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CONSTRUCTIVE APPROXIMATION

Rates of Best Uniform Rational Approximation of Analytic Functions by Ray Sequences of Rational Functions

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Abstract. In this paper, problems related to the approximation of a holomorphic function f on a compact subset E of the complex plane $\mathbf{C}$ by rational functions from the class $\mathcal{R}_{n,m}$ of all rational functions of order (n, m) are considered. Let $\rho_{n,m} = \rho_{n,m}(f; E)$ be the distance of f in the uniform metric on E from the class $\mathcal{R}_{n,m}$. We obtain results characterizing the rate of convergence to zero of the sequence of the best rational approximation $\{\rho_{n,m(n)}\}_{n=0}^{\infty}$, $m(n)/n \to \theta \in (0, 1]$ as $n \to \infty$. In particular, we give an upper estimate for the $\liminf_{n\to\infty} \rho_{n,m(n)}^{1/(n+m(n))}$ in terms of the solution to a certain minimum energy problem with respect to the logarithmic potential. The proofs of the results obtained are based on the methods of the theory of Hankel operators.

1. Introduction

1.1.

Let E be an arbitrary compact set in the complex plane $\mathbf{C}$. Consider a function f holomorphic on the compact set E. For any nonnegative integers, n and m denote by $\mathcal{R}_{n,m}$ the class of all rational functions with complex coefficients of order (n, m):

$$\mathcal{R}_{n,m} = \{r : r = p/q, \deg p \le n, \deg q \le m, q \not\equiv 0\}.$$

The error in best approximation of f in the uniform metric on E in the class $\mathcal{R}_{n,m}$ is denoted by $\rho_{n,m}$:

$$\rho_{n,m} = \rho_{n,m}(f, E) = \inf_{r\in\mathcal{R}_{n,m}} \|f - r\|_E,$$

where $\|\cdot\|_E$ is the supremum norm on E.

Walsh used the methods of the theory of rational interpolation to investigate the convergence of best approximations of analytic functions and to estimate the order of decrease of the sequence $\{\rho_{n,n}\}_{n=0}^{\infty}$. By the well-known theorem of Walsh (see [24] and [2]), if f is holomorphic on $\overline{\mathbf{C}}\setminus F$, where F is a compact set in the extended complex

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plane $\bar{\mathbf{C}}$ such that $F \cap E = \emptyset$, then

$$\limsup_{n\to\infty} \rho_{n,n}^{1/n} \leq \exp(-1/C(E, F)),$$

where $C(E, F)$ denotes the capacity of the condenser (E, F) (see [13], [20], and [22] for the definition and properties of the capacity). We remark that Walsh's inequality is sharp in the class of all functions that are holomorphic on $\bar{\mathbf{C}}\backslash F$.

Gonchar (see, e.g., [6]) conjectured that we also have

$$\liminf_{n\to\infty} \rho_{n,n}^{1/n} \leq \exp(-2/C(E, F)). \tag{1}$$

Parfenov [15] employed methods in the theory of Hankel operators and the Adamyan–Arov–Kreĭn theorem [1] to prove the conjecture of Gonchar for the case when E is the unit disk. In [18], with the aid of the generalization of the Adamyan–Arov–Kreĭn theorem [17], the first author proved Gonchar's conjecture in the general case when E is an arbitrary compact set. We mention that the estimate (1) follows immediately from the inequality

$$\limsup_{n\to\infty} (\rho_{1,1}\rho_{2,2}\cdots\rho_{n,n})^{1/n^2} \leq \exp(-1/C(E, F)).$$

In connection with inequality (1) we point out that for some analytic functions the ordinary limit exists and equality holds in this relation. In particular, for analytic functions having finitely many branch points outside E it has been proved that

$$\lim_{n\to\infty} \rho_{n,n}^{1/n} = \exp(-2/C(E, F)),$$

where the compact set F is uniquely determined by the compact set E and the branch points of f. We mention that the works of Gonchar [7]–[9], Gonchar and Rakhmanov [10]–[11], and Stahl [21] played a leading role in this investigation. The methods used are based on the theory of rational interpolation of analytic functions, including methods of Padé approximants.

The present paper is devoted to an analysis of the rate of decrease of the ray sequence $\{\rho_{n,m(n)}\}_{n=0}^{\infty}$, $m(n)/n \to \theta \in (0, 1]$ as $n \to \infty$, of the Walsh table $\{\rho_{n,m}\}_{m,n=0}^{\infty}$ of the best rational approximations of holomorphic functions. The proof of the results obtained are based on the methods of the theory Hankel operators.

We suppose that $m = m(n)$ and that the sequence $\{m(n)\}$, $n = 0, 1, 2, \ldots,$ satisfies the following conditions:

$$m(n-1) \leq m(n) \leq m(n-1) + 1, \qquad n = 1, 2, \ldots, \tag{2}$$

and

$$\lim_{n\to\infty} \frac{m(n)}{n} = \theta, \qquad 0 < \theta \leq 1. \tag{3}$$

Let E be a compact set with a connected complement in the complex plane $\mathbf{C}$ and let f be holomorphic on $\bar{\mathbf{C}}\backslash F$, where F is a compact set in the extended complex plane $\bar{\mathbf{C}}$ such that $E \cap F = \emptyset$.

We first assume that the compact set E has positive logarithmic capacity.

Let $g(z, \xi)$ be the Green's function of the domain $\bar{\mathbf{C}} \setminus E$ with singularity at the point $\xi \in \bar{\mathbf{C}} \setminus E$. Let $M(F, \theta)$ be the set of all positive Borel measures μ with support $\operatorname{supp} \mu \subseteq F$, satisfying the conditions:

$$\mu(F) = \theta \qquad \text{and} \qquad \int_{|\xi| \geq 1} \log |\xi| \, d\mu(\xi) < +\infty.$$

Denote by $S(E, F, \theta),\ 0 \leq S(E, F, \theta) \leq +\infty$, the extremal constant in the following minimal energy problem:

$$S(E, F, \theta) = \inf_{\mu \in M(F, \theta)} J(\mu),$$

where

$$J(\mu) = \iint g(z, \xi) \, d\mu(\xi) \, d\mu(z) + (1 - \theta) \int g(z, \infty) \, d\mu(z).$$

In the case when logarithmic capacity of the compact set E is equal to zero we set $S(E, F, \theta) = +\infty$.

We note that $S(E, F, \theta) = 1/C(E, F)$ for $\theta = 1$, where $C(E, F)$ is the capacity of the condenser (E, F) (see Section 3 for more details about this minimal energy problem).

The investigation of the asymptotic behavior of the singular numbers of the Hankel operator A_f, constructed from the function f to be approximated, enable us to prove the following theorem characterizing the rate of convergence to zero of the product $\rho_{n,m(n)} \rho_{n-1,m(n)-1} \cdots \rho_{n-m(n),0}$.

Theorem 1. *Let E be an arbitrary compact set with connected complement in* $\mathbf{C}$, *and let f be holomorphic on $\bar{\mathbf{C}} \setminus F$, where F is a compact set in $\bar{\mathbf{C}}$ such that $E \cap F = \emptyset$. Then*

$$\limsup_{n \to \infty} (\rho_{n,m(n)} \rho_{n-1,m(n)-1} \cdots \rho_{n-m(n),0})^{1/nm(n)} \leq \frac{1}{\rho}, \tag{4}$$

where $\rho = \exp(S(E, F, \theta)/\theta)$.

From the theorem stated above, on the basis of the inequalities $\rho_{n,m(n)} \leq \rho_{n-1,m(n)-1} \leq \cdots \leq \rho_{n-m(n),0}$, we obtain the following result:

Corollary 1. *With the assumptions of Theorem* 1,

$$\limsup_{n \to \infty} \rho_{n,m(n)}^{1/(n+m(n))} \leq \left(\frac{1}{\rho} \right)^{1/(1+\theta)}. \tag{5}$$

Theorem 1 also gives us an upper estimate for $\liminf_{n \to \infty} \rho_{n,m(n)}^{1/(n+m(n))}$.

Corollary 2. *With the assumptions of Theorem* 1,

$$\liminf_{n \to \infty} \rho_{n,m(n)}^{1/(n+m(n))} \leq \left(\frac{1}{\rho} \right)^{2/(2-\theta)(1+\theta)}. \tag{6}$$

We single out one more result which is a direct consequence of Theorem 1. This result enables us to make more precise an estimate of $\liminf_{n\to\infty} \rho_{n,m(n)}^{1/(n+m(n))}$ for functions for which equality is attained in (5).

Corollary 3. *If*

$$\limsup_{n\to\infty} \rho_{n.m(n)}^{1/(n+m(n))} = \left(\frac{1}{\rho}\right)^{1/(1+\theta)},$$

then

$$\liminf_{n\to\infty} \rho_{n,m(n)}^{1/(n+m(n))} \leq \left(\frac{1}{\rho}\right)^{1/(1-\theta^2)}.$$

We mention that the behavior of the ray sequences $\{\rho_{n,m(n)}\}_{n=0}^{\infty}$, $m(n)/n \to \theta \in (0, 1]$ as $n \to \infty$, of the best uniform rational approximation of signum-type functions, was investigated by Levin and Saff [14]. Paper [19] is devoted to analogous questions of the best uniform rational approximation of functions of Markov type.

We present the needed auxiliary assertions in Sections 2 and 3. Among them are some questions on the theory of Hankel operators and assertions concerning potential theory. In Section 4 we prove Theorem 1.

2. Auxiliary Results of the Theory of Hankel Operators

2.1.

Let G be a bounded N-connected domain, the boundary Γ of which consists of closed analytic Jordan curves. We assume that Γ is positively oriented with respect to G and $0 \in G$. Fix a nonnegative integer l.

Denote by $L_{2,l}(\Gamma)$ the Hilbert space of functions measurable on Γ, with the inner product

$$(\varphi, \psi) = \int_\Gamma (\varphi\overline{\psi})(\xi)|\xi|^l\,|d\xi|, \qquad \varphi, \psi \in L_{2,l}(\Gamma),$$

and the norm

$$\|\varphi\|_{2,l} = \left(\int_\Gamma |\varphi(\xi)|^2|\xi|^l\,|d\xi|\right)^{1/2}, \qquad \varphi \in L_{2,l}(\Gamma).$$

Denote by $L_\infty(\Gamma)$ the space of the essentially bounded functions, with the norm

$$\|\varphi\|_\infty = \operatorname*{ess\,sup}_{\Gamma} |\varphi(\xi)|.$$

Denote by $E_p(G)$, $1 \leq p \leq \infty$, the Smirnov class of analytic functions on G (see [3], [12], [16], and [23] for more details about the classes $E_p(G)$).

Let H_l be the class of functions q representable in the form $q = \varphi/\xi^l$, where $\varphi \in E_2(G)$. Here and in what follows we will consider H_l and $E_2(G)$ as the subspace of $L_{2,l}(\Gamma)$.

Let a function f be continuous on the boundary Γ of the domain G. The Hankel operator $A_f : E_2(G) \to H_l^{\perp}$ is defined by the formula $A_f q = \mathbf{P}_-(qf),\ q \in E_2(G)$, where $H_l^{\perp}$ is the orthogonal complement of H_l in $L_{2,l}(\Gamma)$, and $\mathbf{P}_-$ is the orthogonal projection of $L_{2,l}(\Gamma)$ onto $H_l^{\perp}$. We note that A_f is a compact operator.

We denote by $\{s_{n,l}\},\ s_{n,l} = s_{n,l}(f; G),\ n = 0, 1, 2, \dots,$ the sequence of singular numbers (with multiplicities counted) of the operator A_f (the $s_{n,l}$ are eigenvalues of the operator $(A_f^* A_f)^{1/2}$, where $A_f^* : H_l^{\perp} \to E_2(G)$ is the adjoint operator of A_f).

We assume that the sequence of singular numbers $\{s_{n,l}\},\ n = 0, 1, 2, \dots$, is nonincreasing (for the properties of singular numbers, see [4]).

There exist $\{q_{n,l}\},\ \{\alpha_{n,l}\},\ n = 0, 1, 2, \dots$ (orthonormal systems of eigenfunctions of the operator $(A_f^* A_f)^{1/2}$ corresponding to the sequence of singular numbers $\{s_{n,l}\}, n = 0, 1, 2, \dots$), such that

$$\int_\Gamma (q_{i,l}\alpha_{j,l} f)(\xi)\xi^l \, d\xi = s_{j,l}\delta_{i,j}, \qquad i, j = 0, 1, 2, \dots,$$

where $\delta_{i,j}$ is the Kronecker symbol (compare with [18]).

Thus, the following formula for the product of singular numbers is valid:

$$s_{0,l}s_{1,l}\cdots s_{k,l} = \left| \int_\Gamma (q_{i,l}\alpha_{j,l} f)(\xi)\xi^l \, d\xi \right|_{i,j=0}^{k}, \qquad k = 0, 1, 2, \dots, \tag{7}$$

(the right-hand side is a determinant of order $k + 1$).

For any nonnegative integer n, denote by $\mathcal{M}_{n+l,n} = \mathcal{M}_{n+l,n}(G)$ the class functions representable in the form $h = p/q\xi^l$, where $p \in E_\infty(G)$ and q is a polynomial of degree at most $n,\ q \not\equiv 0$. Note that $h \in \mathcal{M}_{n+l,n}$ has no more than $n + l$ poles and no more than n free poles.

We denote the error in the best approximation of f in the space $L_\infty(\Gamma)$ in the class $\mathcal{M}_{n+l,n}$ by

$$\Delta_{n+l,n} = \Delta_{n+l,n}(f; G) = \inf_{h \in \mathcal{M}_{n+l,n}} \| f - h\|_\infty.$$

In the case when G is the open unit disk and $l = 0$, Adamyan, Arov, and Kreĭn [1] proved the equalities $s_{n,0} = \Delta_{n,n},\ n = 0, 1, 2, \dots$.

Using the same arguments as in [17] it is not hard to prove a theorem establishing a connection between the singular numbers of the Hankel operator and the best approximations $\Delta_{n+l,n}$ of f. This theorem is a generalization of the Adamyan–Arov–Kreĭn theorem for the case when G is an N-connected domain and $l \geq 0$ (for $l = 0$ see [17]).

Let G be a bounded domain whose boundary Γ consists of N disjoint closed analytic Jordan curves, and let f be a continuous function on Γ. Then

$$\Delta_{n+N-1+l,n+N-1} \leq s_{n,l} \leq \Delta_{n+l,n} \tag{8}$$

for all integers $n \geq N - 1$.

3. Some Results of Potential Theory

3.1.

Let E be a compact subset of the complex plane $\mathbf{C}$ with connected complement and let F be a compact subset of the extended complex plane $\bar{\mathbf{C}}$ such that the sets E and F are disjoint.

Fix a number $\theta \in (0, 1]$. We denote by $M(E, F, \theta)$ the set of all signed measures τ of the form $\tau = \tau_2 - \tau_1$, where τ_1 and τ_2 are positive Borel measures with supports $\operatorname{supp} \tau_1 \subseteq E$, and $\operatorname{supp} \tau_2 \subseteq F$. It will also be assumed that $\tau_1(E) = (1+\theta)/2,\ \tau_2(F) = \theta$, and

(9) $$\int_{|\xi|\geq 1} \log|\xi|\, d\tau_2(\xi) < +\infty.$$

The logarithmic energy of a signed measure $\tau \in M(E, F, \theta)$ is defined as

$$I(\tau) = \iint \log \frac{1}{|\xi - t|} d\tau(\xi)\, d\tau(t)$$

and its logarithmic potential is given by

$$V^{\tau}(z) = \int \log \frac{1}{|z - \xi|} d\tau(\xi).$$

We note that if $\tau \in M(E, F, \theta)$, then $I(\tau) > -\infty$ and $V^{\tau}(z) > -\infty$ for all $z \in \mathbf{C}$.

We first consider the case when the logarithmic capacities of the sets E and F are greater than zero.

The following assertion can be established by well-known methods of potential theory (see, e.g., [13], [20], and [22]):

There is a unique signed measure $\tau^* = \tau_2^* - \tau_1^* \in M(E, F, \theta)$ that minimized the logarithmic energy in the class $M(E, F, \theta)$:

(10) $$I(\tau^*) = \inf_{\tau \in M(E,F,\theta)} I(\tau).$$

There are constants A and B such that the logarithmic potential $V^{\tau^*}(z)$ has the following equilibrium property:

(11) $$V^{\tau^*}(z) \leq B \quad \text{on } \operatorname{supp}(\tau_2^*),$$

(12) $$V^{\tau^*}(z) \geq B \quad \text{q.e. on } F,$$

and

(13) $$V^{\tau^*}(z) = A \quad \text{q.e. on } E,$$

where q.e. (*quasi-everywhere*) *means neglecting sets of zero logarithmic capacity.*

Let $W(E, F, \theta) = B - A$. We remark that $I(\tau^*) = W(E, F, \theta) = 1/C(E, F)$ for $\theta = 1$, where $C(E, F)$ is the capacity of the condenser (E, F).

Denote by $g(z,\xi)$ the Green's function of the domain $\bar{\mathbf{C}}\backslash E$ with singularity at the point $\xi \in \bar{\mathbf{C}}\backslash E$. Let $\mu \in M(F,\theta)$, where $M(F,\theta)$ is the set of all positive Borel measures μ with support $\operatorname{supp}\mu \subseteq F$, satisfying the relation $\mu(F)=\theta$ and condition (9). The Green's potential of the positive measure μ is denoted by

$$V_g^{\mu}(z) = \int g(z,\xi)\,d\mu(\xi), \qquad z \in \bar{\mathbf{C}}\backslash E.$$

We note that the Green's potential can be expressed in the form

$$V_g^{\mu}(z) = V^{\mu-\tilde{\mu}}(z) + \int g(\xi,\infty)\,d\mu(\xi), \qquad z \in \bar{\mathbf{C}}\backslash E, \tag{14}$$

where $\tilde{\mu}$ is balayage of the measure μ on E.

We consider the energy of a measure $\mu \in M(F,\theta)$ with respect to the Green's potential:

$$J(\mu) = \iint g(t,\xi)\,d\mu(\xi)\,d\mu(t) + (1-\theta)\int g(\xi,\infty)\,d\mu(\xi). \tag{15}$$

The following assertion can be proved in the same manner as the assertion stated above:

There is a unique positive measure $\mu^* \in M(F,\theta)$ minimizing the energy expression (15) *in the class $M(F,\theta)$*:

$$J(\mu^*) = S(E,F,\theta) = \inf_{\mu \in M(F,\theta)} J(\mu).$$

There is a constant $W'(E,F,\theta)$ such that the Green's potential $V_g^{\mu^*}(z)$ has the following equilibrium properties:

$$V_g^{\mu^*}(z) + g(z,\infty)(1-\theta)/2 \le W'(E,F,\theta) \quad \textit{on } \operatorname{supp}(\mu^*),$$
$$V_g^{\mu^*}(z) + g(z,\infty)(1-\theta)/2 \ge W'(E,F,\theta) \quad \textit{q.e. on } F.$$

It follows from relations (10)–(14) that the following formulas hold:

$$\tau_2^* = \mu^*, \qquad \tau_1^* = \tilde{\mu}^* + \mu_E(1-\theta)/2,$$

$$W(E,F,\theta) = W'(E,F,\theta),$$

and

$$V_g^{\mu^*}(z) + g(z,\infty)(1-\theta)/2 = V^{\tau^*}(z) - A,$$

where μ_E is the equilibrium measure of mass 1 for the compact set E. We have

$$S(E,F,\theta) = \theta W(E,F,\theta) + \frac{1-\theta}{2}\int g(\xi,\infty)\,d\mu^*(\xi). \tag{16}$$

Note that for $\theta = 1$ we get

$$S(E, F, \theta) = \frac{1}{C(E, F)} = \inf_{\mu \in M(F,\theta)} \iint g(t, \xi)\, d\mu(\xi)\, d\mu(t).$$

In the situation when the logarithmic capacity of the set F is equal to zero, we have $C(E, F) = 0$. Using this fact, on the basis of the relation

$$(17) \qquad \begin{aligned} S(E, F, \theta) &= \inf_{\mu \in M(F,\theta)} J(\mu) \\ &\geq \inf_{\mu \in M(F,\theta)} \iint g(t, \xi)\, d\mu(\xi)\, d\mu(t) = \frac{1}{C(E, F)}, \end{aligned}$$

we get $S(E, F, \theta) = +\infty$.

In the case when logarithmic capacity of the set E is equal to zero we set $S(E, F, \theta) = +\infty$.

We mention that $S(E, F, \theta) < +\infty$ if and only if logarithmic capacities of the sets E and F are greater than zero.

3.2.

In this subsection we present a simple example in which $S(E, F, \theta)$ can be explicitly determined.

Let $E = \{z : |z| \leq 1\}$ and $F = \{z : |z| = \rho\}$, $\rho > 1$. Since

$$\frac{1}{2\pi} \int_0^{2\pi} \ln \frac{1}{|z - re^{i\varphi}|} d\varphi = \begin{cases} \ln 1/|z|, & |z| > r, \\ \ln 1/r, & |z| \leq r, \end{cases}$$

it follows that

$$d\tau_1^* = \frac{1+\theta}{2} \frac{1}{2\pi} d\varphi \quad \text{on } \partial E, \qquad d\tau_2^* = \theta \frac{1}{2\pi} d\varphi \quad \text{on } F,$$

and

$$A = \theta \ln \frac{1}{\rho}, \qquad B = -\frac{1-\theta}{2} \ln \frac{1}{\rho}.$$

Therefore,

$$W(E, F, \theta) = \frac{1+\theta}{2} \ln \rho.$$

Using the formula

$$g(z, \infty) = \ln |z|, \qquad |z| > 1,$$

we get

$$\int g(\xi, \infty)\, d\mu^*(\xi) = \int g(\xi, \infty)\, d\tau_2^*(\xi) = \theta \ln \rho.$$

Then, by (16), we have

$$S(E, F, \theta) = \frac{\theta(1+\theta)}{2} \ln \rho + \frac{\theta(1-\theta)}{2} \ln \rho = \theta \ln \rho.$$

We mention that in this example the quantity $\exp(S(E, F, \theta)/\theta)$ does not depend on θ and equals ρ, where ρ is the radius of the circle F.

3.3.

Let E be a compact set with connected complement in the complex plane $\mathbf{C}$, and let F be a compact set in the extended complex plane $\bar{\mathbf{C}}$ such that the sets E and F are disjoint.

In what follows we will use the following assertion:

Lemma 1. *Suppose that a sequence of condensers* (E_k, F_k), *where* $E_k,\ k = 1, 2, \dots,$ *is the compact set with connected complement in the complex plane* $\mathbf{C}$, *tends monotonically to the condenser* (E, F):

$$E \subset E_k \subset E_{k-1}, \qquad F \subset F_k \subset F_{k-1},$$

$$E = \bigcap_k E_k, \qquad F = \bigcap_k F_k, \qquad E_k \cap F_k = \emptyset.$$

Then

$$S(E_k, F_k, \theta) \to S(E, F, \theta) \qquad \textit{as} \quad k \to +\infty.$$

Proof. We first assume that the logarithmic capacities of E and F are positive.

Since the sequence of condensers $(E_k, F_k),\ k = 1, 2, \dots,$ tends monotonically to (E, F) and the Green's function $g_k(z, \xi)$ of the domain $\bar{\mathbf{C}} \setminus E_k,\ k = 1, 2, \dots,$ is nonincreasing as the compact set E_k expands, it follows from the definition of $S(E_k, F_k, \theta)$ that

$$S(E_k, F_k, \theta) \le S(E_{k+1}, F_{k+1}, \theta) \le S(E, F, \theta), \qquad k = 1, 2, \dots. \tag{18}$$

Let $\mu_k \in M(F_k, \theta),\ k = 1, 2, \dots,$ be a sequence of positive Borel measures for which

$$\iint g_k(t, \xi)\, d\mu_k(t) + (1 - \theta) \int g_k(\xi, \infty)\, d\mu_k(\xi) = S(E_k, F_k, \theta), \qquad k = 1, 2, \dots.$$

We can assume without loss of generality that μ_k tends to a positive measure μ in the weak-star topology on all positive Borel measures on the extended complex plane $\bar{\mathbf{C}}$. It is not hard to see that $\mu \in M(F, \theta)$. We have by Fatou's lemma and the definition of $S(E, F, \theta)$,

$$S(E, F, \theta) \le J(\mu) \le \lim_{k \to \infty} S(E_k, F_k, \theta).$$

Hence by (18)

$$\lim_{k \to \infty} S(E_k, F_k, \theta) = S(E, F, \theta).$$

We now consider the case when the logarithmic capacity of the set F is equal to zero; the case when the logarithmic capacity of the set E is equal to zero can be treated analogously. First, note that in this case $C(E, F) = 0$ and $S(E, F, \theta) = +\infty$. Second, by formula (18), applied to the pair (E_k, F_k), we get

$$S(E_k, F_k, \theta) \ge \frac{1}{C(E_k, F_k)}.$$

Thus, since $C(E_k, F_k) \to C(E, F)$ as $k \to \infty$, it follows from the last inequality

$$\lim_{k \to \infty} S(E_k, F_k, \theta) = S(E, F, \theta). \qquad \blacksquare$$

3.4.

A condenser (E, F) is called *proper* if E and F are bounded by finitely many disjoint closed analytic Jordan curves.

Let (E, F) be an arbitrary proper condenser such that E is a compact set with connected complement in the complex plane $\mathbf{C}$.

Denote by G the domain $\bar{\mathbf{C}}\backslash E$. We remark that the complement of F in the domain G consists of a finite number of connected components. We distinguish components $\{G_i\}$ of $G\backslash F$ such that $\partial G_i \cap E \neq \emptyset$. Let $\tilde{F} = G\backslash \bigcup_i G_i$ and let Γ be a boundary of $\tilde{F}$. Note that the condenser $(E, \tilde{F})$ is proper and that $\Gamma \subseteq F \subseteq \tilde{F}$.

Lemma 2. *The following formulas hold*:

$$S(E, \Gamma, \theta) = S(E, F, \theta) = S(E, \tilde{F}, \theta). \tag{19}$$

Proof. Denote by $\tau^* = \tau_2^* - \tau_1^* \in M(E, \tilde{F}, \theta)$ the extremal signed measure, satisfying relation (10), where F is replaced by $\tilde{F}$. Since the condenser $(E, \tilde{F})$ is proper, we can conclude (see, e.g., [13], [20], and [22]) that

$$V^{\tau^*}(z) = A \quad \text{on } E, \qquad V^{\tau^*}(z) \geq B \quad \text{on } \tilde{F},$$

and

$$V^{\tau^*}(z) = B \quad \text{on } \operatorname{supp} \tau_2^*.$$

To prove (19) it suffices to show that $\operatorname{supp} \tau_2^* \subseteq \Gamma$. The proof of this fact is by contradiction.

We first remark that $\tilde{F}$ is a union of a finite number of closed nonintersecting domain $\bar{U}_j$. Namely, $\tilde{F} = \bigcup_j \bar{U}_j$, and $\Gamma = \bigcup_j \partial U_j$.

Assume that there exist $x \in \operatorname{supp} \tau_2^*$ such that x belongs to an open domain U_{j_0} for some j_0. Since $x \in \operatorname{supp} \tau_2^*$,

$$V^{\tau^*}(x_0) = B.$$

Using the minimum principle for the superharmonic function $V^{\tau^*}(z)$ in the domain U_{j_0}, we get

$$V^{\tau^*}(z) = B, \qquad z \in U_{j_0},$$

so that (see, e.g., [22])

$$\tau_2^*(U_{j_0}) = 0.$$

The last equality contradicts our assumption $\operatorname{supp} \tau_2^* \cap U_{j_0} \neq \emptyset$. Hence, $\operatorname{supp} \tau_2^* \subseteq \Gamma$. ■

4. Proof of Theorem 1

4.1.

It will be assumed in Subsections 4.1 and 4.2 that the pair (E, F) forms a proper condenser. This case is of fundamental importance for the proof of the theorem. It is in this situation that we use results from the theory of Hankel operators to prove estimate (4). The general case is investigated with the help of this particular case (see Subsection 4.3).

Let E_1 and F_1 be the preimages of E and F under the mapping $z = a + 1/\xi$, where a is some fixed interior point in E. Observe that E_1 will contain ∞ and the point 0 belongs to the domain $G = \bar{\mathbf{C}} \setminus E_1$. Let $f_1(\xi) = f(a + 1/\xi)$. It will be assumed that the boundary Γ of the domain G is positively oriented with respect to G.

We mention that the equality $\rho_{n,m}(f; E) = \rho^*_{n,m}(f_1; E_1)$ holds for all nonnegative integers n and m, where

$$\rho^*_{n,m} = \rho^*_{n,m}(f_1; E_1) = \inf_{r \in \mathcal{R}^*_{n,m}} \| f_1 - r \|_{E_1} \tag{20}$$

and

$$\mathcal{R}^*_{n,m} = \{r : r = p/qz^{n-m}, \deg p \le n, \deg q \le m, q \not\equiv 0\}.$$

Denote by $g_1(z, \xi)$ the Green's function for the domain G with singularity at the point $\xi \in G$. We consider the extremal problem

$$S^*(E_1, F_1, \theta) = \inf\left(\iint g_1(z, \xi)\, d\mu(\xi)\, d\mu(z) + (1 - \theta) \int g_1(z, 0)\, d\mu(z) \right),$$

where the infimum is taken over all positive Borel measures μ with support $\operatorname{supp} \mu \subseteq F_1$, satisfying the conditions:

$$\mu(F_1) = \theta \qquad \text{and} \qquad \int_{|\xi| \le 1} \log \frac{1}{|\xi|} d\mu(\xi) < +\infty.$$

Since the Green's function is invariant under linear fractional transformation of the extended complex plane $\bar{\mathbf{C}}$, the following equality holds:

$$S^*(E_1, F_1, \theta) = S(E, F, \theta). \tag{21}$$

We shall prove that

$$\limsup_{n \to \infty} \left(\prod_{j=0}^{m(n)} \rho^*_{n-j,m(n)-j} \right)^{1/nm(n)} \le \exp(-S^*(E_1, F_1, \theta)/\theta), \tag{22}$$

from which, by (20) and (21), we get (4).

In this subsection we establish the inequality

$$\limsup_{n \to \infty} (\Delta_{n,m(n)} \Delta_{n-1,m(n)-1} \cdots \Delta_{n-m(n),0})^{1/nm(n)} \le \exp(-S^*(E_1, F_1, \theta)/\theta), \tag{23}$$

where

$$\Delta_{n-j,m(n)-j} = \Delta_{n-j,m(n)-j}(f_1; G) = \inf_h \| f_1 - h \|_\infty, \qquad j = 0, 1, \ldots, m(n),$$

is the best approximation of f_1 in $L_\infty(\partial G)$ in the class $\mathcal{M}_{n-j,m(n)-j}$. For this it suffices to show (see estimate (8)) that

$$\limsup_{n \to \infty} (s_{0,n-m(n)} s_{1,n-m(n)} \cdots s_{m(n),n-m(n)})^{1/nm(n)} \le \exp(-S^*(E_1, F_1, \theta)/\theta), \tag{24}$$

where $\{s_{k,n-m(n)}\}$, $s_{k,n-m(n)} = s_{k,n-m(n)}(f_1; G)$, $k = 0, 1, 2, \ldots$, is the sequence of singular numbers of the Hankel operator $A_{f_1} : E_2(G) \to H^{\perp}_{n-m(n)}$, constructed from the function f_1.

It is not difficult to pass from the estimate (23) to (22) (see Subsection 4.2); therefore, we now restrict ourselves to proving inequality (24).

First of all we introduce the necessary notation. Let $w(z)$ be the solution of the Dirichlet problem constructed in each of the finite number of domains making up the open set $\bar{\mathbf{C}}\setminus(E_1 \cup F_1)$ with boundary data equal to 1 on ∂F_1 and to 0 on ∂E_1. It will be assumed that $w(z)$ is extended by continuity to $\bar{\mathbf{C}}$: $w(z) = 1$ for $z \in F_1$, and $w(z) = 0$ for $z \in E_1$. For an arbitrary number ε with $0 < \varepsilon < 1$, let $\gamma(\varepsilon) = \{z : w(z) = \varepsilon\}$, $E_1(\varepsilon) = \{z : w(z) \le \varepsilon\}$, $G(\varepsilon) = \{z : w(z) > \varepsilon\}$, and $F_1(\varepsilon) = \{z : w(z) \ge \varepsilon\}$. Note that

$$E_1 \subset E_1(\varepsilon_1) \subset E_1(\varepsilon), \qquad F_1 \subset G(\varepsilon) \subset G(\varepsilon_1) \subset G, \qquad F_1 \subset F_1(\varepsilon) \subset F_1(\varepsilon_1),$$

for $0 < \varepsilon_1 < \varepsilon < 1$, and

$$E_1 = \bigcap_{0<\varepsilon<1} E_1(\varepsilon), \qquad G = \bigcup_{0<\varepsilon<1} G(\varepsilon).$$

Let

$$\tilde{F}_1 = \bigcap_{0<\varepsilon<1} F_1(\varepsilon).$$

Before continuing with the proof of the theorem we note that the quantity $S^*(E_1, F_1, \theta)$ satisfies properties which are analogues of Lemmas 1 and 2. In particular, on the basis of the fact that the Green's function is invariant under linear fractional transformations of the extended complex plane $\bar{\mathbf{C}}$ and with aid of Lemma 1, we get

$$\lim_{\varepsilon\to 1, \varepsilon_1 \to 0} S^*(E_1(\varepsilon_1), F_1(\varepsilon), \theta) = S^*(E_1, \tilde{F}_1, \theta) \tag{25}$$

and

$$\lim_{\varepsilon_1 \to 0} S^*(E_1(\varepsilon_1), F_1, \theta) = S^*(E_1, F_1, \theta). \tag{26}$$

Analogously, using Lemma 2, we obtain

$$S^*(E_1, F_1, \theta) = S^*(E_1, \tilde{F}_1, \theta) \tag{27}$$

and

$$S^*(E_1(\varepsilon_1), \gamma(\varepsilon), \theta) = S^*(E_1(\varepsilon_1), F_1(\varepsilon), \theta) \tag{28}$$

for $0 < \varepsilon_1 < \varepsilon < 1$.

We choose and fix a number ε close enough to 1 so that the contour $\gamma(\varepsilon)$ consists of finitely many closed analytic curves that in total separate E_1 and F_1. It will be assumed that $\gamma(\varepsilon)$ is positively oriented with respect to the open set $G(\varepsilon)$. Fix a positive integer n such that $m(n) \ge 1$.

Let us use formula (7) with $k = m(n),\ l = n - m(n)$. Since the functions $q_{i,n-m(n)}$, $\alpha_{j,n-m(n)}, i,\ j = 0, 1, 2, \ldots$, belong to $E_2(G)$ and f_1 is holomorphic on $\bar{\mathbf{C}}\setminus F_1$, the relation

$$s_{0,n-m(n)}s_{1,n-m(n)}\cdots s_{m(n),n-m(n)} = \left|\int_{\gamma(\varepsilon)}(q_{i,n-m(n)}\alpha_{j,n-m(n)}\, f_1)(\xi)\xi^{n-m(n)}\, d\xi\right|_{i,j=0}^{m(n)}$$

can be written for the product of singular numbers. From the last relation (compare with [5] and [18]),

$$\begin{aligned}(29)\qquad (m(n)+1)!\prod_{i=0}^{m(n)} s_{i,n-m(n)} &= \int_{\gamma(\varepsilon)}\cdots\int_{\gamma(\varepsilon)} f_1(\xi_0)\cdots f_1(\xi_{m(n)})\\ &\times B_1(\xi_0,\ldots,\xi_{m(n)})B_2(\xi_0,\ldots,\xi_{m(n)})\prod_{i=0}^{m(n)}\xi_i^{n-m(n)}d\xi_0\cdots d\xi_{m(n)},\end{aligned}$$

where

$$(30)\qquad B_1(\xi_0,\xi_1,\ldots,\xi_{m(n)}) = |\alpha_{i,n-m(n)}(\xi_j)|_{i,j=0}^{m(n)}$$

and

$$(31)\qquad B_2(\xi_0,\xi_1,\ldots,\xi_{m(n)}) = |q_{i,n-m(n)}(\xi_j)|_{i,j=0}^{m(n)}.$$

Next we estimate the determinants B_1 and B_2. To do this we fix a number ε_1 with $0 < \varepsilon_1 < \varepsilon < 1$. It will be assumed that ε_1 is close enough to 0 that the open set $G(\varepsilon_1)$ is a domain and $0 \in G(\varepsilon_1)$. By the Cauchy formula,

$$\alpha^2_{j,n-m(n)}(\xi)\xi^{n-m(n)} = \frac{1}{2\pi i}\int_\Gamma \frac{\alpha^2_{j,n-m(n)}(t)t^{n-m(n)}}{t-\xi}dt,\qquad \xi\in\gamma(\varepsilon_1),\quad j = 0, 1, \ldots.$$

Since $\|\alpha_{j,n-m(n)}\|_{2,n-m(n)} = 1,\ j = 0, 1, 2, \ldots$, it follows from the last formula that

$$(32)\qquad \max_{\xi\in\gamma(\varepsilon_1)}|\alpha^2_{j,n-m(n)}(\xi)\xi^{n-m(n)}| \le C,\qquad j = 0, 1, \ldots,$$

(here and in what follows $C, C_1, C_2, \ldots$ denote positive quantities not depending on n). Similarly, since $\|q_{j,n-m(n)}\|_{2,n-m(n)} = 1,\ j = 0, 1, 2, \ldots$, it follows that

$$(33)\qquad \max_{\xi\in\gamma(\varepsilon_1)}|q^2_{j,n-m(n)}(\xi)\xi^{n-m(n)}| \le C,\qquad j = 0, 1, 2, \ldots.$$

Using the inequalities (32) and (33), we can write

$$\begin{aligned}(34)\qquad \max_{\xi_i\in\gamma(\varepsilon_1)}&|B_1(\xi_0,\ldots,\xi_{m(n)})B_2(\xi_0,\ldots,\xi_{m(n)})\cdot\xi_0^{n-m(n)}\cdots\xi_{m(n)}^{n-m(n)}|\\ &\le((m(n)+1)!)^2C^{m(n)+1}.\end{aligned}$$

Denote by $g_2(z,\xi)$ the Green's function of the domain $G(\varepsilon_1)$ with singularity at the point $\xi \in G(\varepsilon_1)$. We estimate the product $B_1 B_2 \prod_{i=0}^{m(n)} \xi_i^{n-m(n)}$ in the case when the variables ξ_i, $i = 0,\dots,m(n)$, belong to $\gamma(\varepsilon)$. Using formulas (30) and (31), we find

$$\begin{aligned}(35)\quad D(\xi_0,\dots,\xi_{m(n)}) &:= B_1(\xi_0,\dots,\xi_{m(n)}) B_2(\xi_0,\dots,\xi_{m(n)}) \xi_0^{n-m(n)} \cdots \xi_{m(n)}^{n-m(n)} \\ &= \prod_{0\le i<j\le m(n)} (\xi_i - \xi_j)^2 \cdot \Psi(\xi_0,\dots,\xi_{m(n)}) \xi_0^{n-m(n)} \cdots \xi_{m(n)}^{n-m(n)},\end{aligned}$$

where the function $\Psi(\xi_0,\dots,\xi_{m(n)})$ is holomorphic function of $m(n)+1$ complex variables in the domain $G(\varepsilon_1)\times\cdots\times G(\varepsilon_1)$ ($m(n)+1$ factors in the Cartesian product).

Let us consider the function

$$\ln|D(\xi_0,\xi_1,\dots,\xi_{m(n)})| + 2\sum_{0\le i<j\le m(n)} g_2(\xi_i,\xi_j) + (n-m(n))\sum_{i=0}^{m(n)} g_2(\xi_i,0).$$

We note that this function is subharmonic in the domain $G(\varepsilon_1)$ with respect to the variable ξ_i, $i = 0,\dots,m(n)$, when the remaining variables $\xi_j \in G(\varepsilon_1)$, $j \ne i$, $j \in \{0,1,\dots,m(n)\}$, are fixed.

We use the maximum principle for subharmonic functions successively with respect to each variable, together with (34) and (36), obtaining

$$\begin{aligned}\ln|D(\xi_0,\xi_1,\dots,\xi_{m(n)})| + 2\sum_{0\le i<j\le m(n)} g_2(\xi_i,\xi_j) + (n-m(n))\sum_{i=0}^{m(n)} g_2(\xi_i,0)& \\ \le \ln(((m(n)+1)!)^2 C^{m(n)+1})&,\end{aligned}$$

where $\xi_i \in \gamma(\varepsilon)$, $i = 0,1,\dots,m(n)$.

By the formula for a product of singular numbers (see (29)), this gives us the inequality

(36)

$$\begin{aligned}\prod_{i=0}^{m(n)} s_{i,n-m(n)} &\le (m(n)+1)!\, C_1^{m(n)} \left(\max_{\xi\in\gamma(\varepsilon)} |f_1(\xi)|\right)^{m(n)+1} \\ &\times \exp\left(-\min_{\xi_i\in\gamma(\varepsilon)}\left(2\sum_{0\le i<j\le m(n)} g_2(\xi_i,\xi_j) + (n-m(n))\sum_{i=0}^{m(n)} g_2(\xi_i,0)\right)\right).\end{aligned}$$

Let us estimate the right-hand side of this inequality. We note first that there exists a constant $C_2 > 0$ such that f admits the upper estimate

$$(37)\qquad \max_{\xi\in\gamma(\varepsilon)} |f_1(\xi)| \le C_2.$$

We next use the relation which can be obtained by the well-known methods of potential theory (see, e.g., [13], [20], and [22])

$$\begin{aligned}(38)\quad \frac{1}{n^2}\min_{\xi_i\in\gamma(\varepsilon)}\left(2\sum_{0\le i<j\le m(n)} g_2(\xi_i,\xi_j) + (n-m(n))\sum_{i=0}^{m(n)} g_2(\xi_i,0)\right)& \\ \to S^*(E_1(\varepsilon_1),\gamma(\varepsilon),\theta)&.\end{aligned}$$

From the inequalities (36)–(38) and relations (3) and (28) we obtain

$$\limsup_{n\to\infty}(s_{0,n-m(n)}s_{1,n-m(n)}\cdots s_{m(n),n-m(n)})^{1/nm(n)} \le \exp(-S^*(E_1(\varepsilon_1), F_1(\varepsilon), \theta)/\theta).$$

By the properties of the quantities $S^*(E_1(\varepsilon_1), F_1(\varepsilon), \theta)$ (see (25)), it is possible to pass to the limit on the right-hand side of the last relation as $\varepsilon_1 \to 0$ and $\varepsilon \to 1$, getting

$$\limsup_{n\to\infty}(s_{0,n-m(n)}s_{1,n-m(n)}\cdots s_{m(n),n-m(n))})^{1/nm(n)} \le \exp(-S^*(E_1, \tilde{F}_1, \theta)/\theta).$$

From this, by (27),

$$\limsup_{n\to\infty}(s_{0,n-m(n)}s_{1,n-m(n)}\cdots s_{m(n),n-m(n)})^{1/nm(n)} \le \exp(-S^*(E_1, F_1, \theta)/\theta).$$

As mentioned above, relation (23) is thereby obtained.

4.2.

We now show how to use the estimate (23) to get the inequality (22).

Fix $\varepsilon_1 > 0$ sufficiently close to 0 so that the open set $G(\varepsilon_1)$ is a domain and $0 \in G(\varepsilon_1)$. Here it is assumed that $\gamma(\varepsilon_1)$ is positively oriented with respect to $G(\varepsilon_1)$.

Fix also nonnegative integers n and j, $0 \le j \le m(n)$. For an arbitrary function h representable in the form $h = p/(qz^{n-m(n)})$, where $p \in E_\infty(G(\varepsilon_1))$, q is a polynomial of degree at most $m(n) - j$, with zeros outside $\gamma(\varepsilon_1)$, $q \not\equiv 0$, we have by the Cauchy formula

$$(r' - f_1)(z) + f_1(\infty) = \frac{1}{2\pi i}\int_{\gamma(\varepsilon_1)} \frac{(f_1 - h)(\xi)\, d\xi}{\xi - z}, \qquad z \in E_1, \tag{39}$$

where r' is the sum of the principal parts of h corresponding to poles of h lying in $G(\varepsilon_1)$.

We estimate the integral in (39), getting

$$\| f_1 - f_1(\infty) - r'\|_{E_1} \le C\| f_1 - h\|_\infty, \tag{40}$$

where the positive quantity C is independent of h, n, and j, and $\|\cdot\|_\infty$ is the norm in the space $L_\infty(\gamma(\varepsilon_1))$.

Using now the definition of the quantity $\rho^*_{n-j,m(n)-j}$ and the fact that the rational function $r' + f_1(\infty)$ belongs to the class $\mathcal{R}^*_{n-j,m(n)-j}$, we have from (40) the estimate

$$\rho^*_{n-j,m(n)-j} \le C\| f_1 - h\|_\infty.$$

Next, since h is an arbitrary function in $\mathcal{M}_{n-j,m(n)-j}(G(\varepsilon_1))$

$$\rho^*_{n-j,m(n)-j} \le C \inf_{h\in\mathcal{M}_{n-j,m(n)-j}G(\varepsilon_1)} \| f_1 - h\|_\infty = C\Delta_{n-j,m(n)-j}(f_1; G(\varepsilon_1)).$$

We now use results in Subsection 4.1 (see relation (23)), applied to the pair $(E_1(\varepsilon_1), F_1)$ of the compact sets, to get

$$\limsup_{n\to\infty}(\rho^*_{n,m(n)}\rho^*_{n-1,m(n)-1}\cdots\rho^*_{n-m(n),0})^{1/nm(n)} \le \exp(-S^*(E_1(\varepsilon_1), F_1, \theta)/\theta).$$

It remains to let ε_1 tend to 0, use the limit relation (26) and get the required relation (22).

4.3.

We now get rid of the condition that (E, F) forms a proper condenser, i.e., that E and F are bounded by finitely many disjoint closed analytic Jordan curves. Consider the case when (E, F) is an arbitrary condenser.

We construct a sequence of proper condensers (E'_k, F'_k), $k = 1, 2, \dots,$ that tends monotonically to $(E, F) : E \subset E'_k \subset E'_{k-1},\ F \subset F'_k \subset F'_{k-1},\ E = \bigcap_k E'_k,\ F = \bigcap_k F'_k,\ E'_k \cap F'_k = \emptyset$.

Fix a positive integer k. Since E is contained in the compact set E'_k, we can write

$$\rho_{n,m}(f; E) \le \rho_{n,m}(f; E'_k) \tag{41}$$

for nonnegative integers n and m. We now use the fact that (E'_k, F'_k) is a proper condenser and f is holomorphic on $\bar{\mathbf{C}}\backslash F'_k$, therefore, we can apply the results in the preceding subsection and get the estimate

$$\limsup_{n\to\infty}\left(\prod_{j=0}^{m(n)} \rho_{n-j,m(n)-j}(f; E'_k)\right)^{1/nm(n)} \le \exp(-S(E'_k, F'_k, \theta)/\theta),$$

which implies that (see (41))

$$\limsup_{n\to\infty}\left(\prod_{j=0}^{m(n)} \rho_{n-j,m(n)-j}(f; E)\right)^{1/nm(n)} \le \exp(-S(E'_k, F'_k, \theta)/\theta). \tag{42}$$

By Lemma 1 we can pass to the limit on the right-hand side of (42) as $k \to \infty$, obtaining

$$\limsup_{n\to\infty}\left(\prod_{j=0}^{m(n)} \rho_{n-j,m(n)-j}(f; E)\right)^{1/nm(n)} \le \exp(-S(E, F, \theta)/\theta).$$

Theorem 1 is proved. ■

4.4.

Let us proceed to the proofs of Corollaries 2 and 3 to Theorem 1.

Proof of Corollary 2. The proof of Corollary 2 is by contradiction. Suppose that

$$\liminf_{n\to\infty} \rho_{n,m(n)}^{1/(n+m(n))} \ge \left(\frac{1}{\rho}\right)^{2\lambda/(2-\theta)(1+\theta)},$$

where $0 < \lambda < 1$. From the last relation, we get

$$\liminf_{n\to\infty} \rho_{n,m(n)}^{1/n} \ge \left(\frac{1}{\rho}\right)^{2\lambda/(2-\theta)}. \tag{43}$$

With help of relation (2) we can write

$$m(n) - j \le m(n-j), \qquad j = 0, 1, \dots, m(n).$$

Therefore, we have

$$\rho_{n-j,m(n-j)} \leq \rho_{n-j,m(n)-j}, \qquad j = 0, 1, \ldots, m(n)$$

and

$$\prod_{k=n-m(n)}^{n} \rho_{k,m(k)} = \prod_{j=0}^{m(n)} \rho_{n-j,m(n-j)} \leq \prod_{j=0}^{m(n)} \rho_{n-j,m(n)-j}. \tag{44}$$

It follows from Theorem 1 that

$$\limsup_{n\to\infty} \left(\prod_{j=0}^{m(n)} \rho_{n-j,m(n)-j} \right)^{1/nm(n)} \leq \frac{1}{\rho}. \tag{45}$$

By relations (43) and (44) and

$$\lim_{n\to\infty} \frac{\sum_{k=n-m(n)}^{n} k}{nm(n)} = \frac{2-\theta}{2},$$

we get

$$\liminf_{n\to\infty} \left(\prod_{j=0}^{m(n)} \rho_{n-j,m(n)-j} \right)^{1/nm(n)} \geq \left(\frac{1}{\rho}\right)^{\lambda} > \frac{1}{\rho},$$

which contradicts inequality (45). ■

Proof of Corollary 3. Let Λ be a sequence of positive integers such that

$$\lim_{n\to\infty, n\in\Lambda} \rho_{n,m(n)}^{1/(n+m(n))} = \left(\frac{1}{\rho}\right)^{1/(1+\theta)}.$$

From this we get

$$\lim_{n\to\infty, n\in\Lambda} \rho_{n,m(n)}{}^{1/n} = \frac{1}{\rho}. \tag{46}$$

Fix an arbitrary $1-\theta < \lambda \leq 1$. Denote by $\{k_n\}$, $n = 1, 2, \ldots$, the sequence of integers such that $n - m(n) \leq k_n \leq n$ and $k_n/n \to \lambda$ as $n \to \infty$. Since the sequence $\{\rho_{n,m(n)}\}$, $n = 1, 2, \ldots$, is nonincreasing,

$$\rho_{n,m(n)}^{m(n)+1} \leq \rho_{k_n,m(k_n)}^{k_n-n+m(n)+1} \rho_{n,m(n)}^{n-k_n} \leq \prod_{k=n-m(n)}^{n} \rho_{k,m(k)}.$$

From this and from relations (44), (45), and (46), we get

$$\lim_{\substack{n\to\infty\\ n\in\Lambda}} \rho_{k_n,m(k_n)}^{(k_n-n+m(n)+1)/nm(n)} = \left(\frac{1}{\rho}\right)^{1-(1-\lambda)/\theta},$$

which implies that

$$\lim_{\substack{n\to\infty\\ n\in\Lambda}} \rho_{k_n,m(k_n)}{}^{1/k_n} = \left(\frac{1}{\rho}\right)^{1/\lambda}.$$

It follows from the last relation that

$$\liminf_{n\to\infty} {\rho_{n,m(n)}}^{1/n} \leq \left(\frac{1}{\rho}\right)^{1/\lambda}.$$

It remains to let λ tend to $(1-\theta)$ and obtain

$$\liminf_{n\to\infty} {\rho_{n,m(n)}}^{1/n} \leq \left(\frac{1}{\rho}\right)^{1/(1-\theta)}.$$

Therefore,

$$\liminf_{n\to\infty} \rho_{n,m(n)}^{1/(n+m(n))} \leq \left(\frac{1}{\rho}\right)^{1/(1-\theta^2)}. \qquad \blacksquare$$

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References

1. V. M. ADAMYAN, D. Z. AROV, M. G. KREĬN (1971): *Analytic properties of Schmidt pairs, Hankel operators, and the generalized Schur–Takagi problem*. Mat. Sb., **86**(128):34–75. English translation in Math USSR Sb., **15** (1971).
2. T. BAGBY (1969): *On interpolation by rational functions*. Duke Math. J., **36**:95–104.
3. J. B. GARNETT (1981): Bounded Analytic Functions. New York: Academic Press.
4. I. TS. GOKHBERG [ISRAEL GOHBERG], M. G. KREĬN (1965): Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space. Moscow: Nauka. English translation, Providence, RI: American Mathematical Society, 1969.
5. G. M. GOLUZIN (1966): Geometric Theory of Functions of a Complex Variable, 2nd ed. Moscow: Nauka. English translation, Providence, RI: American Mathematical Society, 1969.
6. A. A. GONCHAR (1984): *Rational approximation of analytic functions*. In: Linear and Complex Analysis Problem Book (V. P. Havin [Khavin] et al., eds.). Lecture Notes in Mathematics, vol. 1043. Berlin: Springer-Verlag, pp. 471–474.
7. A. A. GONCHAR (1984): *On the rate of rational approximation of analytic functions*. Trudy Mat. Inst. Steklov, **166**:52–60: English translation in Proc. Steklov Inst. Math. (1986), **1**(166).
8. A. A. GONCHAR (1986): *Rational approximations of analytic functions*. Proc. Internat. Congr. Math. (Berkeley, CA, 1986), vol. 1. Providence, RI: American Mathematical Society, 1987, pp. 739–748. English translation in Amererican Mathematical Society Translation (2), **147** (1990).
9. A. A. GONCHAR (1978): *On the degree of rational approximation of certain analytic functions*. Mat. Sb., **105**(147):147–163. English translation in Math. USSR Sb., **34** (1978).
10. A. A. GONCHAR, E. A. RAKHMANOV (1984): *The equilibrium measure and the distribution of zeros of extremal polynomials*. Mat. Sb., **125**(167):117–127. English translation in Math. USSR Sb., **53** (1986).
11. A. A. GONCHAR, E. A. RAKHMANOV (1987): *Equilibrium distributions and the degree of rational approximation of analytic functions*, Mat. Sb., **134**(176):306–352. English translation in Math. USSR Sb., **62** (1989).
12. P. KOOSIS (1980): Introduction to H^p Spaces. London Mathematical Society Lecture Note Series, vol. 40. Cambridge, UK: Cambridge University Press.
13. N. S. LANDKOF (1966): Foundations of Modern Potential Theory. Moscow: Nauka. English translation, Berlin: Springer-Verlag, 1972.

14. A. L. LEVIN, E. B. SAFF (1994): *Optimal ray sequences of rational functions connected with the Zolotarev problem*. Constr. Approx., **10**:235–273.
15. O. G. PARFENOV (1986): *Estimates of the singular numbers of a Carleson operator*. Mat. Sb., **131**(173):501–518. English translation in Math. USSR Sb., **59** (1988).
16. I. I. PRIVALOV (1950): Boundary Properties of Analytic Functions, 2nd ed. Moscow: GITTL. German translation, Berlin: VEB Deutscher Verlag Wiss., 1956.
17. V. A. PROKHOROV (1993): *On a theorem of Adamyan, Arov, Kreĭn*. Mat. Sb., **184**:89–104. English translation in Russian Acad. Sci. Sb. Math., **78** (1994).
18. V. A. PROKHOROV (1993): *Rational approximation of analytic functions*. Mat. Sb., **184**:3–32. English translation in Russian Acad. Sci. Sb. Math., **78** (1994).
19. V. A. PROKHOROV (1984): *On Chebyshev rational approximations of analytic functions*. Dokl. Akad. Nauk SSSR, **276**:1321–1324. English translation in Soviet Math. Dokl., **29** (1984).
20. E. B. SAFF, V. TOTIK (1997): Logarithmic Potentials with External Fields. Heidelberg: Springer-Verlag.
21. HERBERT STAHL (1986): *Orthogonal polynomials with complex-valued weight functions*, I, II. Constr. Approx., **2**:225–240, 240–251.
22. M. TSUJI (1959): Potential Theory in Modern Function Theory. Tokyo: Maruzen.
23. G. TS. TUMARKIN, S. YU. KHAVINSON (1958): *On the definition of analytic functions of class E_p in multiply connected domains*. Uspekhi Mat. Nauk, **13**, no. 1 (79):201–206 (Russian).
24. J. L. WALSH (1956): Interpolation and Approximation by Rational Functions in the Complex Domain, 2nd ed. Providence, RI: American Mathematical Society.

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