let $\mathbb{B} \subset \mathbb{V}^*$ be a closed norm-determining*[2] *subset of the dual space, and let $1 \le m < p \le r$ be integers. Let $g_j \in \mathbb{X}$ and $\varphi_j \in \mathbb{V}^*$, $j = \overline{1, m}$, be such that*

$$\varphi_i(g_j) = \delta_{ij} \qquad (i, j = \overline{1, m}),$$

and let $\mathcal{L} : \mathbb{V} \to \mathbb{X}$ be defined by

$$\mathcal{L}(f) := \sum_{j=1}^{m} \varphi_j(f) g_j.$$

Then there exist $\varphi_j \in \mathbb{B}$, $j = \overline{m+1, p}$, and $f_j \in \mathbb{X}$, $j = \overline{1, p}$, such that

$$\varphi_i(f_j) = \delta_{ij} \qquad (i, j = \overline{1, p}), \tag{2.1}$$

$$\|\varphi_j\| = \|f_j\| = 1 \qquad (j = \overline{m+1, p}), \tag{2.2}$$

and the operator $\tilde{\mathcal{L}} : \mathbb{V} \to \mathbb{X}$, given by

$$\tilde{\mathcal{L}}(f) := \sum_{j=1}^{p} \varphi_j(f) f_j, \tag{2.3}$$

satisfies

$$\|\tilde{\mathcal{L}}\| \le (p - m + 1)\|\mathcal{L}\| + (p - m).$$

Proof. Let $\mathbb{Y} := \{f \in \mathbb{X} : \varphi_i(f) = 0, i = \overline{1, m}\}$ be endowed with the same norm as $\mathbb{V}$. Then $\mathbb{Y}$ is an $(r - m)$-dimensional linear space and the restriction $\mathbb{B}|_{\mathbb{Y}}$ is a closed norm-determining set for $\mathbb{Y}$. By Auerbach's theorem (see, e.g., [5]), there exist $f_j \in \mathbb{Y} \subset \mathbb{X}$ and $\tilde{\varphi}_j = \varphi_j|_{\mathbb{Y}} \in \mathbb{B}|_{\mathbb{Y}}$, $j = \overline{m+1, p}$, such that $\|f_j\| = 1$, $\|\tilde{\varphi}_j\| = 1$, and

$$\varphi_i(f_j) = \delta_{ij} \qquad (i, j = \overline{m+1, p}).$$

Notice that, since $\mathbb{B}$ is norm-determining and $\varphi_j \in \mathbb{B}$, we have $\|\tilde{\varphi}_j\| \le \|\varphi_j\| \le 1$, and so $\|\varphi_j\| = 1$ for $j = \overline{m+1, p}$. Thus (2.2) holds.

Next, for $j = \overline{1, m}$, define

$$f_j := g_j - \sum_{k=m+1}^{p} \varphi_k(g_j) f_k.$$

Then, for $j = \overline{1, m}$,

$$\begin{aligned} \varphi_i(f_j) &= \varphi_i(g_j) - \sum_{k=m+1}^{p} \varphi_k(g_j)\varphi_i(f_k) = \varphi_i(g_j) - \sum_{k=m+1}^{p} \varphi_k(g_j)\delta_{ik} \\ &= \begin{cases} \delta_{ij} - \displaystyle\sum_{k=m+1}^{p} \varphi_k(g_j) \cdot 0 = \delta_{ij} & \text{if } i \le m, \\ \varphi_i(g_j) - \varphi_i(g_j) = 0 = \delta_{ij} & \text{if } m < i \le p, \end{cases} \end{aligned}$$

where in the case $i \le m$ we used the fact that $f_k \in \mathbb{Y}$ for $k \ge m + 1$. Thus (2.1) holds.

[2] By "norm-determining," we mean that for each $v \in \mathbb{V}$ there holds $\|v\| = \sup\{|\varphi(v)| : \varphi \in \mathbb{B}\}$.

Let $f \in \mathbb{V}$ with $\|f\| \le 1$. Then from (2.2) we get

$$\begin{aligned}
\|\tilde{\mathcal{L}}(f)\| &= \left\|\sum_{j=1}^{p} \varphi_j(f) f_j\right\| = \left\|\left(\sum_{j=1}^{m} + \sum_{j=m+1}^{p}\right)\varphi_j(f) f_j\right\| \le \left\|\sum_{j=1}^{m} \varphi_j(f) f_j\right\| + (p-m) \\
&= \left\|\sum_{j=1}^{m} \varphi_j(f) g_j - \sum_{j=1}^{m}\sum_{k=m+1}^{p} \varphi_j(f)\varphi_k(g_j) f_k\right\| + (p-m) \\
&= \left\|\mathcal{L}(f) - \sum_{k=m+1}^{p} \varphi_k(\mathcal{L}(f)) f_k\right\| + (p-m) \\
&\le \|\mathcal{L}\| + \sum_{k=m+1}^{p} \|\varphi_k\|\|\mathcal{L}\|\|f_k\| + (p-m) = (p-m+1)\|\mathcal{L}\| + (p-m).
\end{aligned}$$

Hence (2.3) holds. ■

We now apply Proposition 2.1 to a subspace of continuous functions on an infinite compact Hausdorff space $\mathbb{S}$, where the norm is the uniform norm on $\mathbb{S}$. With each $\mathbf{x} \in \mathbb{S}$ we associate $\mathbf{x}^* \in C(\mathbb{S})^*$ by setting $\mathbf{x}^*(f) := f(\mathbf{x})$, $f \in C(\mathbb{S})$. Then $\mathbb{B} := \{\mathbf{x}^* : \mathbf{x} \in \mathbb{S}\}$ is a closed norm-determining set for $C(\mathbb{S})$.

Corollary 2.2. *Let $\mathbb{X}_r \subset C(\mathbb{S})$ be an r-dimensional subspace, and let $1 \le m < p \le r$ be integers. Let $\tau := \{\mathbf{x}_j\}_{j=1}^{m} \subset \mathbb{S}$ be a set of nodes for which there exist $L_i \in \mathbb{X}_r$, $i = \overline{1, m}$, such that*

$$L_i(\mathbf{x}_j) = \delta_{ij} \qquad (i, j = \overline{1, m}),$$

and define the interpolation operator $\mathcal{P} : C(\mathbb{S}) \to \mathbb{X}_r$ by

$$\mathcal{P}(f) = \sum_{i=1}^{m} f(\mathbf{x}_i) L_i. \tag{2.4}$$

Then there exists a set of p interpolation nodes $\tilde{\tau} = \{\mathbf{x}_j\}_{j=1}^{p}$ containing τ and functions $\tilde{L}_i \in \mathbb{X}_r$, $i = \overline{1, p}$, such that

$$\tilde{L}_i(\mathbf{x}_j) = \delta_{ij} \qquad (i, j = \overline{1, p}),$$

and the interpolation operator $\tilde{\mathcal{P}} : C(\mathbb{S}) \to \mathbb{X}_r$ corresponding to $\tilde{\tau}$ satisfies

$$\|\tilde{\mathcal{P}}\| \le (p-m+1)\|\mathcal{P}\| + (p-m). \tag{2.5}$$

Remark 2.3. The norms of the operators $\mathcal{P}$ and $\tilde{\mathcal{P}}$ in (2.4), (2.5) are given by formulas similar to those for the Lebesgue constants defined in (1.2).

Remark 2.4. If $p = r$ and the functions $\{p_i\}_{i=1}^{r}$ form a basis for $\mathbb{X}_r$, then the set $\tilde{\tau}$ can be obtained, for example, by adjoining to τ a set of $r - m$ nodes $\{\mathbf{x}_j^*\}_{j=m+1}^{r}$ maximizing the determinant

$$|(p_i(\mathbf{x}_j))_{1 \le i, j \le r}|$$

over all $(\mathbf{x}_{m+1}, \ldots, \mathbf{x}_r) \in \mathbb{S}^{r-m}$.

Proposition 2.5. *Let $\mathbb{V}$ be a Banach space, $1 \le m < p$ integers, and let $\{\psi_i\}_{i=1}^p \subset \mathbb{V}^*$, $\{f_i\}_{i=1}^p \subset \mathbb{V}$ satisfy*

$$\psi_i(f_j) = \delta_{ij} \qquad (i, j = \overline{1, p}),$$
$$\|\psi_i\| = 1 \qquad (i = \overline{1, p}). \tag{2.6}$$

Let $\mathbb{X}_m$ be an m-dimensional subspace of $\mathbb{Y}_p := \operatorname{span}\{f_1, \dots, f_p\}$. Then there exist a subset $\{\psi_{k_i}\}_{i=1}^m$ of $\{\psi_j\}_{j=1}^p$ and $\{g_j\}_{j=1}^m \subset \mathbb{X}_m$ such that

$$\psi_{k_i}(g_j) = \delta_{ij} \qquad (i, j = \overline{1, m}),$$

and the operator $\hat{\mathcal{L}} : \mathbb{V} \to \mathbb{X}_m$, defined by

$$\hat{\mathcal{L}}(f) := \sum_{i=1}^m \psi_{k_i}(f) g_i, \tag{2.7}$$

satisfies

$$\|\hat{\mathcal{L}}\| \le m \sum_{j=1}^p \|f_j\|. \tag{2.8}$$

Furthermore, if $\|f_j\| \le K$, $j = \overline{1, p}$, then

$$\|\hat{\mathcal{L}}\| \le \min\{\|\mathcal{L}\| + K(m+1)(p-m), Km(p-m+1)\}, \tag{2.9}$$

where $\mathcal{L} : \mathbb{V} \to \mathbb{Y}_p$ is defined by $\mathcal{L}(f) := \sum_{j=1}^p \psi_j(f) f_j$.

Proof. Define on $\mathbb{X}_m$ the new norm $\|\cdot\|_{\mathbb{X}_m}$ by

$$\|f\|_{\mathbb{X}_m} := \max_{i=\overline{1,p}} |\psi_i(f)|.$$

Then $\{\psi_1, \dots, \psi_p\}$ is a closed norm-determining set of linear functionals on $(\mathbb{X}_m, \|\cdot\|_{\mathbb{X}_m})$. Note further that $\|f_k\|_{\mathbb{X}_m} = 1$ for all k.

By Auerbach's theorem, there exist $\{\psi_{k_i}\}_{i=1}^m \subset \{\psi_j\}_{j=1}^p$ and $\{g_i\}_{i=1}^m \subset \mathbb{X}_m$ such that

$$\psi_{k_i}(g_j) = \delta_{ij} \qquad (i, j = \overline{1, m}), \tag{2.10}$$

and

$$\|g_i\|_{\mathbb{X}_m} = \|\psi_{k_i}\|_{\mathbb{X}_m^*} = 1 \qquad (i = \overline{1, m}). \tag{2.11}$$

To simplify notation we assume that $\psi_{k_i} = \psi_i$ for $i = \overline{1, m}$.

Now let $\hat{\mathcal{L}} : \mathbb{V} \to (\mathbb{X}_m, \|\cdot\|)$ be defined by

$$\hat{\mathcal{L}}(f) = \sum_{i=1}^m \psi_i(f) g_i.$$

Since $g_j \in \mathbb{X}_m \subset \mathbb{Y}_p$, $j = \overline{1, m}$, taking into account (2.10), we have

$$g_j = \sum_{i=1}^p \psi_i(g_j) f_i = f_j + \sum_{i=m+1}^p \psi_i(g_j) f_i. \tag{2.12}$$

Let $f \in \mathbb{V}$, $\| f \| \le 1$. Then using (2.6), (2.11), and (2.12) we obtain

$$\|\hat{\mathcal{L}}(f)\| \le \sum_{j=1}^{m} |\psi_j(f)| \|g_j\| \le \sum_{j=1}^{m} |\psi_j(f)| \left(\sum_{i=1}^{p} \| f_i \| \right) \le \sum_{j=1}^{m} \sum_{i=1}^{p} \| f_i \|,$$

and (2.8) follows.

Next assume that $\| f_j \| \le K,\ j = \overline{1, p}$. To establish (2.9) we use (2.12) to obtain

$$\begin{aligned} \hat{\mathcal{L}}(f) &= \sum_{j=1}^{m} \psi_j(f) \left(f_j + \sum_{i=m+1}^{p} \psi_i(g_j) f_i \right) \\ &= \mathcal{L}(f) + \sum_{i=m+1}^{p} \left(\sum_{j=1}^{m} \psi_j(f) \psi_i(g_j) - \psi_i(f) \right) f_i, \end{aligned}$$

and so (2.6) and (2.11) imply that

$$\|\hat{\mathcal{L}}(f)\| \le \|\mathcal{L}(f)\| + \sum_{i=m+1}^{p} \left(\sum_{j=1}^{m} \|g_j\|_{\mathbb{X}_m} + 1 \right) K = \|\mathcal{L}(f)\| + (p-m)(m+1)K.$$

On the other hand,

$$\begin{aligned} \|\hat{\mathcal{L}}(f)\| &= \left\| \sum_{j=1}^{m} \psi_j(f) f_j + \sum_{j=1}^{m} \psi_j(f) \sum_{i=m+1}^{p} \psi_i(g_j) f_i \right\| \\ &\le mK + m(p-m)K = m(p-m+1)K. \end{aligned}$$

Hence (2.9) holds. ■

Analogous to Corollary 2.2 we obtain the following result.

Corollary 2.6. *Let $1 \le m < p$ be integers, let $\mathbb{Y}_p$ be a p-dimensional subspace of $C(\mathbb{S})$, and let $\tau := \{\mathbf{x}_j\}_{j=1}^{p} \subset \mathbb{S}$ be a set of interpolation nodes for which the interpolation problem is solvable in $\mathbb{Y}_p$. Let $\mathbb{X}_m$ be an m-dimensional subspace of $\mathbb{Y}_p$. Then there exists a subset $\hat{\tau}$ of τ consisting of m interpolation nodes such that the interpolation problem is solvable in $\mathbb{X}_m$ and the corresponding interpolation operator satisfies*

$$\|\widehat{\mathcal{P}}\| \le \|\mathcal{P}\| \min\{(m+1)(p-m)+1, m(p-m+1)\}. \tag{2.13}$$

Remark 2.7. In numerical applications, if $B = \{p_i\}_{i=1}^{m}$ is a basis for $\mathbb{X}_m$, the set $\widehat{\tau} \subset \tau$ can be found by maximizing the absolute value of the determinant

$$|(p_i(\mathbf{x}_{k_j}))_{1 \le i, j \le m}| \tag{2.14}$$

over all $\binom{p}{m}$ possible choices of $\{\mathbf{x}_{k_j}\}_{j=1}^{m} \subset \tau$.

3. Unions of Sets Having AIMs

We first prove that the AIM property is not affected by the addition of finitely many points.

Proposition 3.1. *Let $E \subset \mathbb{R}^s$ be an infinite compact set, and let $F \subset \mathbb{R}^s$ be a finite set. Then E has an* AIM *if and only if $E \cup F$ has an* AIM. *Moreover, if E has an* AIM, *then for every finite set F*:

$$\mu^E = \mu^{E\cup F}.$$

Proof. It suffices to prove the result for a singleton set $F = \{\mathbf{z}\}$ with $\mathbf{z} \notin E$. Let $\widetilde{E} := E \cup \{\mathbf{z}\}$.

Suppose first that E has an AIM, and let $\tilde{\tau}_n = \{\tilde{\mathbf{x}}_{1,n}, \ldots, \tilde{\mathbf{x}}_{d_n^{\tilde{E}},n}\} \subset \widetilde{E}$ be a sequence of interpolation points with Lebesgue constants $\tilde{\lambda}_n$ (on $\widetilde{E}$) satisfying

$$\lim_{n\to\infty} (\tilde{\lambda}_n)^{1/n} = 1. \tag{3.1}$$

Suppose that the points are ordered such that $\tilde{\mathbf{x}}_{i,n} \neq \mathbf{z}$, $i = \overline{1, d_n^{\widetilde{E}} - 1}$. Denote by $L_{i,n}^{\widetilde{E}}$ the corresponding fundamental Lagrange polynomials (see (1.1)).

Case 1: $\tilde{\mathbf{x}}_{d_n^{\tilde{E}},n} = \mathbf{z}$ and $d_n^{\widetilde{E}} = d_n^E$. Since

$$\max_{\mathbf{x}\in E} \sum_{i=1}^{d_n^E-1} |L_{i,n}^{\widetilde{E}}(\mathbf{x})| \le \max_{\mathbf{x}\in \widetilde{E}} \sum_{i=1}^{d_n^E} |L_{i,n}^{\widetilde{E}}(\mathbf{x})| = \tilde{\lambda}_n,$$

using Corollary 2.2 with $\mathbb{S} = E$, $m = d_n^E - 1$, $p = r = d_n^E$, $\mathbb{X}_r = \mathbf{\Pi}_n(E)$, $\tau = \tilde{\tau}_n \backslash \{\mathbf{z}\}$, and $L_i = L_{i,n}^{\widetilde{E}}$, we can choose a point $\mathbf{x}_{d_n^E,n} \in E$ such that the fundamental Lagrange polynomials $L_{i,n}^E \in \mathbf{\Pi}_n(E)$ for

$$\tau_n := \{\mathbf{x}_{1,n}, \ldots, \mathbf{x}_{d_n^E,n}\} := \{\tilde{\mathbf{x}}_{1,n}, \ldots, \tilde{\mathbf{x}}_{d_n^{\tilde{E}}-1,n}, \mathbf{x}_{d_n^E,n}\},$$

satisfy

$$\lambda_n := \max_{\mathbf{x}\in E} \sum_{i=1}^{d_n^E} |L_{i,n}^E(\mathbf{x})| \le 2\tilde{\lambda}_n + 1.$$

Case 2: "Otherwise." Define $\tau_n := \tilde{\tau}_n|_E$. Then $\lambda_n \le \tilde{\lambda}_n$ for the corresponding Lebesgue constants over E and $\widetilde{E}$, respectively.

In either case, we have $\lambda_n \le 2\tilde{\lambda}_n + 1$, and thus the Lebesgue constants associated with τ_n (for E) satisfy a limit condition similar to (3.1). By assumption, E has an AIM, so

$$\frac{1}{d_n^E} \sum_{\mathbf{x}\in\tau_n} \delta_{\mathbf{x}} \to \mu^E.$$

Furthermore, it is easy to show that

$$d_n^E \le \tilde{d}_n := d_n^{\widetilde{E}} \le d_n^E + 1. \tag{3.2}$$

Consequently, $\tilde{d}_n \to \infty$ as $n \to \infty$ (recall that E is infinite), and it follows that

$$\frac{1}{\tilde{d}_n} \sum_{\mathbf{x} \in \tilde{\tau}_n} \boldsymbol{\delta}_{\mathbf{x}} \to \mu^E.$$

Thus, $\widetilde{E}$ has an AIM and $\mu^{\widetilde{E}} = \mu^E$.

Conversely, suppose that $\widetilde{E}$ has an AIM. Note first that

$$\operatorname{supp} \mu^{\widetilde{E}} \subseteq E. \tag{3.3}$$

Let

$$\hat{\boldsymbol{\Pi}}_n(E) := \{p|_E : p \in \boldsymbol{\Pi}_n(\widetilde{E}) \text{ and } p(\mathbf{z}) = 0\}.$$

Clearly, $\hat{\boldsymbol{\Pi}}_n(E) \subseteq \boldsymbol{\Pi}_n(E)$ and, moreover,

$$d_n^E - 1 \leq \hat{d}_n := \dim \hat{\boldsymbol{\Pi}}_n(E) \leq d_n^E. \tag{3.4}$$

Indeed, let the polynomials $p_k \in \boldsymbol{\Pi}_n(\widetilde{E})$, $k = \overline{1, d_n^E}$, form a basis for $\boldsymbol{\Pi}_n(E)$ when restricted to E. If, for some j, $p_j(\mathbf{z}) \neq 0$, then the polynomials

$$q_k(\mathbf{x}) := p_k(\mathbf{x}) - p_j(\mathbf{x}) \frac{p_k(\mathbf{z})}{p_j(\mathbf{z})} \qquad (k = \overline{1, d_n^E}, k \neq j), \tag{3.5}$$

are linearly independent in $\hat{\boldsymbol{\Pi}}_n(E)$.

Now, let $\tau_n = \{\mathbf{x}_{1,n}, \ldots, \mathbf{x}_{d_n^E,n}\} \subset E$ be a sequence of interpolation points with Lebesgue constants λ_n (on E) satisfying

$$\lim_{n\to\infty} (\lambda_n)^{1/n} = 1. \tag{3.6}$$

If $\hat{d}_n < d_n^E$, we apply Corollary 2.6 with $m = \hat{d}_n$, $p = d_n^E$, $\mathbb{X}_m = \hat{\boldsymbol{\Pi}}_n(E)$, $\mathbb{Y}_p = \boldsymbol{\Pi}_n(E)$, and $\tau = \tau_n$ to get a subset $\hat{\tau}_n$ of τ_n consisting of $\hat{d}_n$ interpolation nodes for which the interpolation problem is solvable in $\hat{\boldsymbol{\Pi}}_n(E)$ and the corresponding Lebesgue constant satisfies

$$\widehat{\lambda_n} \leq (d_n^E + 1)\lambda_n.$$

(If $\hat{d}_n = d_n^E$, we simply set $\hat{\tau}_n := \tau_n$.) Next, if $\tilde{d}_n > \hat{d}_n$, setting $m = \hat{d}_n$, $p = r = \tilde{d}_n$, $\mathbb{X}_r = \boldsymbol{\Pi}_n(\widetilde{E})$, and $\tau := \hat{\tau}_n$, we use Corollary 2.2 to adjoin $\tilde{d}_n - \hat{d}_n \leq 2$ nodes to $\hat{\tau}_n$ and obtain a complete set $\tilde{\tau}_n$ of interpolation nodes on $\widetilde{E}$ with Lebesgue constant satisfying

$$\widetilde{\lambda_n} \leq 3\widehat{\lambda_n} + 2 \leq 3(d_n^E + 1)\lambda_n + 2.$$

So, using (3.6), we conclude that

$$\lim_{n\to\infty} (\widetilde{\lambda_n})^{1/n} \leq 1.$$

By the assumption that $\widetilde{E}$ has an AIM we get

$$\frac{1}{\tilde{d}_n} \sum_{\mathbf{x} \in \tilde{\tau}_n} \boldsymbol{\delta}_{\mathbf{x}} \to \mu^{\widetilde{E}}.$$

Note that

$$\frac{1}{\tilde{d}_n}\sum_{\mathbf{x}\in\tilde{\tau}_n}\delta_{\mathbf{x}} = \frac{d_n^E}{\tilde{d}_n}\left(\frac{1}{d_n^E}\sum_{\mathbf{x}\in\tau_n}\delta_{\mathbf{x}}\right) + \frac{1}{\tilde{d}_n}\sum_{\mathbf{x}\in(\tilde{\tau}_n\setminus\tau_n)}\delta_{\mathbf{x}}. \tag{3.7}$$

Since, thanks to (3.2), we have

$$\lim_{n\to\infty}\frac{d_n^E}{\tilde{d}_n} = 1,$$

and since the total mass of the second sum in the right-hand side of (3.7) is at most $3/d_n^E$, it follows that

$$\frac{1}{d_n^E}\sum_{\mathbf{x}\in\tau_n}\delta_{\mathbf{x}} \to \mu^{\widetilde{E}},$$

which together with (3.3) completes the proof. ∎

The main goal of this section is to establish the following result concerning algebraic curves in $\mathbb{R}^2$.

Theorem 3.2. *Let $V \subset \mathbb{R}^2$ be an algebraic variety consisting of distinct algebraic curves $V^{(j)}$, $j = \overline{1,m}$, generated by irreducible polynomials $Q^{(j)}$ of respective degrees $d^{(j)}$, and set $d^V := \sum_{j=1}^m d^{(j)}$. In addition, let $E^{(j)} \subset V^{(j)}$, $j = \overline{1,m}$, be infinite compact sets, and define $E := \bigcup_{j=1}^m E^{(j)}$. Then if each $E^{(j)}$ has an* AIM*, so does E. Moreover,*

$$\mu^E = \sum_{j=1}^m \frac{d^{(j)}}{d^V}\mu^{E^{(j)}}. \tag{3.8}$$

Proof. Since we are interested in the limiting behavior of interpolation nodes, we can assume that $n \geq d^V$. For any j, since card $E^{(j)} = \infty$ we have by Bézout's theorem $d_n^{E^{(j)}} = d_n^{V^{(j)}}$, and so (see, e.g., [2]):

$$\begin{aligned} d_n^{(j)} \;:=\; d_n^{E^{(j)}} &= \binom{n+2}{2} - \binom{n-d^{(j)}+2}{2} \\ &= d^{(j)}n - \frac{d^{(j)}(d^{(j)}-3)}{2} =: d^{(j)}n - c^{(j)} \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} d_n^E \;=\; d_n^V &= \binom{n+2}{2} - \binom{n-d^V+2}{2} \\ &= d^V n - \frac{d^V(d^V-3)}{2} =: d^V n - c^V. \end{aligned} \tag{3.10}$$

Note that $d_n^E = O(n)$ as $n \to \infty$.

Let $\tau_n = \{\mathbf{x}_{k,n}\}_{k=1}^{d_n^E} \subset E$, $n = 0, 1, \ldots,$ be sets of interpolation nodes satisfying (1.3), $\tau_n^{(j)} := \tau_n \cap E^{(j)}$. Assuming that $\tau_n^{(j)} \neq \emptyset$ and denoting $n_\tau^{(j)} :=$ card $\tau_n^{(j)}$, we represent the set $\tau_n^{(j)}$ in the form

$$\tau_n^{(j)} = \{\mathbf{x}_{k,n}^{(j)}\}_{k=1}^{n_\tau^{(j)}},$$

where $\mathbf{x}_{k,n}^{(j)} = \mathbf{x}_{l(k),n}$ for some $l(k)$, $k = \overline{1, n_\tau^{(j)}}$.

Although the curves $V^{(j)}$, $j = \overline{1, m}$, are not necessarily pairwise disjoint, by Bézout's theorem (see, e.g., [14, Th. 3.1]):

$$\operatorname{card}(V^{(j)} \cap V^{(i)}) \le d^{(j)} d^{(i)} \qquad \text{for} \quad i \ne j.$$

Hence,

$$d_n^E \le \sum_{j=1}^m n_\tau^{(j)} \le d_n^E + C^V, \tag{3.11}$$

where the constant C^V is given by

$$C^V := \sum_{1 \le i < j \le m} d^{(i)} d^{(j)}. \tag{3.12}$$

Since, for each j, the corresponding interpolation problem on the set $\tau_n^{(j)}$ is solvable in $\mathbf{\Pi}_n(E^{(j)})$, we conclude that

$$n_\tau^{(j)} \le d_n^{(j)} \qquad (j = \overline{1, m}).$$

It follows that, for any i,

$$\begin{aligned} n_\tau^{(i)} \ge d_n^E - \sum_{j \ne i} n_\tau^{(j)} \ &\ge \ d_n^E - \sum_{j \ne i} d_n^{(j)} = (d^V n - c^V) - \left(\left(\sum_{j \ne i} d^{(j)} \right) n - \sum_{j \ne i} c^{(j)} \right) \\ &= \left(d^V - \sum_{j \ne i} d^{(j)} \right) n - \left(c^V - \sum_{j \ne i} c^{(j)} \right) = d^{(i)} n - \left(c^V - \sum_{j \ne i} c^{(j)} \right) \\ &= d_n^{(i)} - \left(c^V - \sum_{j=1}^m c^{(j)} \right) = d_n^{(i)} - C^V, \end{aligned}$$

and so

$$0 \le d_n^{(i)} - n_\tau^{(i)} \le C^V. \tag{3.13}$$

Our first purpose is to obtain the limiting distribution of the normalized counting measure

$$\nu(\tau_n) := \frac{1}{d_n^E} \sum_{k=1}^{d_n^E} \delta_{\mathbf{x}_{k,n}}$$

assuming that each set $E^{(j)}$, $j = \overline{1, m}$, has an AIM.

Fix j and, for n large enough, consider the interpolation problem on $E^{(j)}$ with nodes

$$\tau_n^{(j)} = \{\mathbf{x}_{k,n}^{(j)}\}_{k=1}^{n_\tau^{(j)}}.$$

(Note that for n large enough $\tau_n^{(j)} \ne \emptyset$ thanks to (3.13) and (3.9).)

If $n_\tau^{(j)} < d_n^{(j)}$, denote by $L_{i,n}^{(j)}$, $i = \overline{1, n_\tau^{(j)}}$, the polynomials $L_{l(i),n}^E$ restricted to $E^{(j)}$. Then $L_{i,n}^{(j)} \in \mathbf{\Pi}_n(E^{(j)})$ and, clearly,

$$\max_{\mathbf{x}\in E^{(j)}} \sum_{i=1}^{m} |L_{i,n}^{(j)}(\mathbf{x})| \le \lambda_n^E, \tag{3.14}$$

where λ_n^E denotes the Lebesgue constant associated with τ_n. We now use Corollary 2.2 with $\mathbb{S} = E^{(j)}$, $m = n_\tau^{(j)}$, $p = r = d_n^{(j)}$, $\mathbb{X}_r = \mathbf{\Pi}_n(E^{(j)})$, $\tau = \tau_n^{(j)}$, and $L_i = L_{i,n}^{(j)}$ to complete the set $\tau_n^{(j)}$ by adjoining points $\{\mathbf{x}_{k,n}^{(j)}\}_{k=n_\tau^{(j)}+1}^{d_n^{(j)}}$. Then, for $E^{(j)}$, the Lebesgue constants $\lambda_n^{E^{(j)}}$ associated with the nodes $\{\mathbf{x}_{k,n}^{(j)}\}_{k=1}^{d_n^{(j)}}$ satisfy

$$\lambda_n^{E^{(j)}} \le (d_n^{(j)} - n_\tau^{(j)} + 1)\lambda_n^E + (d_n^{(j)} - n_\tau^{(j)}) \le (C^V + 1)\lambda_n^E + C^V,$$

thanks to (3.14) and (3.13). From (1.3) it then follows that

$$\limsup_{n\to\infty}(\lambda_n^{E^{(j)}})^{1/n} \le \limsup_{n\to\infty}((C^V + 1)\lambda_n^E + C^V)^{1/n} \le 1.$$

Therefore, (1.3), with E replaced by $E^{(j)}$, is also satisfied. In addition, since $E^{(j)}$ has an AIM, we get

$$\nu(\tau_n^{(j)}) \to \mu^{E^{(j)}} \qquad \text{as} \quad n \to \infty,$$

in the weak-star topology, where $\nu(\tau_n^{(j)})$ is the normalized counting measure in the points $\{\mathbf{x}_{k,n}^{(j)}\}_{k=1}^{d_n^{(j)}}$.

Note that by (3.9) and (3.10),

$$\lim_{n\to\infty} \frac{d_n^{(j)}}{d_n^E} = \frac{d^{(j)}}{d^V}. \tag{3.15}$$

Thus, we have

$$\begin{aligned}\lim_{n\to\infty} \nu(\tau_n) &= \lim_{n\to\infty} \frac{1}{d_n^E}\left[\sum_{j=1}^{m}\sum_{k=1}^{d_n^{(j)}} \boldsymbol{\delta}_{\mathbf{x}_{k,n}^{(j)}} - \sum_{j=1}^{m}\sum_{k=n_\tau^{(j)}+1}^{d_n^{(j)}} \boldsymbol{\delta}_{\mathbf{x}_{k,n}^{(j)}}\right. \\ &\qquad \left. + \left(\sum_{k=1}^{d_n^E} \boldsymbol{\delta}_{\mathbf{x}_{k,n}} - \sum_{j=1}^{m}\sum_{k=1}^{n_\tau^{(j)}} \boldsymbol{\delta}_{\mathbf{x}_{k,n}^{(j)}}\right)\right] \\ &= \lim_{n\to\infty}\left[\sum_{j=1}^{m} \frac{d_n^{(j)}}{d_n^E}\nu\left(\tau_n^{(j)}\right) - \frac{1}{d_n^E}\sum_{j=1}^{m}\sum_{k=n_\tau^{(j)}+1}^{d_n^{(j)}} \boldsymbol{\delta}_{\mathbf{x}_{k,n}^{(j)}}\right. \\ &\qquad \left. + \frac{1}{d_n^E}\left(\sum_{k=1}^{d_n^E} \boldsymbol{\delta}_{\mathbf{x}_{k,n}} - \sum_{j=1}^{m}\sum_{k=1}^{n_\tau^{(j)}} \boldsymbol{\delta}_{\mathbf{x}_{k,n}^{(j)}}\right)\right].\end{aligned} \tag{3.16}$$

Regarding the second measure we note that its total mass satisfies

$$\left\| \frac{1}{d_n^E} \sum_{j=1}^{m} \sum_{k=n_\tau^{(j)}+1}^{d_n^{(j)}} \delta_{\mathbf{x}_{k,n}^{(j)}} \right\| = \frac{1}{d_n^E} \sum_{j=1}^{m} (d_n^{(j)} - n_\tau^{(j)}) = \frac{1}{d_n^E} \left[\left(n \sum_{j=1}^{m} d^{(j)} - \sum_{j=1}^{m} c^{(j)} \right) - \sum_{j=1}^{m} n_\tau^{(j)} \right]$$

$$= \frac{1}{d_n^E} \left(d^V n - \sum_{j=1}^{m} c^{(j)} - d_n^E \right) = \frac{C^V}{d_n^E} \to 0 \qquad \text{as} \quad n \to \infty.$$

The total mass of the third term can be estimated using (3.11):

$$\left\| \frac{1}{d_n^E} \left(\sum_{k=1}^{d_n^E} \delta_{\mathbf{x}_{k,n}} - \sum_{j=1}^{m} \sum_{k=1}^{n_\tau^{(j)}} \delta_{\mathbf{x}_{k,n}^{(j)}} \right) \right\| \leq \frac{1}{d_n^E} \left(\sum_{j=1}^{m} n_\tau^{(j)} - d_n^E \right) \leq \frac{C^V}{d_n^E} \to 0 \qquad \text{as} \quad n \to \infty.$$

For the first measure on the right-hand side of (3.16), using (3.15) and (1.4) we get

$$\lim_{n\to\infty} \sum_{j=1}^{m} \frac{d_n^{(j)}}{d_n^E} \nu(\tau_n^{(j)}) = \sum_{j=1}^{m} \lim_{n\to\infty} \left(\frac{d_n^{(j)}}{d_n^E} \nu(\tau_n^{(j)}) \right) = \frac{1}{d^V} \sum_{j=1}^{m} d^{(j)} \mu^{E^{(j)}}.$$

Consequently,

$$\lim_{n\to\infty} \nu(\tau_n) = \frac{1}{d^V} \sum_{j=1}^{m} d^{(j)} \mu^{E^{(j)}},$$

and (3.8) is proved. ■

Remark 3.3. The converse of Theorem 3.2 also holds provided the set $\bigcup_{i\neq j}(E^{(i)} \cap E^{(j)})$ of intersection points of the $E^{(j)}$'s has zero μ^E-measure. In this case, the existence of μ^E implies that each $\mu^{E^{(j)}}$ exists and is given by

$$\mu^{E^{(j)}} = \frac{d^V}{d^{(j)}} \mu^E \Big|_{E^{(j)}}.$$

This can be shown by applying Corollaries 2.6 and 2.2 to a set $\tau_n^{(j)}$ of "good nodes" on $E^{(j)}$ to get a complete set of nodes on E and then using the AIM property of E along with simple arguments regarding weak-star convergence of restricted measures.

V. Totik [12] has constructed the following example which shows that Theorem 3.2 is not true for the union of arbitrary compact sets $E^{(j)}$.

Example 3.4. First we list some simple assertions which will be used in the construction:

(i) If a set E contains infinitely many triangles, then for all n the dimension of $\Pi_n(E)$ is maximal, i.e., $d_n^E = (n+1)(n+2)/2$.

(ii) If E consists of finitely many segments and of a disk, and if for a system of $(n+1)(n+2)/2$ points in E the interpolation problem is solvable for polynomials of degree $\leq n$, then $n^2/2 - O(n)$ of the interpolation points lie in the disk. In fact, each segment can only contain at most $n+1$ interpolation points.

(iii) If E consists of an interval $[a, b]$ and infinitely many triangles $T_1, T_2, \ldots$, such that these triangles converge to one of the endpoints, say to a (i.e., every neighborhood of a contains all but finitely many of the T_j's), then the limit distribution for nearly optimal interpolation points is $\boldsymbol{\delta}_a$. Indeed, by (i) the dimension of the polynomials on E is maximal, and, as we have just mentioned, each segment of E can only contain at most $n+1$ interpolation points. Thus, there are at most $O(n)$ interpolation points outside any neighborhood of a.

(iv) If τ_n, card $\tau_n = (n+1)(n+2)/2$, is a set of interpolation nodes on a set E for degree n with Lebesgue constant λ_n and if $x \in E$ is any point, then we can place a small disk $B(x, \varepsilon)$ around x so that the Lebesgue constant for τ_n on the set $B(x, \varepsilon) \cup E$ is at most $(1 + 1/n)\lambda_n$.

We now give the construction of two disjoint compact sets $E^{(1)}$ and $E^{(2)}$ with the AIM property for which the union $E^{(1)} \cup E^{(2)}$ does not have this property. $E^{(1)}$ will consist of the segment $[0, 1]$ together with infinitely many triangles converging to 0, and $E^{(2)}$ will consist of the segment $[2, 3]$ together with infinitely many triangles converging to 3. By (iii) these sets have the AIM property.

Let $E_0 = [0, 1] \cup [2, 3]$, and suppose we have already constructed E_{m-1} that consists of E_0 and some finitely many triangles. Suppose also that we have already defined two positive radii r_{m-1} and ρ_{m-1}. If m is even (odd), then consider the union $E_{m-1} \cup B(0, r_m)$ $(E_{m-1} \cup B(3, \rho_m))$ of E_m with the closed disk $B(0, r_m)$ $(B(3, \rho_m))$ of radius $r_m \leq r_{m-1}/m$ $(\rho_m \leq \rho_{m-1}/m)$ with center at 0 (at 3). For each n, take an optimal set of interpolation nodes $\tau_{n,m}$ for this set $E_{m-1} \cup B(0, r_m)$ $(E_{m-1} \cup B(3, \rho_m))$. Then $\tau_{n,m}$ contains $(n+1)(n+2)/2$ points, and by (ii) most of them lie in $B(0, r_m)$ $(B(3, \rho_m))$. Hence, if we choose n_m sufficiently large, then there holds

$$\nu(\tau_{n_m,m})(\mathbb{C} \setminus B(0, r_m)) < 1/m \qquad \text{if } m \text{ is even,} \tag{3.17}$$

$$\nu(\tau_{n_m,m})(\mathbb{C} \setminus B(3, \rho_m)) < 1/m \qquad \text{if } m \text{ is odd.} \tag{3.18}$$

By (iv), if ρ_m (r_m) is sufficiently small, then the Lebesgue constant for the nodes $\tau_{n_m,m}$ on the set $E_{m-1} \cup B(0, r_m) \cup B(3, \rho_m)$ is at most $(1 + 1/n)$-times the corresponding Lebesgue constant on the set $E_{m-1} \cup B(0, r_m)$ $(E_{m-1} \cup B(3, \rho_m))$. This means that the sequence $\{\tau_{n_m,m}\}_{m=1}^{\infty}$ is a nearly optimal sequence of interpolation nodes *for any compact set* H with the property $H \subset E_{m-1} \cup B(0, r_m) \cup B(3, \rho_m)$ and $\tau_{n_m,m} \subset H$ for all m.

Now the set $\tau_{n_m,m} \setminus E_{m-1}$ consists of finitely many points lying in $B(0, r_m)$ $(B(3, \rho_m))$. For each such point select a small triangle that passes through that point and is contained in $B(0, r_m)$ $(B(3, \rho_m))$. Let E_m be the union of all these triangles with E_{m-1}.

This completes the definition of the sets E_m and the sequences $\{r_m\}$ and $\{\rho_m\}$. We define the set $E := \bigcup_{m=0}^{\infty} E_m$. It is clear that E consists of $[0, 1] \cup [2, 3]$ and infinitely many triangles converging either to 0 or 3. Since the construction gives that $E \subset E_{m-1} \cup B(0, r_m) \cup B(3, \rho_m)$ for every m, the sequence $\{\tau_{n_m,m}\}_{\infty}$ is a nearly optimal sequence of interpolation nodes for E. But the sequence $\{\nu(\tau_{n_{2k},2k})\}_{k=1}^{\infty}$ converges to $\boldsymbol{\delta}_0$ (see (3.17)), while $\{\nu(\tau_{n_{2k+1},2k+1})\}_{k=1}^{\infty}$ converges to $\boldsymbol{\delta}_3$ (see (3.18)), so the set E does not have the AIM property.

Finally, let $E^{(1)}$ (resp., $E^{(2)}$) be the portion of E lying to the left (resp., to the right) of the line $x = 3/2$.

4. Algebraic Curves of Genus 0

In this section we shall show that compact subsets of planar irreducible algebraic curves of genus 0 have an asymptotic interpolation measure and we find a formula for this measure.

Irreducible algebraic curves of genus 0 can be characterized by the following.

Theorem 4.1 ([8, Th. 5.27]). *An irreducible algebraic curve in $\mathbb{R}^2$ is rational if and only if it has genus* 0.

Let $I = \mathbb{R}$ or $I = \mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$. Suppose $L := \{(x, y) \in \mathbb{R}^2 : Q(x, y) = 0\}$ is an irreducible (over the complex field) real algebraic curve of genus 0, and

$$x = x(t) = \frac{p_1(t)}{q(t)}, \qquad y = y(t) = \frac{p_2(t)}{q(t)} \qquad (t \in I), \tag{4.1}$$

is its *proper* rational parametrization with (possibly complex) polynomials p_1, p_2, q in the sense that for all but at most a finite number of points $(x, y) \in L$, there is a unique $t \in I$ such that $x = x(t)$, $y = y(t)$ and, conversely, for all but at most finitely many $t \in I$, $(x(t), y(t))$ is a point on L. Note that by Lüroth's theorem (see, e.g., [14, Ch. V, Th. 7.3]), every irreducible rationally parametrizable curve also has such a proper parametrization. Moreover, we may of course, in the following, assume that the greatest common divisor of the parametrizing polynomials satisfies

$$\gcd(p_1, p_2, q) = 1. \tag{4.2}$$

By [6, Th. 4.4], if L does not only consist of a single point, this implies that

$$\max(\deg(p_1), \deg(p_2), \deg(q)) = \deg(Q) = d^L =: d. \tag{4.3}$$

Let $E \subset L$ be a compact set of positive logarithmic capacity. According to (3.9), for $n \geq d$, the dimension of the linear space $\Pi_n(E)$ is given by

$$d_n^E = dn - c^L. \tag{4.4}$$

If $n < d$, then $\Pi_n(E)$ has full dimension, i.e.,

$$d_n^E = \binom{n+2}{2} = \frac{(n+2)(n+1)}{2}.$$

Note that in either case

$$\dim \Pi_{dn}(I) - c^L - 1 \leq d_n^E < \dim \Pi_{dn}(I) = dn + 1 \qquad \text{if} \quad d \geq 3. \tag{4.5}$$

(For $d = 1, 2$ one has $d_n^E = \dim \Pi_{dn}(I) = dn + 1$ for all n.)

Suppose that a scheme of interpolation points $\tau_n := \{(x_{i,n}, y_{i,n})\}_{i=1}^{d_n^E}$, $n = 0, 1 \ldots,$ is given on E such that the interpolation problem is solvable on each τ_n and the corresponding Lebesgue constants λ_n^E satisfy (1.3). According to Proposition 3.1, we can assume without loss of generality (w.l.o.g.) that E does not contain isolated points of the algebraic curve (if such points exist). Denoting by L_s the set of points of self-intersection of L we can also assume that

$$E' := \overline{\{t \in I : (x(t), y(t)) \in E \setminus L_s\}} \tag{4.6}$$

is compact. This is clearly the case if $I = \mathbb{S}^1$; otherwise, it can be easily established by reparametrizing via the circle (see Example 6.2). So for any $\left(x_{i,n}, y_{i,n}\right) \in \tau_n$, $i = \overline{1, d_n^E}$, there exists $t_{i,n} \in E'$ such that

$$(x_{i,n}, y_{i,n}) = \left(\frac{p_1(t_{i,n})}{q(t_{i,n})}, \frac{p_2(t_{i,n})}{q(t_{i,n})} \right),$$

and $t_{i,n}$ is unique except, possibly, for the case when $(x_{i,n}, y_{i,n})$ is a point of self-intersection of the curve (in such a case, any of the preimages can be chosen).

We define a discrete measure $\nu(\tilde{\tau}_n)$ on E' associated with $\nu(\tau_n)$ as the normalized counting measure of the set $\tilde{\tau}_n := \{t_{i,n}\}_{i=1}^{d_n^E}$, i.e.,

$$\nu(\tilde{\tau}_n) := \frac{1}{d_n^E} \sum_{i=1}^{d_n^E} \delta_{t_{i,n}}.$$

The subsequent results are formulated in terms of potential theoretic notions, such as *weighted equilibrium measure, Robin equilibrium measure, balayage*. For their introduction and discussion, the reader is referred to [10].

Theorem 4.2. *If E' is compact and conditions* (1.3) *and* (4.3) *hold, then the weak-star limit as $n \to \infty$ of the measures $\nu(\tilde{\tau}_n)$ exists, and it is the weighted equilibrium measure μ_w on E' with the weight*

$$w(t) = \frac{1}{|q(t)|^{1/d}}. \tag{4.7}$$

More precisely,

$$\lim_{n\to\infty} \nu(\tilde{\tau}_n) = \mu_w = \frac{\deg(q)}{d} \widehat{\nu_q} + \left(1 - \frac{\deg(q)}{d}\right) \omega_{E'}, \tag{4.8}$$

where $\omega_{E'}$ is the Robin equilibrium distribution on E' and $\widehat{\nu_q}$ denotes the balayage of the normalized counting measure ν_q of the zeros of q onto E'.

Corollary 4.3. *If E is a compact subset of an algebraic curve of genus* 0 *and E has positive logarithmic capacity, then E has an* AIM. *Moreover, μ^E is given by*

$$\mu^E(B) = \mu_w(B'), \qquad B' := \{t \in E' : (x(t), y(t)) \in B\}, \tag{4.9}$$

for any Borel subset B of E, where μ_w is as in (4.8).

Remark 4.4. Frequently, it is possible to find the density of μ_w explicitly. For instance,

(a) if E' is an interval $[a, b]$ and q has only real roots $t_1, \ldots, t_{d_q}$, $d_q := \deg(q)$, which due to (4.2) are not in $[a, b]$, then μ_w has density

$$\frac{1}{\pi d} \frac{1}{\sqrt{(t-a)(b-t)}} \left(\sum_{j=1}^{d_q} \frac{\sqrt{(t_j - a)(t_j - b)}}{|t - t_j|} + d - d_q \right)$$

with respect to the Lebesgue measure on $[a, b]$ (see [10, p. 122]);

(b) if the curve L is compact (as in Examples 6.4 and 6.5 of Section 6) and $E = L$, using the complex parametrization and denoting the roots of q by $\zeta_j := r_j e^{i\theta_j}$, $j = \overline{1, d_q}$, for the density of μ_w with respect to the arclength on the unit circle, we obtain the formula

$$\frac{1}{d} \left(\sum_{j=1}^{d_q} |P(z, \zeta_j)| + \frac{d - d_q}{2\pi} \right) \frac{1}{2\pi d} \left(\sum_{j=1}^{d_q} \frac{|1 - r_j^2|}{1 - 2r_j \cos(\varphi - \theta_j) + r_j^2} + d - d_q \right),$$

$$z := e^{i\varphi},$$

where $P(\cdot, \cdot)$ is the Poisson kernel.

The proof of Theorem 4.2 (see below) can be carried over to dimensions $s > 2$ and yields the following.

Theorem 4.5. *Suppose the curve L in $\mathbb{R}^s$ is rationally parametrizable via*

$$x_1 = \frac{p_1(t)}{q(t)}, \ldots, x_s = \frac{p_s(t)}{q(t)} \qquad (t \in I), \tag{4.10}$$

and assume that for a compact set $E \subset L$, whose preimage has positive capacity, we have

$$d := \max\{\deg(p_1), \ldots, \deg(p_s), \deg(q)\} = \frac{\dim \boldsymbol{\Pi}_n(E)}{n} + O\left(\frac{1}{n}\right).$$

Then the weak-star limit of the normalized counting measures associated with interpolation nodes on E having Lebesgue constants of polynomial growth can be characterized as the image under the transformation (4.10) *of the $|q|^{-1/d}$-weighted equilibrium measure on the* (*w.l.o.g. compact*) *preimage of E under* (4.10).

Proof of Theorem 4.2. As in Section 3, $\{L_{i,n}^E(x, y)\}_{i=1}^{d_n^E}$ denotes the basis of Lagrange polynomials associated with τ_n. We have

$$L_{i,n}^E \left(\frac{p_1(t)}{q(t)}, \frac{p_2(t)}{q(t)} \right) = \frac{P_{i,dn}(t)}{q(t)^n}$$

with polynomials $P_{i,dn}(t)$ in t of degree at most dn.

If $d \geq 3$, then $d_n^E < dn+1$ (see (4.5)) and we use Corollary 2.2 with $\mathbb{S} = E'$, $m = d_n^E$, $p = r = dn+1$, $\mathbb{X}_r = q^{-n}\Pi_{dn}(I)$, the weighted space of polynomials of degree at most dn on I, $\tau = \tilde{\tau}_n$, and $L_i = q^{-n}P_{i,dn}$ to find points $t_{i,n} \in E'$, $i = \overline{d_n^E + 1, dn+1}$, with Lagrange fundamental functions $l_i(t) \in \mathbb{X}_r$, $i = \overline{1, dn+1}$, satisfying

$$\begin{aligned}\sum_{i=1}^{dn+1} |l_i(t)| \;&\leq\; (dn - d_n^E + 2)\max_{t\in E'} \sum_{i=1}^{d_n^E} \frac{|P_{i,dn}(t)|}{|q(t)|^n} + (dn - d_n^E + 1)\\ &\leq\; (c^L + 2)\max_{\mathbf{x}\in E} \sum_{i=1}^{d_n^E} |L_{i,n}^E(\mathbf{x})| + (c^L + 1) \leq 2(c^L + 2)\lambda_n^E \qquad (t \in E').\end{aligned}$$

For $d = 2$ or $d = 1$ we have $d_n^E = dn+1$, and there is no need to adjoin additional points. Next, with w defined as in (4.7), consider the sequence of w-weighted kth Chebyshev polynomials T_k^w on E' which by definition are the monic polynomials of degree k with minimal weighted norm $\|w^k T_k^w\|_{E'}$ (see [10, p. 163]). Using the standard estimation of the interpolation error by the interpolation norm and error in best approximation, we find

$$\left\| w(t)^{dn} \prod_{j=1}^{dn+1} (t - t_{j,n}) \right\|_{E'} \leq \left(1 + \sup_{t\in E'} \sum_{j=1}^{dn+1} |l_j(t)|\right) \|w^{dn} T_{dn+1}^w\|_{E'}.$$

By assumption (1.3) and estimate (4.11) it follows that

$$\limsup_{n\to\infty} \left\| w(t)^{dn} \prod_{j=1}^{dn+1} (t - t_{j,n}) \right\|_{E'}^{1/(dn)} \leq \limsup_{n\to\infty} \|w^{dn} T_{dn+1}^w\|_{E'}^{1/(dn)},$$

which implies that the weak-star limit distribution of the normalized counting measures of the points $\{t_{i,n}\}_{i=1}^{dn+1}$ is the w-weighted equilibrium distribution on E' (combine [10, Th. III.3.1] and [10, Th. III.4.2]). Finally we remark that removing the previously added points $t_{i,n}$, $i = \overline{d_n^E + 1, dn+1}$, which are of uniformly bounded cardinality, does not change the weak-star limiting behavior. ∎

Combining Theorems 4.2 and 3.2 we obtain

Corollary 4.6. *Let V and E be as in Theorem* 3.2, *and assume that the curves $V^{(j)}$, $j = \overline{1, m}$, are rational. Then E has the* AIM *property and μ^E is given by* (3.8), *where, for each j, the measure $\mu^{E^{(j)}}$ is defined on $E^{(j)}$ via* (4.9).

5. Constructing "Good Points" for Interpolation

Now we prove an inverse statement to Theorem 4.2. Namely, assuming that the asymptotic interpolation measure μ^E is known for a compact subset E of an algebraic curve of genus 0, our purpose is to show how one can easily obtain "good points" for interpolation on E in the sense that the corresponding sequence of Lebesgue constants satisfies (1.3).

For a set τ of m points on I we set

(5.1) $$\varepsilon_\tau := \min_{\substack{u,v\in\tau\\ u\neq v}} |u-v| \qquad \text{and} \qquad \nu(\tau) := \frac{1}{m}\sum_{t\in\tau} \boldsymbol{\delta}_t.$$

Recall that the *discrepancy* of a signed Borel measure σ on I with compact support $S \subset I$ is defined by

$$D[\sigma] := \sup_J |\sigma(J)|,$$

where the supremum is taken over all intervals (arcs) $J \subset I$ (see [10, Sec. VIII.7]).

We need the following simple inequality (which is analogous to Koksma's inequality [7, p. 143]) regarding the discrepancy of signed measures:

Let $S \subset \mathbb{R}$ be a compact set, let σ be a signed measure on S, and let $f(t) \geq 0$ be monotonic and continuous on S. Then

(5.2) $$\left|\int_S f(t)\, d\sigma(t)\right| \leq D[\sigma] \max_{t\in S} |f(t)|.$$

For a discrete measure $\sigma := \sum_{k=1}^{p} a_k \boldsymbol{\delta}_{t_k}$ the estimate (5.2) follows immediately from Abel's identity

$$\sum_{k=1}^{p} a_k b_k = \sum_{k=1}^{p-1} A_k(b_k - b_{k+1}) + A_p b_p, \qquad A_k := \sum_{j=1}^{k} a_j,$$

and the general case can now be verified by discretizing σ.

The estimate (5.2) also holds for $S \subset \mathbb{S}^1$ with a suitable definition of monotonicity.

Lemma 5.1. *Let $S \subset I$ be a compact set of positive logarithmic capacity, and let μ_w denote the weighted equilibrium measure on S for the continuous positive weight $w(t)$. Suppose that*:

(a) $S = S_w := \operatorname{supp} \mu_w$; *and*
(b) *there exist constants $C > 0$, $\rho > 0$, and $c \in (0,1)$ such that, for any Borel set $U \subset S$ with one-dimensional Lebesgue measure $|U| \leq c$,*

$$\mu_w(U) \leq C(-\log |U|)^{-(1+\rho)}.$$

Let a sequence $\gamma_k = \{t_{i,k}\}_{i=1}^{k+1} \subset S$, $k = 0, 1, \ldots$, be such that:

(c) $D[\nu(\gamma_k) - \mu_w] \log \varepsilon_{\gamma_k} \to 0$ *as $k \to \infty$.*

Then the sequence of Lebesgue constants λ_k corresponding to γ_k in the space $w^k\boldsymbol{\Pi}_k(I)$, $k = 0, 1, \ldots$, satisfies

(5.3) $$\limsup_{k\to\infty} \lambda_k^{1/k} \leq 1.$$

Proof. Note that condition (c) implies $\nu(\gamma_k) \to \mu_w$ in the weak-star sense as $k \to \infty$. Moreover, for fixed i, the sets $\gamma_{i,k} := \gamma_k \backslash \{t_{i,k}\}$ have the same limit distribution as the γ_k's, i.e.,

$$\nu_{i,k} := \frac{1}{k} \sum_{t \in \gamma_{i,k}} \delta_t \to \mu_w \qquad \text{as} \quad k \to \infty.$$

We consider the case when $I = \mathbb{R}$. For a parametrization using the circle $\mathbb{S}^1$ the proof is similar.

It is easy to verify from the definition of discrepancy, and the fact that μ_w is absolutely continuous with respect to Lebesgue measure, that

$$D[\nu(\gamma_k) - \mu_w] \geq \frac{1}{k+1}. \tag{5.4}$$

In particular, it follows from (c) that $\log(1/\varepsilon_{\gamma_k}) = o(k)$ as $k \to \infty$, i.e., points in γ_k cannot become exponentially close to each other. Also, since

$$\nu_{i,k} - \mu_w = (\nu(\gamma_k) - \mu_w) + (\nu(\gamma_k) - \delta_{t_{i,k}})/k,$$

using (5.4) we get

$$D[\nu_{i,k} - \mu_w] \leq D[\nu(\gamma_k) - \mu_w] + 2/k \leq (3 + 2/k) D[\nu(\gamma_k) - \mu_w]. \tag{5.5}$$

The weighted fundamental Lagrange polynomials corresponding to γ_k are given by

$$L_{i,k}(x) := \frac{w(x)^k}{w(t_{i,k})^k} \prod_{j \neq i} \frac{x - t_{j,k}}{t_{i,k} - t_{j,k}} \qquad (i = \overline{1, k+1},\ k = 0, 1, \ldots).$$

Let $x_{i,k} \in S_w$ be a point where $|L_{i,k}(x)|$ attains its maximum on S_w. Then

$$\begin{aligned} I_i := \log(\|L_{i,k}\|_{S_w}^{1/k}) \ &= \ \log\left|\frac{w(x_{i,k})}{w(t_{i,k})}\right| + \frac{1}{k} \sum_{j \neq i} \log\left|\frac{x_{i,k} - t_{j,k}}{t_{i,k} - t_{j,k}}\right| \\ &= \ -Q(x_{i,k}) + Q(t_{i,k}) + \int_{S_w} \log\left|\frac{x_{i,k} - t}{t_{i,k} - t}\right| d\nu_{i,k}(t) \\ &= \ -Q(x_{i,k}) + Q(t_{i,k}) - U^{\mu_w}(x_{i,k}) + U^{\mu_w}(t_{i,k}) \\ &\qquad + \int_{S_w} \log\left|\frac{x_{i,k} - t}{t_{i,k} - t}\right| (d\nu_{i,k} - d\mu_w)(t), \end{aligned}$$

where $Q(x) := -\log w(x)$ and U^{μ_w} denotes the logarithmic potential of μ_w.

It can be shown that (b) implies that U^{μ_w} is continuous on I, and so

$$U^{\mu_w}(x) + Q(x) \equiv \text{const} \quad \text{on } S_w$$

(see [10, Th. I.4.4]). Write $\varepsilon_k := \varepsilon_{\gamma_k}$. Then putting $B_{i,k} := B(t_{i,k}, \varepsilon_k) \cup B(x_{i,k}, \varepsilon_k/2)$, where $B(x, r) := (x - r, x + r)$, we get

$$\begin{aligned} I_i &= \int_{S_w} \log\left|\frac{x_{i,k} - t}{t_{i,k} - t}\right| (d\nu_{i,k} - d\mu_w)(t) \\ &= \left(\int_{S_w \setminus B_{i,k}} + \int_{B_{i,k}}\right) \log\left|\frac{x_{i,k} - t}{t_{i,k} - t}\right| (d\nu_{i,k} - d\mu_w)(t) \\ &= \int_{S_w \setminus B_{i,k}} \log\left|\frac{x_{i,k} - t}{t_{i,k} - t}\right| (d\nu_{i,k} - d\mu_w)(t) + \int_{B_{i,k}} \log\left|\frac{t_{i,k} - t}{x_{i,k} - t}\right| d\mu_w(t) \\ &\quad + \int_{B(x_{i,k}, \varepsilon_k/2)} \log\left|\frac{x_{i,k} - t}{t_{i,k} - t}\right| d\nu_{i,k}(t) =: I_i^{(1)} + I_i^{(2)} + I_i^{(3)}. \end{aligned}$$

The second integral can be estimated by

$$I_i^{(2)} \le \int_{B(x_{i,k}, \varepsilon_k)} \log\left|\frac{t_{i,k} - t}{x_{i,k} - t}\right| d\mu_w(t) \le \int_{B(x_{i,k}, \varepsilon_k)} \log\frac{\operatorname{diam} S}{|x_{i,k} - t|}\, d\mu_w(t).$$

Then by partitioning $B(x_{i,k}, \varepsilon_k)$ via the intervals $\{t : 2^{m+1} \log \varepsilon_k \le \log|x_{i,k} - t| < 2^m \log \varepsilon_k\}$, $m = 0, 1, \ldots$, and using (b) we get

$$I_i^{(2)} \le C_1 \left(\log \frac{C_2}{\varepsilon_k}\right)^{-\rho} \tag{5.6}$$

with constants $C_1, C_2 > 0$ independent of i, k.

For the third integral we have

$$I_i^{(3)} = \begin{cases} 0 & \text{if } \operatorname{dist}(x_{i,k}, \gamma_{i,k}) \ge \varepsilon_k/2, \\ \dfrac{1}{k} \log\left|\dfrac{x_{i,k} - t^*}{t_{i,k} - t^*}\right| < -\dfrac{\log 2}{k} & \text{if } \operatorname{dist}(x_{i,k}, \gamma_{i,k}) < \varepsilon_k/2, \end{cases} \tag{5.7}$$

where t^* denotes the point of $\gamma_{i,k}$ closest to $x_{i,k}$.

Finally, assuming for definiteness that $t_{i,k} < x_{i,k}$ and applying (5.2) to $S \cap \{t \le t_{i,k} - \varepsilon_k\}$, $S \cap \{t \ge x_{i,k} + \varepsilon_k/2\}$, and $S \cap \{t_{i,k} + \varepsilon_k \le t \le x_{i,k} - \varepsilon_k/2\}$ (separately for $t \le (x_{i,k} + t_{i,k})/2$ and $t \ge (x_{i,k} + t_{i,k})/2$) if $x_{i,k} - t_{i,k} \ge 3\varepsilon_k/2$ we obtain, from (5.5), that

$$|I_i^{(1)}| \le 4 \log\left(\frac{2 \operatorname{diam} S}{\varepsilon_k}\right) D[\nu_{i,k} - \mu_w] \le C_3 \log\left(\frac{C_4}{\varepsilon_k}\right) D[\nu(\gamma_k) - \mu_w], \tag{5.8}$$

where C_3 and C_4 are constants independent of i, k.

So it follows from (5.6), (5.7), (5.8), and (5.2) that

$$\begin{aligned} \log(\lambda_k^{1/k}) &\le \frac{1}{k} \log\left((k+1) \max_{i=\overline{1,k+1}} \|L_{i,k}\|_{S_w}\right) \\ &= \frac{\log(k+1)}{k} + \max_{i=\overline{1,k+1}} I_i \le \frac{\log(k+1)}{k} + \max_{i=\overline{1,k+1}} (I_i^{(1)} + I_i^{(2)}) \\ &\le \frac{\log(k+1)}{k} + C_3 \log\left(\frac{C_4}{\varepsilon_k}\right) D[\nu(\gamma_k) - \mu_w] + C_1 \left(\log\frac{C_2}{\varepsilon_k}\right)^{-\rho} \to 0 \end{aligned}$$

as $k \to \infty$, which implies (5.3). ■

Theorem 5.2. *Let L be a rational curve parametrized by* (4.1), *let d denote the degree of L as defined in* (4.3), *let $E \subset L$ be a compact subset of positive logarithmic capacity, let μ^E be the limit measure determined in Corollary* 4.3, *and let $E_{\mu^E} := \operatorname{supp} \mu^E$. Suppose $\rho := \operatorname{card}(E \setminus E_{\mu^E}) < \infty$ and that the corresponding weighted measure μ_w on I satisfies condition* (b) *of Lemma* 5.1. *For $n = 0, 1, \ldots$, let $\gamma_n = \{t_{i,n}\}_{i=1}^{dn+1} \subset S_w$ be a set of interpolation nodes satisfying* (c) *of Lemma* 5.1, *and define $\tau_n := \{(x(t), y(t)) : t \in \gamma_n\}$. Then there exist sets $\tilde{\tau}_n$, $n \geq d^L$, of interpolation nodes on E such that*:

(i) $\operatorname{card} \tilde{\tau}_n = d_n^E$ *and* $\operatorname{card}(\tau_n \setminus \tilde{\tau}_n) \leq \rho + c^L + 1$, *where c^L is defined in* (4.4); *and*
(ii) *the interpolation problem is solvable on $\tilde{\tau}_n$ and the corresponding Lebesgue constants satisfy*

$$\limsup_{n \to \infty} \tilde{\lambda}_n^{1/n} \leq 1.$$

Proof. By Corollary 4.3, for the set $(E_{\mu^E})'$ defined for E_{μ^E} via (4.6) we have

$$S_w := \operatorname{supp} \mu_w = (E_{\mu^E})'.$$

So according to Lemma 5.1 (with $S = S_w$), the Lebesgue constants λ_n corresponding to γ_n satisfy (5.3).

If $\rho > 0$, denote by $\hat{\Pi}_n(E)$ the subspace of $\Pi_n(E)$ consisting of polynomials vanishing on $E \setminus E_{\mu^E}$. Then

$$d_n^E - \rho \leq \hat{d}_n^E := \dim \hat{\Pi}_n(E) \leq d_n^E.$$

(For $\rho = 1$ we have (3.4), and for $\rho > 1$ it can be shown by induction.)

If $d \geq 3$ or $d < 3$ and $\rho > 0$, for each n, let B_n be a basis in $\hat{\Pi}_n(E)$, and denote

$$B_n' := \{p(x(t), y(t)) : p(x, y) \in B_n\}, \qquad \mathcal{B}_n' := \operatorname{span} B_n'. \tag{5.9}$$

Applying Corollary 2.6 with $\mathbb{S} = S_w$, $m = \hat{d}_n^E$, $p = dn + 1$, $\tau = \gamma_n$, $\mathbb{Y}_p = q^{-n}\Pi_{dn}(I)$, and $\mathbb{X}_m = \mathcal{B}_n'$ we get a subset $\hat{\gamma}_n = \{\hat{t}_{i,n}\}_{i=1}^{\hat{d}_n^E}$ of γ_n such that its Lebesgue constant $\hat{\lambda}_n$ satisfies (2.13), i.e.,

$$\hat{\lambda}_n \leq ((\hat{d}_n^E + 1)(dn + 1 - \hat{d}_n^E) + 1)\lambda_n \leq ((d_n^E + 1)(c^L + \rho + 1) + 1)\lambda_n \leq C_1 n \lambda_n,$$

where $C_1 > 0$ is a constant independent of n. Hence, if

$$\hat{L}_{l,dn}(t) = \sum_{p_j \in B_n} c_{l,j}\, p_j(x(t), y(t)) \qquad (l = \overline{1, \hat{d}_n^E}),$$

are the fundamental Lagrange functions corresponding to $\widehat{\gamma}_n$, then the polynomials

$$\hat{L}_{l,n}(x, y) := \sum_{p_j \in B_n} c_{l,j}\, p_j(x, y) \in \hat{\Pi}_n(E) \tag{5.10}$$

are the fundamental Lagrange polynomials corresponding to $\hat{\tau}_n := \{(x(t), y(t)) : t \in \hat{\gamma}_n\} \subset \tau_n$ having the same Lebesgue constant $\hat{\lambda}_n$ on E_{μ^E} and, hence, on E.

Now, if $\hat{d}_n^E < d_n^E$, we use Corollary 2.2 with $\mathbb{S} = E$, $m = \hat{d}_n^E$, $p = r = d_n^E$, $\mathbb{X}_r = \Pi_n(E)$, and $\tau = \hat{\tau}_n$ to get a complete set $\tilde{\tau}_n$ of d_n^E nodes on E and the fundamental Lagrange polynomials $\tilde{L}_{i,n}^E$, $i = \overline{1, d_n^E}$, with the Lebesgue constants satisfying

$$\tilde{\lambda}_n \leq (d_n^E - \hat{d}_n^E + 1)\hat{\lambda}_n + (d_n^E - \hat{d}_n^E) \leq C_2 n \lambda_n.$$

Thus, (ii) follows from Lemma 5.1. Clearly,

$$\operatorname{card}(\tau_n \backslash \tilde{\tau}_n) \leq (dn + 1) - \hat{d}_n^E \leq \rho + c^L + 1. \qquad \blacksquare$$

Remark 5.3. If $d \geq 3$, then $d_n^E < dn + 1$ (see (4.5)). This fact is essential for the solvability statement in (ii). Furthermore, the use of Corollary 2.6 in choosing a set $\hat{\gamma}_n$ of "good nodes" for E among $dn + 1$ of those on E' is extremely helpful. We illustrate these assertions in Example 5.4.

Theorem 4.2 and the proof of Theorem 5.2 suggest the following algorithm for the construction of "good" interpolation points on a compact subset E of an algebraic curve L of genus 0 (provided, of course, that the set E itself is "good enough" in the sense that its AIM measure μ^E is known and satisfies the conditions of Theorem 5.2).

- Given n, choose a set γ_n of $d^L n+1$ points on E' such that condition (c) of Lemma 5.1 is satisfied. To make such a choice, one can, say, discretize μ_w using the method described in [10, Sec. VI.4]. Alternatively, one can use zeros of weighted Chebyshev polynomials (or other weighted monic polynomials not growing exponentially fast on E').
- With E_{i} denoting the set of isolated points of E and $\rho := \operatorname{card} E_{\mathrm{i}}$, choose a set B_n of $d_n^E - \rho$ linearly independent polynomials in $\Pi_n(E)$ vanishing on E_{i} (this can be done inductively using (3.5)) and use Remark 2.7 (with $B = B_n'$ defined in (5.9) and $\mathbb{X}_m = \mathcal{B}_n'$) to select a subset $\hat{\gamma}_n$ of γ_n consisting of $d_n^E - \rho$ points.
- Apply Remark 2.4 with $\mathbb{X}_r = \Pi_n(E)$, $\tau = \hat{\tau}_n := \{(x(t), y(t)) : t \in \hat{\gamma}_n\}$, and any (say, monomial) basis $\{p_i(x, y)\}_{i=1}^{d_n^E}$ in $\Pi_n(E)$ to add ρ missing nodes, thereby constructing a complete set $\tilde{\tau}_n$ of d_n^E nodes on E.

Then, for any n, the interpolation problem with the set of nodes $\tilde{\tau}_n$ is solvable in $\Pi_n(E)$ and the sequence λ_n^E, $n \geq d^L$, of the corresponding Lebesgue constants satisfies (1.3), i.e., points in the sequence $\{\tilde{\tau}_n\}$ are "good points" for interpolation. Actually, the above algorithm is designed not only to achieve (1.3) but to preserve the slow growth of the Lebesgue constants for the parametric interval problem.

Example 5.4. Let $E := \{(x, y) : y = x^3, x \in [-1, 1]\}$ be the subarc of the cubic curve with the natural parametrization $x = t$, $y = t^3$, $t \in \mathbb{R}$. Then $d_n^E = 3n$, $\rho = 0$, $E' = [-1, 1]$, $w(t) \equiv 1$, and $\mu := \mu_w$ is the Robin (arcsine) measure on $[-1, 1]$. It is well-known that the zeros of the Chebyshev polynomials $T_k(t) = \cos(k \arccos t)$, $k = 1, 2, \ldots$, are uniformly distributed with respect to μ on E' in the weak-star sense and, clearly, satisfy the discrepancy-separation condition (c) of Lemma 5.1. Denote by $\hat{\gamma}_n$ the set of zeros of T_{3n}, and let $\hat{\tau}_n := \{(t, t^3) : t \in \hat{\gamma}_n\}$. Since, for any n, $T_{3n} \in \mathcal{B}_n' = \operatorname{span}\{1, t, \ldots, t^{3n-2}, t^{3n}\}$, the polynomial $\hat{T}_n \in \mathcal{B}_n (= \Pi_n(E))$ corresponding to T_{3n} via

(5.10) is identically zero on $\hat{\tau}_n$. Therefore, the interpolation problem is *not* solvable on τ_n for any $n = 1, 2, \ldots$. So we should start with a set γ_n of $3n+1$ nodes on $[-1, 1]$ and then apply Remark 2.7 in order to choose a node to be omitted.

It is worth mentioning that the procedure described in Remark 2.7 can be easily implemented in this particular case. Indeed, for a set γ of k points on $[-1, 1]$, let V_γ denote the usual kth order Vandermonde determinant corresponding to γ. We shall choose $\gamma_n = \{t_{k,n}\}_{k=1}^{3n+1}$ to be the set of min/max points of T_{3n} on $[-1, 1]$, i.e., $t_{k,n} := \cos((k-1)\pi/(3n))$. Denoting by Δ_l, $l = \overline{1, 3n+1}$, the determinants presented in (2.14), we note that Δ_l can be obtained from V_{γ_n} by omitting the $3n$th row and the lth column. Let $P(t, l) = c_{l,0}t^{3n} + c_{l,1}t^{3n-1} + \cdots$ be the polynomial obtained by replacing in V_{γ_n} the node $t_{l,n}$ by t. Then $\Delta_l = (-1)^{3n+l}c_{l,1}$. Since $P(t_{k,n}, l) = 0$ for $k \neq l$ and $P(t_{l,n}, l) = V_{\gamma_n}$, we have

$$P(t, l) = \frac{V_{\gamma_n}\Omega_{3n+1}(t)}{\Omega'_{3n+1}(t_{l,n})(t - t_{l,n})} = \frac{2^{3n-1}3nV_{\gamma_n}}{\Omega'_{3n+1}\left(t_{l,n}\right)} \prod_{\substack{1 \le k \le 3n+1 \\ k \neq l}} (t - t_{k,n})$$

$$= c_{l,0}t^{3n} + \frac{2^{3n-1}3nV_{\gamma_n}t_{l,n}}{\Omega'_{3n+1}(t_{l,n})}t^{3n-1} + \cdots,$$

where $\Omega_{3n+1}(t) = (t^2 - 1)T'_{3n}(t)$. Consequently,

$$\Delta_l = (-1)^{3n+l}\frac{2^{3n-1}3nV_{\gamma_n}t_{l,n}}{\Omega'_{3n+1}\left(t_{l,n}\right)} \qquad (l = \overline{1, 3n+1}).$$

With $t = \cos\theta$, we have $\Omega_{3n+1}(t) = 3n\sin\theta\sin(3n\theta)$, and so

$$\Omega'_{3n+1}(t) = -\frac{3n}{\sin\theta}(\cos\theta\sin(3n\theta) + 3n\sin\theta\cos(3n\theta)).$$

Thus

$$\Delta_l = \frac{(-1)^{3n}2^{3n-1}V_{\gamma_n}t_{l,n}}{3nm_l},$$

where $m_l = 2$ if $l = 1, 3n+1$, and $m_l = 1$ otherwise. So the node to be excluded is $t_{2,n} = \cos(\pi/3n)$ (or $-t_{2,n} = t_{3n,n}$).

Similar arguments can be applied in the case of an arbitrary compact subset E on the cubic curve without isolated points. Given $\gamma_n = \{t_{k,n}\}_{k=1}^{3n+1} \subset E'$, we have

$$|\Delta_l| = \left|V_{\gamma_n}\frac{t_{l,n} - s_{\gamma_n}}{\Omega'_{3n+1}(t_{l,n})}\right|,$$

where $\Omega_{3n+1}(t) := \prod_{k=1}^{3n+1}(t - t_{k,n})$ and $s_{\gamma_n} := \sum_{k=1}^{3n+1} t_{k,n}$. So the problem of maximizing the determinants is an easy task.

Now let V be as defined in Theorem 3.2. Assuming that its components $V^{(j)}$, $j = \overline{1, m}$, are rational, by Corollary 4.6 we know the limiting distribution of "good" interpolation points on V. This fact along with Theorem 5.2 allows us to generate such points.

Theorem 5.5. *With the notation of Theorem* 3.2, *suppose that* $m \geq 2$ *and that each curve* $V^{(j)}$, $j = \overline{1, m}$, *has genus* 0 *and satisfies the assumptions of Theorem* 5.2. *For each* j, *let* $\gamma_n^{(j)} = \{t_{i,n}^{(j)}\}_{i=1}^{d^{(j)}n+1} \subset S_{w^{(j)}}$ *be sets of interpolation nodes satisfying condition* (c) *of Lemma* 5.1, *and define* $\tau_n^{(j)} := \{(x^{(j)}(t), y^{(j)}(t))\colon t \in \gamma_n^{(j)}\}$, $\tau_n := \bigcup_{j=1}^m \tau_n^{(j)}$. *Then there exist sets* $\tilde{\tau}_n$, $n \geq d^V$, *of interpolation nodes on* E *such that*:

(i) $\operatorname{card} \tilde{\tau}_n = d_n^E$ *and* $\operatorname{card}(\tau_n \setminus \tilde{\tau}_n) \leq \sum_{j=1}^m \rho^{(j)} + c^V + m$; *and*
(ii) *the interpolation problem is solvable on* $\tilde{\tau}_n$ *and the corresponding sequence of Lebesgue constants satisfies*

$$\limsup_{n\to\infty} \tilde{\lambda}_n^{1/n} \leq 1. \tag{5.11}$$

Proof. For each j, we apply Theorem 5.2 to the set $E^{(j)}$ in order to determine sets of nodes $\tilde{\tau}_n^{(j)}$, $n \geq d^{(j)}$, satisfying

$$\operatorname{card} \tilde{\tau}_n^{(j)} = d_n^{(j)}, \qquad \operatorname{card}(\tau_n^{(j)} \setminus \tilde{\tau}_n^{(j)}) \leq \rho^{(j)} + c^{(j)} + 1, \tag{5.12}$$

where $c^{(j)}$ is defined in (3.9), and

$$\limsup_{n\to\infty} (\tilde{\lambda}_n^{(j)})^{1/n} \leq 1. \tag{5.13}$$

With $Q^{(k)}$ defined as in Theorem 3.2, for $n \geq d^V$ denote

$$\hat{\mathbf{\Pi}}_n(E^{(j)}) := \mathbf{\Pi}_{n-d^V+d^{(j)}}(E^{(j)}) \prod_{\substack{1\leq k\leq m\\ k\neq j}} Q^{(k)}|_{V^{(j)}}.$$

Then $\hat{\mathbf{\Pi}}_n(E^{(j)})$ is a $d_{n-d^V+d^{(j)}}^{(j)}$-dimensional subspace of $\mathbf{\Pi}_n(E^{(j)})$. We consider polynomials in $\hat{\mathbf{\Pi}}_n(E^{(j)})$ as polynomials in $\mathbf{\Pi}_n(E)$ vanishing on $E \setminus E^{(j)}$. Applying Corollary 2.6 with $p = d_n^{(j)}$, $m = m^{(j)} := d_{n-d^V+d^{(j)}}^{(j)}$, $\mathbb{Y}_p = \mathbf{\Pi}_n(E^{(j)})$, $\mathbb{X}_m = \hat{\mathbf{\Pi}}_n(E^{(j)})$, and $\tau = \tilde{\tau}_n^{(j)}$ we find a set $\widehat{\tilde{\tau}_n^{(j)}} \subset \tilde{\tau}_n^{(j)}$ of $m^{(j)}$ interpolation nodes such that the interpolation problem is solvable in $\hat{\mathbf{\Pi}}_n(E^{(j)})$ and the corresponding Lebesgue constant satisfies

$$\widehat{\tilde{\lambda}_n^{(j)}} \leq \tilde{\lambda}_n^{(j)}((m^{(j)}+1)(p-m^{(j)})+1) \leq (d^V)^2 d_n^{(j)} \tilde{\lambda}_n^{(j)}.$$

Set

$$\hat{\tau}_n := \bigcup_{j=1}^m \widehat{\tilde{\tau}_n^{(j)}}.$$

Then using (3.9), (3.10), and (3.12), after some computations, we get

$$\begin{aligned} \operatorname{card} \hat{\tau}_n &= \sum_{j=1}^m m^{(j)} = \sum_{j=1}^m d_{n-d^V+d^{(j)}}^{(j)} \\ &= \sum_{j=1}^m [d^{(j)}(n-d^V+d^{(j)}) - c^{(j)}] = d_n^E - C^V, \end{aligned} \tag{5.14}$$

where C^V is defined by (3.12).

Further, with $\hat{L}_{k,n}^{(j)} \in \hat{\Pi}_n(E^{(j)})$, $k = \overline{1, m^{(j)}}$, denoting the fundamental Lagrange polynomials corresponding to $\widehat{\tilde{\tau}_n^{(j)}}$, we define the fundamental Lagrange polynomials $\hat{L}_{i,n}$ for $\hat{\tau}_n$ by setting

$$\hat{L}_{i,n} := \hat{L}_{k,n}^{(j)}$$

if the corresponding node $\mathbf{x}_i$ from $\hat{\tau}_n$ satisfies $\mathbf{x}_i = \mathbf{z}_k \in \widehat{\tilde{\tau}_n^{(j)}}$. Then the Lebesgue constant $\hat{\lambda}_n$ (on E) for the set of nodes $\hat{\tau}_n$ satisfies

$$\hat{\lambda}_n \leq \max_{1 \leq j \leq m} \widehat{\tilde{\lambda}_n^{(j)}} \leq (d^V)^2 d_n^V \max_{1 \leq j \leq m} \tilde{\lambda}_n^{(j)} \leq 2(d^V)^3 n \max_{1 \leq j \leq m} \tilde{\lambda}_n^{(j)}.$$

Finally, taking into account (5.14), we use Corollary 2.2 with $\mathbb{X}_r = \Pi_n(E), r = p = d_n^E$, $m = d_n^E - C^V$, and $\tau = \hat{\tau}_n$ to obtain a set $\tilde{\tau}_n$ of d_n^E nodes for which the interpolation problem is solvable and its Lebesgue constant $\tilde{\lambda}_n$ satisfies

$$\tilde{\lambda}_n \leq (C^V + 1)\hat{\lambda}_n + C^V \leq C_1(V) n \max_{1 \leq j \leq m} \tilde{\lambda}_n^{(j)}.$$

Now (5.11) follows from (5.13). For (i), using (5.12) we obtain

$$\operatorname{card}(\tau_n \backslash \tilde{\tau}_n) \leq \sum_{j=1}^{m} \rho^{(j)} + \sum_{j=1}^{m} c^{(j)} + m + C^V = \sum_{j=1}^{m} \rho^{(j)} + c^V + m. \qquad \blacksquare$$

6. Examples

Example 6.1. Suppose the curve L is the graph of $y = p(x)/q(x)$, where p, q are real polynomials having no common factor. From the natural parametrization, $x = tq(t)/q(t)$, $y = p(t)/q(t)$, we see then that Theorem 4.2 applies with $d = \max(\deg(p), \deg(q)+1)$. For the special case $y = x^m$ with a positive integer m, the observation in [3] concerning the distribution of Fekete points can thus also be obtained by means of Theorem 4.2.

Example 6.2. Suppose L is the unit circle $x^2 + y^2 = 1$. Then we may parametrize L in two essentially different ways: in a complex setting via

$$x = \frac{z^2+1}{2z}, \qquad y = \frac{z^2-1}{2iz} \qquad (|z| = 1),$$

which follows from the familiar trigonometric representation, or in a real setting via

$$x = \frac{2t}{t^2+1}, \qquad y = \frac{t^2-1}{t^2+1} \qquad (t \in \mathbb{R}),$$

which one obtains from its stereographic representation. Using the first parametrization, it is easily seen that the limiting distribution of interpolation points on $E' \subset L$ with Lebesgue constants of polynomial growth is just the classical Robin equilibrium distribution for the set E'. It is shown in [9, Th. 14.7] that Auerbach–Fekete points for the unit circle are exactly the equally spaced points.

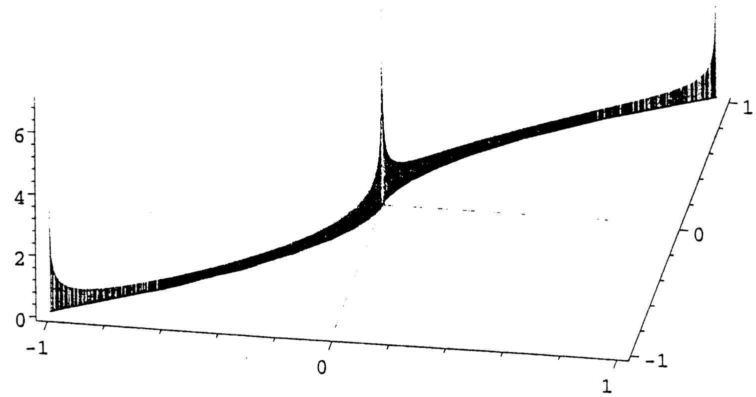

Fig. 6.1. AIM distribution for $E = \{(x, y) : x^3 = y^5, -1 \le x \le 1\}$.

Example 6.3. Suppose L is the graph of an equation of the form $x^m = y^p$ with m, p integers (not necessarily positive). Here, a parametrization is

$$x = t^p, \qquad y = t^m \qquad (t \in \mathbb{R}).$$

This example already shows that contrary to the behavior of "good" interpolation points in polynomial interpolation in one complex variable, the problem in the bivariate case depends on the precise algebraic structure of the curve.

For instance, if $p, m \ge 2$ are such that $x^m - y^p$ is irreducible and if the origin is an interior point of E, then the density of μ^E with respect to arclength

$$\frac{d\mu^E}{ds} = \frac{d\omega_{E'}}{dt} \Big/ \frac{ds}{dt}$$

will blow up at the origin, since $d\omega_{E'}/dt$ remains bounded from below by a positive constant in a neighborhood of $t = 0$, while $ds/dt = \sqrt{t^{2p-2} + t^{2m-2}}$ tends to zero as $t \to 0$. This effect is illustrated in Figure 6.1 where E is the subset of the curve $x^3 = y^5$ for $x \in [-1, 1]$. Note that as, for instance, $p = m + 2 \to \infty$, the curve $x^m = y^p$ looks locally more and more like a straight line. Thus "geometry" is less an issue than the fine algebraic structure.

With the aid of Examples 6.1–6.3 and the invariance of the AIM property under affine transformations, it is possible to solve the problem for all compact sets E of positive capacity, which are subsets of conic sections.

Example 6.4. Suppose that the curve L is the image of the unit circle $\mathbb{S}^1$ under a polynomial mapping $P_2(z) = \alpha + \beta z + \gamma z^2$, $|\beta| + |\gamma| \neq 0$. Then the equations

$$\begin{aligned} x = x(z) &:= \Re(P_2(z)) = \frac{1}{2}\,\frac{\gamma z^4 + \beta z^3 + 2\Re(\alpha) z^2 + \bar{\beta} z + \bar{\gamma}}{z^2}, \\ y = y(z) &:= \Im(P_2(z)) = \frac{1}{2i}\,\frac{\gamma z^4 + \beta z^3 + 2i\Im(\alpha) z^2 - \bar{\beta} z - \bar{\gamma}}{z^2}, \end{aligned} \tag{6.1}$$

$z \in \mathbb{S}^1$, give a complex parametrization of L. It can be easily verified that the algebraic equation determining this curve is

$$Q(x, y) := (|P_2 - \alpha|^2 - |\gamma|^2)^2 - |\beta|^2|P_2 - \alpha|^2 - |\beta\gamma|^2 - 2\Re(\bar{\beta}^2\gamma(P_2 - \alpha)) = 0.$$

Assuming that $\gamma \neq 0$ and $\beta \neq 0$ (so that L is not just a circle or a circle passed twice), one can conclude that $Q(x, y)$ is irreducible over $\mathbb{C}$. So L is a rational algebraic curve of order 4, and (6.1) represents its proper parametrization. It follows from Remark 4.4 that, for $E = L$, the density of μ_w on $\mathbb{S}^1$ is $1/2\pi$, i.e., μ_w is the uniform (Robin) measure on $\mathbb{S}^1$.

Note that if $P_2(z)$ is one-to-one on $\mathbb{S}^1$ (i.e., L is a Jordan curve), then it maps the open unit disk conformally and one-to-one onto the interior of L. Thus the AIM in this case is the image of the uniform measure on $\mathbb{S}^1$ under an *interior* Riemann mapping. We remark that for the complex polynomial interpolation the limit distribution of "good" interpolation nodes is given by the Robin measure on L which is the image of the uniform measure on $\mathbb{S}^1$ under an *exterior* Riemann mapping function. This fact demonstrates the substantial difference in the distribution of "good" nodes for bivariate and complex polynomial interpolation.

Example 6.5. Due to applications in computer graphics and geometric modeling, there is a substantial interest in feasible algorithms for the parametrization of algebraic curves (see, for instance, [1], [11], [13]). Maple contains subroutines (based on [13]) with the aid of which one can check if an algebraic curve has genus 0, and find appropriate parametrizations. For example, the trisectrix $(x^2 + y^2 - 2x)^2 - x^2 - y^2 = 0$, which is a particular case of the *limaçon of Pascal*, can be parametrized via

$$x(t) = \frac{2t(t^2 + 4t + 1)}{(t^2 + 1)^2}, \qquad y(t) = \frac{(t^2 - 1)(t^2 + 4t + 1)}{(t^2 + 1)^2} \qquad (t \in \mathbb{R}). \tag{6.2}$$

(In fact, Maple gives a more complicated but equivalent representation for this curve.) But if E is the whole curve, a complex parametrization should be used instead. From (6.2), one can easily verify that this curve is the image of $\mathbb{S}^1$ under the mapping $P_2(z) = z^2 + z + 1$. Hence, it follows from the previous example that μ_w is just a normalized arclength on

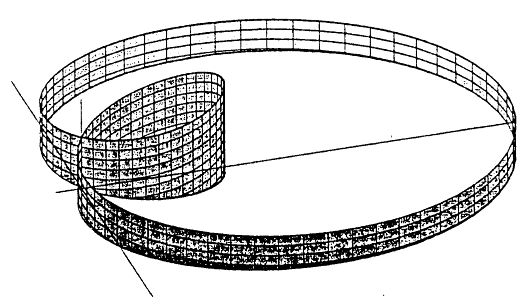

Fig. 6.2. AIM distribution for the trisectrix $(x^2 + y^2 - 2x)^2 - x^2 - y^2 = 0$.

the unit circle and so the density of the AIM on the trisectrix is given by

$$\frac{d\mu^E(S)}{dS} = \frac{1}{2\pi}\frac{1}{|2z+1|} = \frac{1}{2\pi}\frac{1}{\sqrt{5+4\cos s}} \qquad (z = e^{is},\, s \in [0, 2\pi)),$$

where S denotes the length of the arc connecting $P_2(1) = 3$ and $P_2(z)$ (see Figure 6.2).

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