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CONSTRUCTIVE APPROXIMATION

Potential and Discrepancy Estimates for Weighted Extremal Points

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Abstract. We derive estimates for Green and logarithmic potentials of measures associated with extremal points. These results are applied to obtain discrepancy estimates for weighted Fekete and Tsuji points on quasiconformal arcs or curves.

1. Introduction

Much research has been devoted to Fekete as well as Tsuji points and estimates for the respective logarithmic and Green potentials (see, for instance, [PS31], [Po65], [KK83], [Me83], [Kl84], and [AB97]). These points can be viewed as the special case of extremal points for a discrete weighted energy problem with weight equal to 1. Allowing an arbitrary weight is the starting point for this paper.

Suppose $X \subset \mathbf{C}$ is a simply connected domain. If $X \neq \mathbf{C}$ we denote by $k(x, y) = g(x, y)$ the Green function of X. In the case $X = \mathbf{C}$, $k(x, y)$ will stand for the logarithmic kernel $\log |x - y|^{-1}$.

Let L be a compact Jordan arc or Jordan curve in X and suppose that f is continuous on L.

It is known [ST97, Theorems I.1.3 and II.5.10] that there exists a unique unit measure μ_f on L minimizing the Gauss–Frostman energy functional

$$I_f(\mu) = \iint k(x, y)\, d\mu(x)\, d\mu(y) - 2 \int f\, d\mu$$

with respect to all unit Borel measures μ on L. Its (Green or logarithmic) potential

$$k_{\mu_f}(x) = \int k(x, y)\, d\mu_f(y) \qquad (x \in X),$$

has the property that

$$k_{\mu_f}(x) \begin{cases} \geq f(x) + c_f & \text{for every } x \in L, \\ = f(x) + c_f & \text{for every } x \in \operatorname{supp}(\mu_f), \end{cases} \tag{1}$$

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where $\operatorname{supp}(\mu_f)$ denotes the support of μ_f and the constant c_f is given by

$$c_f = \iint k(x, y)\, d\mu_f(x)\, d\mu_f(y) - \int f\, d\mu_f.$$

We are interested in approximating this measure μ_f by certain point measures which we introduce now. Let $n \geq 2$ and denote by F_n a set of points $x_1^{(n)}, \ldots, x_n^{(n)} \in L$ having minimal discrete weighted energy in the sense that

$$\frac{1}{n^2}\sum_{\substack{k,j=1\\k\neq j}}^{n} k(x_k^{(n)}, x_j^{(n)}) - \frac{2}{n}\sum_{k=1}^{n} f(x_k^{(n)}) = \inf_{y_1,\ldots,y_n\in L} \frac{1}{n^2}\sum_{\substack{k,j=1\\k\neq j}}^{n} k(y_k, y_j) - \frac{2}{n}\sum_{k=1}^{n} f(y_k). \tag{2}$$

Denote by μ_{F_n} the unit measure associating the mass $1/n$ with each point $x_i^{(n)}$ and by $I_f^*(\mu_{F_n})$ the discrete weighted energy appearing on the left-hand side of (2). F_n will be referred to as a set of nth weighted extremal points for the function f. In the case $X = \mathbf{C}$ the points $x_j^{(n)}$ reduce to the well-investigated weighted Fekete points.

Example. Let $X = \mathbf{C}$, $L = [-1, 1]$, $a, b > 0$. The nth weighted Fekete points associated with the weight

$$\omega(x) = \exp(f(x)) = (1-x)^a(1+x)^b \qquad (x \in [-1, 1]),$$

are given by the zeros of the Jacobi polynomial $P_n^{(\tilde{a},\tilde{b})}$ with the parameters $\tilde{a} = 2a(n-1) - 1$ and $\tilde{b} = 2b(n-1) - 1$ (see [ST97, p. 187]).

It is shown in [ST97] and [FS99] that the measures μ_{F_n} converge to μ_f in the weak-star sense. In particular, the corresponding potentials

$$k_{\mu_{F_n}}(x) = \int k(x, y)\, d\mu_{F_n}(y)$$

will converge to k_{μ_f} locally uniformly in $X\backslash L$. It is worth pointing out that if L is a closed curve and $\operatorname{supp}(\mu_f) = L$, then the potentials $k_{\mu_{F_n}}$ will be approximations to the solution of the Dirichlet problem in the domain interior to L with boundary values $f + c_f$.

We raise the question of how to quantify these two types of convergence.

We shall assume throughout this paper that f is Hölder continuous on L.

In the case $X = \mathbf{C}$ quantitative estimates for the potential difference between the logarithmic potential of μ_f and μ_{F_n} were given by Kleiner [Kl64]. Under more restrictive assumptions (L a $\mathcal{C}^2$-curve, $\operatorname{supp}(\mu_f) = L$, $f \in \mathcal{C}^1(L)$) his results contain estimates of order $O(\log n/\sqrt{n})$, that we are able to improve significantly.

2. Statement of the Results

Theorem 1. *There exists a constant $C_0 > 0$ depending only on X, L, and f, such that for all $n \geq 2$,*

$$-C_0\frac{\log n}{n} \leq k_{\mu_{F_n}}(x) - k_{\mu_f}(x) \leq C_0\frac{\log n}{n} + \frac{C_0}{n}\log\frac{1}{\min(\frac{1}{2}, \mathrm{d}(x, L))} \qquad (x \in X),$$

where $\mathrm{d}(x, A)$ *denotes the distance from a point x to a set $A \subset \mathbf{C}$.*

One should compare Theorem 1 with estimates for unweighted points due to Pommerenke [Po65], Blatt and Mhaskar [BM93], Monterie [Mo95, Part A], Korevaar and Monterie [KM98, Remark 5.1], and Kloke [Kl84, Satz 3.5], respectively. See also the papers of Korevaar and Kortram [KK83] as well as Korevaar and Monterie [KM99] containing a discussion of when in the unweighted case the $\log n/n$-term can be replaced by $1/n$. For potential estimates for Fekete points in higher dimensions we refer to Korevaar and Monterie [KM98] as well as [Gö98].

The proof of Theorem 1 is based on the following separation result, which is of independent interest. In the unweighted case ($f = 0$), it already appears in the paper of Kövari and Pommerenke [KP68, Theorem 1] (cf. a similar estimate in [Kl84, Satz 3.3]).

Theorem 2. *There exist constants $C_0 > 0$, $\gamma_0 > 0$ depending only on X, L, and f, such that for all $n \geq 2$:*

$$\min_{1 \leq i < j \leq n} |x_i^{(n)} - x_j^{(n)}| \;\geq\; \frac{C_0}{n^{1/\gamma_0}}.$$

Remark. The proof of Theorem 2 together with well-known facts concerning the boundary behavior of harmonic functions actually reveals the following: Suppose that L is a smooth curve or arc of class $\mathcal{C}^{1+\varepsilon}$.

(i) If f is Hölder continuous with Hölder constant β ($1 > \beta > 0$), then one can choose $\gamma_0 = \beta$ in the case of a curve and $\gamma_0 = \min(\beta, \frac{1}{2})$ in the case of an arc.

(ii) If f is smooth of class $\mathcal{C}^{1+\varepsilon}$, then

$$\min_{\substack{1 \leq i \leq n \\ i \neq j}} |x_i^{(n)} - x_j^{(n)}| \;\geq\; C_0\, d_{j,n},$$

where $d_{j,n}$ denotes the distance of a point $x_j^{(n)}$ from the level curve $L_{1/n}$ (see (3)) and $C_0 = C_0(f, L, X)$. In particular, if $L = [-1, 1]$, this yields an estimate in the form of Lemma 2.15 in [DKM98] and, consequently (see [DKM98, p. 413]), μ_f has bounded density with respect to the arcsine distribution on $[-1, 1]$.

To state the discrepancy results we need to introduce some notions and definitions.

Denote by Δ the exterior of the closed unit disk with respect to $\overline{\mathbf{C}}$ and consider $\Phi : \text{ext}(L) \to \Delta$ mapping the exterior of L (with respect to $\overline{\mathbf{C}}$) conformally and univalently onto Δ with standard normalization

$$\Phi(\infty) = \infty, \qquad \Phi'(\infty) := \lim_{z\to\infty} \frac{\Phi(z)}{z} > 0.$$

Consider the level lines

$$L_r := \{z \in \mathbf{C}\colon\ |\Phi(z)| = 1 + r\} \qquad (r > 0), \tag{3}$$

and choose $\frac{1}{2} \geq r_0 > 0$ such that $L_{r_0} \subset X$.

Finally, denote by $\mu_L = \mu_{L,\mathbf{C}}$ the equilibrium distribution of L (with respect to $\mathbf{C}$), which is the unit measure on L minimizing the Gauss–Frostman energy in the special case of the external field f being identically equal to 0 and k being the logarithmic kernel.

Now, suppose that L is a quasiconformal curve or arc in X, which by definition is the image of the unit circle or of a subarc of the unit circle under a quasiconformal mapping of the extended complex plane $\overline{\mathbf{C}}$ onto itself (see the standard work [Ah66] for the definition of a quasiconformal mapping).

The following metric characterization (see [LV71, p. 100]) shows that, in particular, convex curves, curves of bounded variation without cusps and rectifiable Jordan curves which have locally the same order of arc- and chord-length are quasiconformal.

A Jordan curve $\Gamma \subset \mathbf{C}$ *is quasiconformal if and only if there is a* $c \geq 1$ *such that for all* $z_1,\ z_2 \in \Gamma$, $z_1 \neq z_2$:

$$\min(\operatorname{diam}(\Gamma(z_1, z_2), \operatorname{diam}(\Gamma(z_2, z_1)) \leq c|z_1 - z_2|,$$

where $\Gamma(z_1, z_2),\ \Gamma(z_2, z_1)$ *denote the two subarcs of* Γ *having endpoints* z_1, z_2.

We now want to formulate a corresponding discrepancy estimate. For two (unit) measures μ and ν on L we call

$$D[\mu - \nu] := \sup\{|\mu(l) - \nu(l)| : l \text{ a subarc of } L\}$$

the *discrepancy* between μ and ν.

A connection between double-sided bounds for logarithmic potentials and discrepancy on quasiconformal curves was given by Andrievskii and Blatt. Analogous results related to Green potentials were obtained by the first author of this paper in his PhD thesis. We combine the statements of these results in the following:

Theorem ([AB97, Theorem 1], [Gö98, Satz 4.2.1]). *Let* L *be a quasiconformal curve or arc. Let* μ *be a unit measure on* L *and suppose that* c *and* ρ *are positive constants such that, for all subarcs* l *of* L:

$$\mu(l) \leq c(\mu_L(l))^{\rho}. \tag{4}$$

Then there exist positive constants τ *and* c_0, *depending only on* c, ρ, X, *and* L, *such that, for all unit measures* ν *on* L *and all* $0 < r \leq r_0$:

$$D[\mu - \nu] \leq c_0(\varepsilon(r)\log(1/r) + r^{\tau}), \tag{5}$$

where

$$\varepsilon(r) = \varepsilon(r, \mu, \nu) := \sup_{z \in L_r} |k_\mu(z) - k_\nu(z)|.$$

By Theorem 1 the quantity $\varepsilon(r, \mu_f, \mu_{F_n})$ can be estimated as follows. Since for $0 < r \leq r_0$, there holds

$$c_1 r^2 \leq \mathrm{d}(x, L) \qquad (x \in L_r), \tag{6}$$

with $c_1 > 0$ independent of r [ABD95, Corollary 2.7], it follows that

$$\sup_{z \in L_r} |k_{\mu_f}(z) - k_{\mu_{F_n}}(z)| \le C_0 \frac{\log n}{n} + \frac{C_0}{n} \log \frac{1}{r} \qquad (0 < r \le r_0).$$

Inserting this estimate into (5) one can deduce:

Theorem 3. *Suppose L is a quasiconformal curve or arc. Then there exists a constant $C_0 > 0$ such that, for all $n \ge 2$:*

$$D[\mu_f - \mu_{F_n}] \le C_0 \frac{(\log n)^2}{n}.$$

In the unweighted case this statement coincides with those of Andrievskii and Blatt for Fekete points [AB97, Theorem 3] and the first author for Tsuji points. But it also generalizes these results to Fekete and Tsuji points defined for more general sets. More precisely, if K is a compact subset of a quasiconformal curve L and such that the equilibrium potential is Hölder continuous in a neighborhood of L, then for the unweighted extremal points for K, a discrepancy estimate of order $O((\log n)^2/n)$ holds. In fact, such points can be interpreted as weighted extremal points under the influence of a Hölder continuous external field that coincides with the equilibrium potential on K and is smaller on $L \backslash K$. A particularly interesting case is when K consists of a finite number of mutually disjoint quasiconformal arcs.

Finally, note that Kleiner uses an approach contrary to ours: He first derives a discrepancy estimate of order $O(\log n/\sqrt{n})$ and from this he obtains the potential estimates.

3. Proofs

For the proofs of our estimates for weighted extremal points we may without loss of generality assume that the constant c_f is equal to 0 and thus $f = k_{\mu_f}$ on the support of μ_f. Indeed, for every constant c we have $\mu_{f+c} = \mu_f$, and the extremal points also remain unchanged when adding in (2) a constant to f.

Occasionally we will use that

$$k(x, y) = \log \frac{1}{|x - y|} - h(x, y) \qquad (x, y \in X),$$

with a function h harmonic in both variables.

For a Jordan curve or arc Γ, we denote by $\operatorname{ext}(\Gamma)$ the domain exterior to Γ with respect to $\overline{\mathbf{C}}$. In addition, if Γ is a curve, then $\operatorname{int}(\Gamma)$ stands for the domain interior to Γ.

Let C_0 denote a positive constant depending at most on X, L, and f, possibly different at different occurrences. The numbers $\gamma_0, \gamma_1, \ldots$ will be positive numbers depending only on X, L, and f. For convenience, when dealing with the extremal points, we omit the upper index n and write x_i instead of $x_i^{(n)}$.

The proofs will be partly based on properties of the modulus of continuity of the solution to certain Dirichlet problems in terms of the modulus of continuity of the boundary values.

Here, we refer to a result of Johnston [Jo80, Corollary 3] that for our purposes we want to formulate in the following way:

Lemma. *Suppose that G is a simply connected, bounded domain. If u is a function continuous on $\overline{G}$, harmonic in G, and Hölder continuous on ∂G, then u is Hölder continuous on $\overline{G}$.*

By means of an inversion $z \mapsto 1/(z - z_0)$ this result can also be formulated for simply connected domains in $\overline{\mathbf{C}}$ containing the point at infinity and such that the complement has nonempty interior.

We will make use of the fact that this lemma is actually rather of a local character. More precisely, we have:

Lemma 1. *If for a neighborhood U of the Jordan curve or arc L, u is continuous in U, harmonic in $U \setminus L$, and Hölder continuous on L, then u is Hölder continuous in a neighborhood of L.*

Proof. Without loss of generality we may assume U to be bounded and that u is harmonic even in a neighborhood of ∂U. We want to show that for $x \in U$ and $y \in U$ there holds

$$|u(x) - u(y)| \leq C_0 |x - y|^{\gamma_0}.$$

Here, it suffices to consider only the case that $x \in L$. To justify this, let $\delta > 0$ and consider the modulus of continuity

$$\omega(\delta) := \sup_{\substack{|z-\zeta| \leq \delta \\ z, \zeta \in \overline{U}}} |u(z) - u(\zeta)|.$$

Let $z, \zeta \in \overline{U}$ be such that $\omega(\delta) = |u(z) - u(\zeta)|$. Set $s := \min(\mathrm{d}(z, L \cup \partial U), \mathrm{d}(\zeta, L \cup \partial U))$, say $s = |\zeta - \zeta^*|$ with $\zeta^* \in L \cup \partial U$. If $s > 0$, consider the auxiliary function

$$h(\xi) := \frac{u(z + s\xi) - u(\zeta + s\xi)}{|u(z) - u(\zeta)|} \qquad (\xi \in \overline{\mathbf{D}}).$$

This function is harmonic in the open unit disk $\mathbf{D}$, continuous and in modulus bounded by 1 on $\overline{\mathbf{D}}$. By the mean value principle,

$$1 = |h(0)| = \left| \frac{1}{2\pi} \int_0^{2\pi} h(0 + se^{i\theta})\, d\theta \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |h(0 + se^{i\theta})|\, d\theta \leq 1.$$

Thus, $|h| = 1$ on $\partial\mathbf{D}$ and, in particular,

$$|u(z - \zeta + \zeta^*) - u(\zeta^*)| = |u(z) - u(\zeta)|.$$

This equality trivially also holds in the case that $s = 0$. Now, ζ^* is a point on $L \cup \partial U$ and

$z - \zeta + \zeta^*$ a point from $\overline{U}$. It remains to note that we have assumed u to be harmonic in a neighborhood of ∂U. Thus, it suffices to consider only $x \in L$.

Now that we have reduced our problem we first consider the case when L is a curve. Let $x \in L$ and $y \in \mathrm{ext}(L) \cap \overline{U}$. Denote by u_{μ_L} the logarithmic potential of the equilibrium distribution of L with respect to $\mathbf{C}$. For the so-called Robin constant $V_L = -\log \mathrm{cap}(L, \mathbf{C})$, we have that $u_{\mu_L} = V_L$ on L and $u_{\mu_L} < V_L$ in $\mathrm{ext}(L)$. Now, denoting by h_u the solution to the Dirichlet problem in $\mathrm{ext}(L)$ with boundary values u on L, the maximum principle yields that for a sufficiently large constant $C_0 > 0$:

$$h_u - C_0\,(V_L - u_{\mu_L}) \le u \le h_u + C_0(V_L - u_{\mu_L}) \quad \text{in} \quad \overline{U} \cap \mathrm{ext}(L).$$

But h_u is Hölder continuous by the lemma of Johnston, and it is well known that u_{μ_L} is Hölder continuous in $\overline{U}$ with exponent $\frac{1}{2}$ (this follows, e.g., from (6)). Thus,

$$|u(x) - u(y)| \le C_0 |u_{\mu_L}(y) - V_L| + C_0 |x - y|^{\gamma_1} \le C_0 |x - y|^{\gamma_1}.$$

If, however, $y \in \mathrm{int}(L) \cap \overline{U}$, then one may use a suitable inversion $z \mapsto 1/(z - z_0)$ to transform the problem to the situation that we just dealt with.

Suppose now that L is an arc. Denote by $z_1,\ z_2$ its endpoints. Choose a closed proper subarc L_1 of L containing z_1 and consider a simply connected domain G containing $L \backslash \{z_2\}$ such that $L \cap \partial G = \{z_2\}$ and $G^* := G \backslash L$ is simply connected. Then choose a Hölder continuous continuation u^* of u to ∂G^* and denote by h_{u^*} the solution to the Dirichlet problem in G^* with these boundary values u^*. Then, by the maximum principle, if C_0 is chosen sufficiently large, $h_{u^*} + C_0(V_{L_1} - u_{\mu_{L_1}})$ will be a Hölder continuous majorant for u in $\overline{G^*}$ and $h_{u^*} - C_0(V_{L_1} - u_{\mu_{L_1}})$ will be a minorant. For $x \in L_1$ one can now reason as in the case of a curve. If $x \in L_2 := \overline{L \backslash L_1}$, a similar construction leads to the desired estimate. ∎

We do not dwell on how the Hölder exponents of the solutions to the Dirichlet problem depend on the Hölder exponents of the boundary values. For a discussion, see [Jo80], and [An88], [An90a] concerning quasiconformal curves and arcs.

Lemma 2. *If f is Hölder continuous on L, then the potential k_{μ_f} is Hölder continuous in a neighborhood of L.*

Proof. For abbreviation, set $u := k_{\mu_f}$. By Lemma 1, it suffices to show that u is Hölder continuous on L, i.e.,

$$|u(x) - u(y)| \le c_0\,|x - y|^{\gamma} \qquad (x, y \in L), \tag{7}$$

with $c_0,\ \gamma > 0$. Let $x, y \in L$. Since u coincides with the Hölder continuous function f on $S_f := \mathrm{supp}(\mu_f)$, it is enough to establish (7) for $y \notin S_f$. In addition, reasoning as in the proof of Lemma 1, where it was shown that (roughly speaking) the modulus of continuity is attained on the boundary of the region of harmonicity, we see that it suffices to prove (7) for $x \in S_f$ only.

Let $r := \mathrm{d}(y, S_f)$ and choose $y^* \in S_f$ such that $r = |y - y^*|$. If we denote by β_1 the Hölder exponent of f on L, then

$$|u(x) - u(y)| \leq C_0|x - y^*|^{\beta_1} + |u(y) - u(y^*)| \leq C_0 2^{\beta_1} |x - y|^{\beta_1} + |u(y) - u(y^*)|.$$

Therefore, it is enough to prove (7) with x replaced by y^*.

Now that we have reduced our problem, let Ω be a bounded (simply or doubly connected) domain such that $L \subset \Omega \subset X$. We may certainly assume that r is so small that $\{z\colon |z - y^*| \leq 2r\} \subset \Omega$. Denote by g_f the solution to the Dirichlet problem in $\Omega\backslash L$ with boundary values f on L and u on $\partial\Omega$. By Lemma 1, g_f is Hölder continuous in a neighborhood of L with Hölder exponent $\beta_2 > 0$. According to the maximum principle, $u \geq g_f$ in $\overline{\Omega}$, and, consequently,

$$u(y^*) = g_f(y^*) \leq u(z) + C_0|y^* - z|^{\beta_2} \qquad (z \in \overline{\Omega}). \tag{8}$$

In particular, this estimate holds for $z = y$. In what follows we establish the desired reverse inequality.

Since u is harmonic in $\{z\colon |z - y| < r\}$:

$$u(y) = \frac{1}{\pi r^2} \int_{\{|z-y|<r\}} u(z)\, dm_2(z),$$

where m_2 denotes the planar Lebesgue measure. Since u is superharmonic, the mean-value inequality property implies that

$$u(y^*) \geq \frac{1}{4\pi r^2} \int_{\{|z-y^*|<2r\}} u(z)\, dm_2(z).$$

Thus, taking into account the estimate (8):

$$u(y^*) \geq \frac{4\pi r^2 - \pi r^2}{4\pi r^2} (u(y^*) - C_0 r^{\beta_2}) + \frac{\pi r^2}{4\pi r^2} u(y),$$

which implies that

$$u(y^*) \geq u(y) - C_0 r^{\beta_2} = u(y) - C_0|y - y^*|^{\beta_2}.$$

The proof of Lemma 2 is complete. ■

Lemma 2 allows us to henceforth assume that L is a curve. In fact, if L is an arc, then it can be continued to a Jordan curve $\tilde{L}$ lying in X, and one may extend f Hölder continuously to a function $\tilde{f}$ on $\tilde{L}$ such that $\tilde{f} < k_{\mu_f}$ on $\tilde{L}\backslash L$. Then by the uniqueness result [ST97, Theorems I.3.3 and II.5.12], $\mu_f = \mu_{\tilde{f}}$. But extremal points for f on L are extremal points for $\tilde{f}$ on $\tilde{L}$ and vice versa, as follows from:

Lemma 3. *All extremal points $x_i^{(n)}$ are contained in the set*

$$\{z \in L\colon k_{\mu_f}(z) = c_f + f(z)\}.$$

That $k_{\mu_f}(x_i^{(n)}) \ge c_f + f(x_i^{(n)})$ is clear, since the extremal points are supposed to lie on L (see (1)). In the case of logarithmic potentials, the reverse inequality is given in [ST97, Theorem III.1.2]. We will present a sketch of a proof in the case that $X \neq \mathbf{C}$ at the end of this paper.

Proof of Theorem 2. Suppose first that $X \neq \mathbf{C}$. Then the kernel $k(\cdot, \cdot)$ is given by the Green function $g(\cdot, \cdot)$. Making use of a conformal mapping of X onto the unit disk $\mathbf{D} = \{z\colon |z| < 1\}$ and taking into account the invariance of the Green function under such transformations in both arguments we may without loss of generality assume that X is the unit disk. For the Green function of $\mathbf{D}$ we have the formula

$$g(z, \zeta) = \log \left| \frac{1 - z\bar{\zeta}}{z - \zeta} \right| \qquad (z, \zeta \in \mathbf{D}).$$

Fix $1 \le j \le n$ and consider the rational function

$$q_j(z) := \prod_{\substack{k=1\\k\neq j}}^{n} \frac{z - x_k}{1 - z\overline{x_k}} \Bigg/ \prod_{\substack{k=1\\k\neq j}}^{n} \frac{x_j - x_k}{1 - x_j\overline{x_k}} \qquad (z \in \mathbf{C}),$$

having poles exactly in the points $x_k/|x_k|^2$, $k \neq j$. From the extremal property of the weighted extremal points:

$$|q_j(z)| = \exp\left\{ \sum_{\substack{k=1\\k\neq j}}^{n} g(x_j, x_k) - \sum_{\substack{k=1\\k\neq j}}^{n} g(z, x_k) \right\} \le \exp\{n(f(x_j) - f(z))\} \qquad (z \in L).$$

Denote by $G(\cdot, \cdot)$ the Green function of the exterior $\mathrm{ext}(L)$ of the curve L with respect to $\overline{\mathbf{C}}$. Denote by $h_f(z)$ the solution to the Dirichlet problem in $\mathrm{ext}(L)$ with boundary values f. The function

$$\log|q_j(z)| - \sum_{\substack{k=1\\k\neq j}}^{n} G\left(z, \frac{x_k}{|x_k|^2}\right) - n(f(x_j) - h_f(z))$$

is subharmonic (even harmonic) in $\mathrm{ext}(L)$ with boundary values ≤ 0 on L. Consequently, this function is nonpositive in $\mathrm{ext}(L)$ which implies that

$$|q_j(z)| \le \exp\left\{ \sum_{\substack{k=1\\k\neq j}}^{n} G\left(z, \frac{x_k}{|x_k|^2}\right) + n(f(x_j) - h_f(z)) \right\} \qquad (z \in \mathrm{ext}(L)).$$

Since $h_f(z)$ and the functions $G(z, x_k/|x_k|^2)$ are Hölder continuous in a neighborhood of L relative to $\overline{\mathrm{ext}(L)}$ with the same Hölder constant and same Hölder exponent γ_2 we find that

$$|q_j(z)| \le \exp\left\{ n\,\frac{C_0}{n} \right\} \le C_0 \qquad (z \in \overline{\mathrm{ext}(L)},\ |z - x_j| \le 1/n^{1/\gamma_2}).$$

Similarly,

$$|q_j(z)| \le C_0 \qquad (z \in \overline{\mathrm{int}(L)},\ |z - x_j| \le 1/n^{1/\gamma_3}).$$

Thus, setting $\gamma_4 := \min(\gamma_2, \gamma_3)$, then for ζ with $|x_j - \zeta| \le 1/(2n^{1/\gamma_4})$:

$$|q_j'(\zeta)| \le \frac{1}{2\pi} \int_{|z-x_j|=1/n^{1/\gamma_4}} \frac{|q_j(z)|}{|z-\zeta|^2}\,|dz| \le C_0\, n^{1/\gamma_4}.$$

Now, let $k \ne j$. If $|x_j - x_k| \ge 1/(2n^{1/\gamma_4})$ nothing is to be proved. Otherwise, $|x_j - x_k| < 1/(2n^{1/\gamma_4})$ and (using the trick in [KP68, p. 71]) it follows that

$$1 = |q_j(x_j) - q_j(x_k)| = \left| \int_{[x_j, x_k]} q_j'(\zeta)\, d\zeta \right| \le C_0 |x_j - x_k|\, n^{1/\gamma_4}.$$

This is the assertion in the case $X \ne \mathbf{C}$.

The proof in the case $X = \mathbf{C}$ follows the same lines. However, instead of the rational function q_j one has to consider the polynomials

$$p_j(z) = \prod_{\substack{k=1\\k\ne j}}^{n} \frac{z - x_k}{x_j - x_k}.$$

The details are left to the reader. ■

Proof of Theorem 1. From the minimality of the weighted extremal points it follows that for all $i = 1, \ldots, n$:

$$2\sum_{\substack{j=1\\j\ne i}}^{n} k(x_i, x_j) - 2nf(x_i) \le 2\sum_{\substack{j=1\\j\ne i}}^{n} k(x, x_j) - 2nf(x) \qquad (x \in L).$$

Adding these n inequalities and denoting by

$$I^*(\mu_{F_n}) := I_f^*(\mu_{F_n}) + \frac{2}{n}\sum_{i=1}^{n} f(x_i)$$

the unweighted discrete energy of μ_n, we obtain that

$$I^*(\mu_{F_n}) - \int f\, d\mu_{F_n} \le \frac{n-1}{n} k_{\mu_{F_n}}(x) - f(x) \le k_{\mu_{F_n}}(x) - f(x) + \frac{C_0}{n} \qquad (x \in L). \tag{9}$$

*Lower Potential Bounds From Energy Estimates**

Let $r > 0$ be sufficiently small and denote by $\delta_{x_i}^r$ the unit mass, uniformly spread on the circle of radius r, centered at x_i. Set $\mu_n^r := (1/n)\sum_{i=1}^{n} \delta_{x_i}^r$.

* One of the referees has pointed out that the following technique of sweeping to small circles is due to Tsuji and further developed by Siciak.

By Lemma 2,

(10) $$|k_{\mu_f}(x) - k_{\mu_f}(x_i)| \leq C_0 r^{\gamma_5} \qquad (|x - x_i| \leq r,\ i = 1, \ldots, n),$$

and, consequently,

$$\left| \int k_{\mu_f} d\mu_{F_n} - \int k_{\mu_f} d\mu_n^r \right| \leq C_0 r^{\gamma_5}.$$

From the superharmonicity of the kernel $k(\cdot, \cdot)$ it follows that

$$\begin{aligned} I^*(\mu_{F_n}) &\geq \frac{1}{n^2} \sum_{\substack{i,j=1 \\ j \neq i}}^{n} \iint k(x, y)\, d\delta_{x_i}^r(x)\, d\delta_{x_j}^r(y) \\ &\geq \iint k(x, y)\, d\mu_n^r(x)\, d\mu_n^r(y) - \frac{C_0}{n} \log \frac{1}{r}. \end{aligned}$$

The energy principle [ST97, Lemma I.1.8, Theorem II.5.6] implies that

$$\iint k(x, y)\, d\mu_n^r(x)\, d\mu_n^r(y) + \iint k(x, y)\, d\mu_f(x)\, d\mu_f(y)$$

$$\geq 2 \iint k(x, y)\, d\mu_f(x)\, d\mu_n^r(y).$$

Combining the last three inequalities and inserting $r = 1/n^{1/\gamma_5}$ gives

$$I^*(\mu_{F_n}) - \int k_{\mu_f}\, d\mu_{F_n} \geq \int k_{\mu_f}\, d\mu_{F_n} - \int k_{\mu_f}\, d\mu_f - C_0 \frac{\log n}{n}.$$

Taking into account (9) as well as

$$\int k_{\mu_f}\, d\mu_{F_n} - \int f\, d\mu_{F_n} = 0$$

(see Lemma 3) and denoting by ε_n the residue

$$\varepsilon_n := \int k_{\mu_f}\, d\mu_{F_n} - \int k_{\mu_f}\, d\mu_f,$$

we thus obtain the lower potential bound

(11) $$k_{\mu_{F_n}}(x) - k_{\mu_f}(x) = k_{\mu_{F_n}}(x) - f(x) \geq \varepsilon_n - C_0 \frac{\log n}{n} \qquad (x \in \operatorname{supp}(\mu_f)).$$

If we denote by ν the equilibrium measure for $\operatorname{supp}(\mu_f)$ with external field 0, then for some constant c, $k_\nu \leq c$ in X with equality μ_f-almost-everywhere. Now, (11) implies

(12) $$0 \geq \int k_\nu\, d\mu_{F_n} - \int k_\nu\, d\mu_f = \int k_{\mu_{F_n}}\, d\nu - \int k_{\mu_f}\, d\nu \geq \varepsilon_n - C_0 \frac{\log n}{n}.$$

Therefore, the principle of domination is applicable so that the estimate (11) holds in all of X:

(13) $$k_{\mu_{F_n}}(x) - k_{\mu_f}(x) \geq \varepsilon_n - C_0 \frac{\log n}{n} \qquad (x \in X).$$

Upper Potential Bounds

To some extent we follow an idea of Dahlberg [Da78]. Because of Theorem 2, one can determine a constant $1 > c_0 > 0$ so that

$$\min_{\substack{i,j=1,\ldots,n\\ i\neq j}} |x_j - x_i| \geq 4c_0/n^{1/\gamma_0} \qquad (j = 1, \ldots, n,\ n \geq 2). \tag{14}$$

Assume without loss of generality that $c_0 \leq R_0/2$ for all $n \geq 2$, where $0 < 4R_0$ is chosen smaller than the distance from L to the boundary of X. For $1 \leq i \leq n$ we consider the function

$$h_i(x) := \frac{1}{n} \sum_{\substack{j=1\\ j\neq i}}^{n} k(x, x_j) \qquad (x \in X). \tag{15}$$

The norm

$$|\nabla h_i| := \sum_{j=1}^{2} |\partial h_i/\partial y_j|$$

of its first partial derivatives satisfies the estimate

$$|\nabla h_i(x)| \leq C_0 \sum_{\substack{k=1\\ k\neq i}}^{n} \frac{1}{n} \frac{1}{|x - x_k|} + C_0 \qquad (x \in X,\ \mathrm{d}(x, L) \leq R_0). \tag{16}$$

Taking into account (14), it follows that

$$\sup_{|x-x_i|\leq c_0/n^{1/\gamma_0}} |\nabla h_i(x)| \leq C_0 n^{1/\gamma_0}. \tag{17}$$

Next, we claim that

$$k_{\mu_f}(x_i) + \varepsilon_n + \frac{C_0}{n} \geq h_i(x_i) \geq k_{\mu_f}(x_i) + \varepsilon_n - C_0 \frac{\log n}{n}. \tag{18}$$

Indeed, from (1) and the minimizing property of the weighted extremal points we have

$$h_i(x_i) \leq k_{\mu_f}(x_i) + h_i(x) - k_{\mu_f}(x) \qquad (x \in \operatorname{supp}(\mu_f)),$$

so that after integrating with respect to μ_f:

$$h_i(x_i) \leq k_{\mu_f}(x_i) + \int (h_i - k_{\mu_f})\, d\mu_f \leq k_{\mu_f}(x_i) + \varepsilon_n + \frac{C_0}{n}.$$

On the other hand, for $|x - x_i| = 1/n^{1/\gamma_6}$, where $\gamma_6 := \min(\gamma_0, \gamma_5)$, it follows from (10) and (13) that

$$\begin{aligned} h_i(x) &= k_{\mu_{F_n}}(x) - \frac{1}{n} \log \frac{1}{|x - x_i|} + \frac{1}{n} h(x, x_i) \\ &\geq k_{\mu_f}(x) + \varepsilon_n - C_0 \frac{\log n}{n} - C_0 \frac{1}{n} \\ &\geq k_{\mu_f}(x_i) + \varepsilon_n - C_0 \frac{\log n}{n} - C_0 \frac{1}{n}. \end{aligned}$$

Hence, by the mean-value inequality property for superharmonic functions:

$$h_i(x_i) \geq k_{\mu_f}(x_i) + \varepsilon_n - C_0 \frac{\log n}{n}.$$

Let $0 < \alpha \leq c_0/n^{1+1/\gamma_6}$ and x be such that $|x - x_i| = \alpha$. From (17) it follows that

$$|h_i(x) - h_i(x_i)| \leq C_0 \alpha \sup_{|y-x_i|\leq\alpha} |\nabla h_i(y)| \leq C_0 \alpha n^{1/\gamma_0} \leq \frac{C_0}{n}. \tag{19}$$

Since

$$h_i(y) = k_{\mu_{F_n}}(y) - \frac{\log(1/\alpha)}{n} + \frac{1}{n} h(x_i, y) \qquad (y : |y - x_i| = \alpha),$$

it follows from (18) and (19) that

$$|k_{\mu_{F_n}}(x) - k_{\mu_f}(x_i) - \varepsilon_n| \leq C_0 \left(\frac{\log(1/\alpha)}{n} + \frac{\log n}{n} \right) \leq C_0 \frac{\log(1/\alpha)}{n}.$$

Hence, taking into account (10):

$$|k_{\mu_{F_n}}(x) - k_{\mu_f}(x) - \varepsilon_n| \leq C_0 \left(\frac{1}{n} + \frac{\log(1/\alpha)}{n} \right) \leq C_0 \frac{\log(1/\alpha)}{n}.$$

We have shown that

$$k_{\mu_{F_n}}(x) - k_{\mu_f}(x) - \varepsilon_n \leq C_0 \frac{\log(1/\alpha)}{n} \qquad \left(x \in \bigcup_{i=1}^{n} \{y : |y - x_i| = \alpha\} \right).$$

Reasoning as in (12) with ν replaced by the unweighted equilibrium measure of the union of the circles $\{y : |y - x_i| = \alpha\}$ we find that

$$\varepsilon_n \geq -C_0 \frac{\log(1/\alpha)}{n}. \tag{20}$$

Therefore, the maximum principle applied to the subharmonic function $k_{\mu_{F_n}} - k_{\mu_f}$ yields

$$k_{\mu_{F_n}}(x) - k_{\mu_f}(x) \leq \varepsilon_n + C_0 \frac{\log(1/\alpha)}{n} \qquad \left(x \in \bigcap_{i=1}^{n} \{y \in X : |y - x_i| \geq \alpha\} \right). \tag{21}$$

In addition, by virtue of (12) and (20):

$$|\varepsilon_n| \leq C_0 \frac{\log n}{n}. \tag{22}$$

The assertion of Theorem 1 in the case that $\mathrm{d}(x, L) =: \alpha \leq c_0/n^{1+1/\gamma_6}$ follows from

(13), (21), and (22). In the other case, the assertion also follows from (21), namely, by merely inserting $\alpha := c_0/n^{1+1/\gamma_6}$. ■

Proof of Theorem 3. The assertion of Theorem 3 follows by inserting $\mu = \mu_f$, $\nu = \mu_{F_n}$ as well as the double-sided potential bounds of Theorem 1 into (5) and choosing $r = n^{-1/\tau}$. However, it has to be checked that an estimate of the form (4) holds.

For $0 < r \le r_0$ denote by $\mu_f^{(r)}$ the balayage measure of μ_f to L_r, i.e., $\mu_f^{(r)}$ is the unique unit measure on L_r satisfying

$$k_{\mu_f^{(r)}} = k_{\mu_f} \qquad \text{on} \quad \overline{\text{ext}(L_r)} \cap X. \tag{23}$$

It is well known that as $r \to 0^+$, the measures $\mu_f^{(r)}$ converge to μ_f in the weak-star sense.

Now, let l be an arbitrary subarc of L with endpoints, say, z_1 and z_2. Write $\Phi(z_j) = e^{i\theta_j}$, $j = 1, 2$, where $0 \le \theta_1 < \theta_2 < 2\pi$. Since the equilibrium measure μ_L on L is given by the image measure of the normalized arclength on $\partial\mathbf{D}$ under Φ^{-1}, we have $\mu_L(l) = (\theta_2 - \theta_1)/(2\pi)$. Thus, there is no loss in generality, if we henceforth assume that $\theta_2 - \theta_1 \le \pi/2$. Set

$$A(l) := \{z \in \overline{\text{ext}(L)} : \Phi(z) = (1+r)\, e^{i\theta},\ \ \theta_1 \le \theta \le \theta_2,\ 0 \le r \le r_0\}.$$

Then $l_r := A(l) \cap L_r$ is a subarc of L_r, and we denote its endpoints by $z_1^{(r)}$ and $z_2^{(r)}$. Since $\mu_f^{(r)}(l_r) \to \mu_f(l)$, we have in some sense reduced our problem to the estimation of a measure on a smooth (even analytic) curve. Hence, we can use the representation [ST97, Theorem II.1.5] to find that

$$\mu_f^{(r)}(l_r) = -\frac{1}{2\pi}\int_{l_r} \frac{\partial}{\partial n_+} k_{\mu_f}\, ds - \frac{1}{2\pi}\int_{l_r} \frac{\partial}{\partial n_-} k_{\mu_f^{(r)}}\, ds. \tag{24}$$

Here, $\partial/\partial n_+$ ($\partial/\partial n_-$, respectively) denotes differentiation in the direction of the outward unit normal (inward unit normal, respectively), and we have made use of (23). Note that the representation [ST97, Theorem II.1.5] is stated only for logarithmic potentials, but is of course valid also for Green potentials, since these potentials differ only by a harmonic function in a neighborhood of the sets under consideration.

The first integral in (24) can be estimated in the following way. Consider the four annular sectors

$$S_j := \left\{ w : w = (1+r)\, e^{i\theta},\ 0 \le \theta - j\frac{\pi}{2} \le \pi,\ 0 \le r \le r_0 \right\} \qquad (j = 0, 1, 2, 3).$$

Since $0 \le \theta_2 - \theta_1 \le \pi/2$, both, θ_1 and θ_2, will be in the parameter set of one of these sectors. For convenience, assume that this is S_0.

Now, $Q_0 := \Phi^{-1}(\text{int}(S_0))$ is a simply connected region bounded by a quasiconformal curve. By a result of Gehring and Martio [GM83, Cor. 2.3], Privaloff's theorem is valid in such domains: The harmonic conjugate $\widetilde{k}_{\mu_f}$ of k_{μ_f} in Q_0 is also Hölder continuous, even with the same Hölder exponent as that of k_{μ_f}.

Thus, denoting by $\partial/\partial\tau$ tangential differentiation and using the Cauchy–Riemann conditions:

$$\left|\frac{1}{2\pi}\int_{l_r}\frac{\partial}{\partial n_+}k_{\mu_f}\,ds\right| = \left|\frac{1}{2\pi}\int_{l_r}\frac{\partial}{\partial\tau}\widetilde{k}_{\mu_f}\,ds\right| = |\widetilde{k}_{\mu_f}(z_2^{(r)}) - \widetilde{k}_{\mu_f}(z_1^{(r)})| \leq C_0|z_2^{(r)} - z_1^{(r)}|^{\gamma_7}. \tag{25}$$

Now, we turn to an estimation of the second integral in (24). The mapping Φ can be extended to a K^2-quasiconformal mapping of the extended complex plane onto itself. In particular, the sets L_r are level lines of this K^2-quasiconformal mapping. The aforementioned Privaloff-type result of Gehring and Martio states that the harmonic conjugates $\widetilde{k}_{\mu_f^{(r)}}$ of $k_{\mu_f^{(r)}}$ in $\mathrm{int}(L_r)$ are Hölder continous with the same exponent as that of $k_{\mu_f^{(r)}}$ and the Hölder constant depending only on that of $k_{\mu_f^{(r)}}$ and on K^2. But the Hölder exponent of $k_{\mu_f^{(r)}}$ in $\mathrm{int}(L_r)$ depends only on that of the boundary values k_{μ_f} on L_r, and the Hölder constant depends only on that of the boundary values and on the diameter of L_r (see [Jo80, Cor. 3]). We have thus seen that it is possible to choose all Hölder parameters independent of r.

Now, as in the previous reasoning,

$$\left|\frac{1}{2\pi}\int_{l_r}\frac{\partial}{\partial n_+}k_{\mu_f}\,ds\right| \leq C_0|z_2^{(r)} - z_1^{(r)}|^{\gamma_8}. \tag{26}$$

Inserting (25) and (26) into (24), setting $\gamma_9 := \min(\gamma_7, \gamma_8)$ and taking the limit as $r \to 0^+$ yields

$$\mu_f(l) \leq C_0|z_1 - z_2|^{\gamma_9} \leq C_0(\mu_L(l))^{\gamma_{10}},$$

where the last inequality follows, for instance, from a combination of [ABD95, Theorems 2.1 and 4.2]. ■

Sketch of the Proof of Lemma 3. Let $1 \leq i \leq n$ and consider the function h_i as defined in (15). Since we have assumed that $c_f = 0$ so that k_{μ_f} coincides with f on the support of μ_f:

$$h_i(x) - f(x) \geq \inf_{z\in L}(h_i(z) - f(z)) + k_{\mu_f}(x) - f(x) \qquad (x \in \mathrm{supp}(\mu_f)). \tag{27}$$

Reasoning as in (12) we deduce that

$$\inf_{z\in L}(h_i(z) - f(z)) \leq 0.$$

Therefore, after a cancellation of $f(x)$ on both sides of (27), we may apply the principle of domination to obtain that

$$h_i(x) \geq \inf_{z\in L}(h_i(z) - f(z)) + k_{\mu_f}(x) \qquad (x \in X).$$

In particular,

$$h_i(x_i) - f(x_i) - \inf_{z\in L}(h_i(z) - f(z)) \geq k_{\mu_f}(x_i) - f(x_i). \tag{28}$$

But the left-hand side of inequality (28) is equal to zero by the definition of the extremal points. ■

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References

[An88] V. V. ANDRIEVSKII (1988): *Constructive presentation of classes of harmonic functions with singularities on a quasiconformal arc*. Ukrainian Math. J., **40**:1–4.

[An90a] V. V. ANDRIEVSKII (1990): *A constructive characterization of harmonic functions in domains with quasiconformal boundaries*. Math. USSR-Izv., **34**:441–454.

[An90b] V. V. ANDRIEVSKII (1990): *Body contour properties of harmonic functions*. J. Soviet Math., **34**:14–23.

[ABD95] V. V. ANDRIEVSKII, V. I. BELYI, V. K. DZJADYK (1995): Conformal Invariants in Constructive Theory of Functions of Complex Variable. Atlanta, GA: World Fed. Pub.

[AB97] V. V. ANDRIEVSKII, H.-P. BLATT (1997): *A discrepancy theorem on quasiconformal curves*. Constr. Approx., **13**:363–379.

[Ah66] L. V. AHLFORS (1966): Lectures on Quasiconformal Mappings. Princeton, NJ: Van Nostrand.

[BM93] H.-P. BLATT, H. N. MHASKAR (1993): *A general discrepancy theorem*. Ark. Mat., **31**:219–246.

[Da78] B. E. J. DAHLBERG (1978): *On the distribution of Fekete points*. Duke Math. J., **45**:537–542.

[DKM98] P. DEIFT, T. KRIECHERBAUER, K. T.-R. MCLAUGHLIN (1998): *New results on the equilibrium measure for logarithmic potentials in the presence of an external field*. J. Approx. Theory, **95**:388–475.

[FS99] S. D. FISHER, E. B. SAFF (1999): *The asymptotic distribution of zeros of minimal Blaschke products*. J. Approx. Theory, **98**:104–116.

[GM83] F. W. GEHRING, O. MARTIO (1983): *Quasidisks and the Hardy–Littlewood property*. Complex Variables **2**:67–78.

[Gö98] M. GÖTZ (1998): Diskrepanzabschätzungen für Masse und Quantitative Verteilungsaussagen für Energieextremale Punktsysteme. Dissertation. Kath. Univ. Eichstätt. Aachen: Shaker Verlag.

[Jo80] E. H. JOHNSTON (1980): *The boundary modulus of continuity of harmonic functions*. Pacific J. Math., **90**:87–98.

[Kl64] W. KLEINER (1964): *Sur les approximations de M. Leja dans le probléme plan de Dirichlet*. Ann. Polon. Math., **15**:85–90.

[KP68] T. KÖVARI, CH. POMMERENKE (1968): *On the distribution of Fekete points*. Mathematika, **15**:70–75.

[KK83] J. KOREVAAR, R. A. KORTRAM (1983): *Equilibrium distributions of electrons on smooth plane conductors*. Indag. Math., **45**:203–219.

[KM98] J. KOREVAAR, M. A. MONTERIE (1998): *Approximation of the equilibrium distribution by distributions of equal point charges with minimal energy*. Trans. Amer. Math. Soc., **350**:2329–2348.

[KM99] J. KOREVAAR, M. A. MONTERIE (1999): *Fekete polynomials and potentials for continua*. Technical Report 99-07, Wisk. Rapportenseries, Fac. WINS, Univ. van Amsterdam.

[Kl84] H. KLOKE (1984): Punktsysteme mit extremalen Eigenschaften für ebene Kondensatoren. Dissertation Universität Dortmund.

[LV71] O. LEHTO, K. I. VIRTANEN (1971): Quasiconformal Mappings in the Plane, 2nd ed. Berlin: Springer-Verlag.

[Me83] K. MENKE (1983): *On Tsuji points in a continuum*. Complex Variables, **2**:165–175.
[Mo95] M. A. MONTERIE (1995): Studies in Potential Theory. PhD thesis, Free University, Amsterdam.
[PS31] G. PÓLYA, G. SZEGŐ (1931): *Über den transfiniten Durchmesser (Kapazitätskonstante) von ebenen und räumlichen Punktmengen*. J. Reine Angew. Math., **165**:4–49.
[Po65] C. POMMERENKE (1965): *Polynome und konforme Abbildung*. Monatsh. Math., **69**:58–61.
[ST97] E. B. SAFF, V. TOTIK (1997): Logarithmic Potentials with External Fields. Heidelberg: Springer–Verlag.

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