ORTHOGONAL POLYNOMIALS FROM A COMPLEX PERSPECTIVE

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ABSTRACT. Complex function theory and its close companion - potential theory - provide a wealth of tools for analyzing orthogonal polynomials and orthogonal expansions. This paper is designed to show how the complex perspective leads to insights on the behavior of orthogonal polynomials. In particular, we discuss the location of zeros and the growth of orthogonal polynomials in the complex plane. For some of the basic results we provide proofs that are not typically found in the standard literature on orthogonal polynomials.

1 Introduction

That the theory of complex variables can provide deeper understanding and useful techniques for analyzing real-variable problems should not be surprising to the reader. The computation of (real) integrals via Cauchy's Residue theorem and the analysis of power series are but two instances where the broader view from the complex plane **C** is an invaluable aid. Potential theory in the plane, which is a blend of real and complex analysis, provides an even greater resource for attacking "real problems"; particularly the behavior of orthogonal polynomials and orthogonal expansions. For example, the analysis of polynomials orthogonal on the whole real line **R** with respect to an exponential weight took a quantum step forward when x was replaced by z and potential theoretic arguments were introduced (cf. [MS1],[LS],[R],[GR1]).

Our goal is to illustrate how complex and potential theoretic results can be used to analyze orthogonal polynomials and orthogonal expansions. We assume that the reader has little background in potential theory; consequently we introduce some basic facts from this subject as well as provide references for further study. In Sections 2 and 5 we discuss the location and asymptotic behavior of the zeros of orthogonal polynomials. Bounds for the modulus of these polynomials are considered in Sections 6 and 7.

[1]Research supported, in part, by the National Science Foundation under grant DMS-890-6815

P. Nevai (ed.), Orthogonal Polynomials, 363–393.

Throughout, μ denotes a finite, positive measure on the Borel subsets of the complex plane $\mathbf{C}$. We assume that

$$\mathcal{S} = \mathcal{S}(\mu) := \operatorname{supp}(\mu)$$

is compact and contains infinitely many points.

The measure μ gives rise to the inner product

$$(f, g) := \int f(z)\overline{g(z)}d\mu \tag{1.1}$$

for functions $f, g \in L_2(\mu)$. Since $\mathcal{S}$ is infinite, the monomials $1, z, \ldots, z^n$ are linearly independent in $L_2(\mu)$ for every $n \geq 0$. Hence, by the Gram-Schmidt orthogonalization process, there exist unique polynomials

$$p_n(z) = p_n(z; \mu) = \gamma_n z^n + \cdots \in \mathcal{P}_n, \quad \gamma_n > 0, \tag{1.2}$$

satisfying

$$(p_m, p_n) = \delta_{m,n}, \quad m, n = 0, 1, \ldots,$$

where $\delta_{m,n} = 0$ if $m \neq n$, $\delta_{m,m} = 1$, and $\mathcal{P}_n$ denotes the collection of all polynomials (with complex coefficients) having degree at most n.

As we shall see, several basic properties of the polynomials $p_n(z)$ are simple consequences of the following extremal property which characterizes orthogonal polynomials.

Theorem 1.1. *The polynomial*

$$P_n(z) := \frac{1}{\gamma_n} p_n(z; \mu) = z^n + \cdots$$

is the unique monic polynomial of degree n of minimal $L_2(\mu)$–norm; that is, P_n solves the extremal problem

$$\min_{z^n + \cdots \in \mathcal{P}_n} \int |z^n + \cdots|^2 d\mu.$$

The proof of this result is straightforward and can be found in [Sz,§2.2].

2 Basic Properties of Zeros

The zeros of orthogonal polynomials play an important role in quadrature formulae, interpolation theory, spectral theory for certain linear operators, and the design of digital filters. Thus it is fundamental to ask: What can be said about the location (in the complex plane) of the n zeros of $p_n(z;\mu)$? As a simple consequence of Theorem 1.1 we shall prove the following result due to Fejér:

Theorem 2.1. *All the zeros of $p_n(z;\mu)$ lie in the convex hull of the support* $\mathcal{S} = \text{supp}(\mu)$.

By the convex hull $\text{Co}(\mathcal{S})$ of $\mathcal{S}$ we mean the intersection of all closed half-planes containing $\mathcal{S}$.

Proof of Theorem 2.1. It is more convenient to work with the monic orthogonal polynomial $P_n = p_n/\gamma_n$ which has the same zeros as p_n.

Suppose, to the contrary, that $P_n(z_0) = 0$ with $z_0 \notin \text{Co}(\mathcal{S})$. Write $P_n(z) = (z - z_0)q(z)$, where $q \in \mathcal{P}_{n-1}$ is monic. Since $z_0 \notin \text{Co}(\mathcal{S})$, there exists a line $\mathcal{L}$ separating z_0 and $\mathcal{S}$. Let $\hat{z}_0$ be the orthogonal projection of z_0 on $\mathcal{L}$ (see Figure 1). Then

$$|z - \hat{z}_0| < |z - z_0| \qquad \forall \quad z \in \mathcal{S}.$$

Hence

$$|(z - \hat{z}_0)q(z)| < |(z - z_0)q(z)| = |P_n(z)|$$

for all $z \in \mathcal{S}\backslash\{\text{zeros of } q\}$. Since $\mathcal{S} = \text{supp}(\mu)$ is infinite, it follows that

$$\int |(z - \hat{z}_0)q(z)|^2 d\mu < \int |P_n(z)|^2 d\mu,$$

which contradicts Theorem 1.1. ■

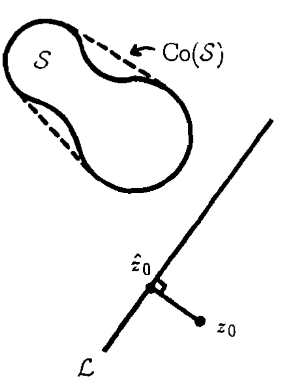


Figure 1

A nice treatment of a generalized version of Theorem 2.1 can be found in the text by P. Davis [D,§10.2].

As the next example illustrates, the zeros of p_n do not, in general, all lie on $\mathcal{S}$ (even when $\mathcal{S} \subset \mathbf{R}$).

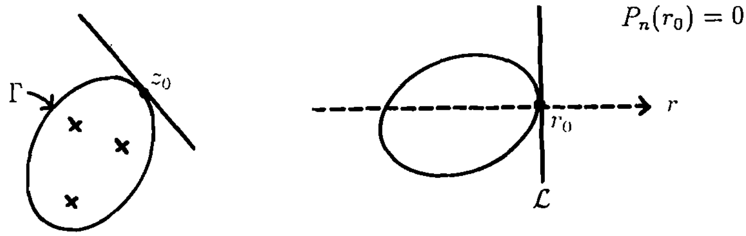


Figure 2

Example 2.A. Let $d\mu = w(x)dx$, where $w(x)$ is a positive, even, continuous function on $[-2,-1] \cup [1,2]$. Then Theorem 2.1 asserts that all the zeros of $p_n(z;\mu)$ lie in $\mathrm{Co}(\mathcal{S}) = [-2,2]$. Now since w is even, it follows from the uniqueness property of orthonormal polynomials that $p_{2n+1}(z;\mu)$ is an odd function for each $n = 0,1,\ldots$. Thus p_{2n+1} vanishes at $z = 0 \notin \mathcal{S}$.

Except when $\mathrm{Co}(\mathcal{S})$ is an interval, we can strengthen Theorem 2.1 by asserting that the zeros of p_n lie strictly inside the convex hull of $\mathcal{S}$.

Theorem 2.2. *If* $\mathrm{Co}(\mathcal{S}(\mu))$ *is not a line segment, then all the zeros of* $p_n(z;\mu)$ *lie in the interior of* $\mathrm{Co}(\mathcal{S}(\mu))$.

Proof. By Theorem 2.1 we need only show that no zeros of $P_n = p_n/\gamma_n$ lie on the boundary Γ of $\mathrm{Co}(\mathcal{S}(\mu))$.

Suppose that $P_n(z_0) = 0$ for some $z_0 \in \Gamma$. By performing a rotation and translation (see Figure 2) we assume, without loss of generality, that $z_0 = r_0$ is real and that all the points of $\mathrm{Co}(\mathcal{S}(\mu))$ lie on or to the left of the vertical line $\mathcal{L}$ through r_0; i.e. $\mathcal{L}$ is a support line for $\mathrm{Co}(\mathcal{S}(\mu))$. Write $P_n(z) = (z - r_0)q(z)$, $q \in P_{n-1}$, and for $r \in \mathbf{R}$ set

$$\begin{aligned} I(r) &:= \int |z-r|^2 |q(z)|^2 d\mu \\ &= \int (|z|^2 + r^2 - 2r \,\mathrm{Re}\, z)|q(z)|^2 d\mu. \end{aligned}$$

By the extremal property of Theorem 1.1, we have

$$I'(r_0) = \int 2(r_0 - \mathrm{Re}\, z)|q(z)|^2 d\mu = 0.$$

But $r_0 - \mathrm{Re}\, z \geq 0$ for all $z \in \mathcal{S}(\mu)$ and so

$$(r_0 - \mathrm{Re}\, z)|q(z)|^2 = 0 \quad d\mu - \text{a.e.}$$

This implies that only finitely many points of $\mathcal{S}(\mu)$ lie to the left of $\mathcal{L}$ and, moreover, q must vanish in these points. Since $\mathrm{Co}(\mathcal{S}(\mu))$ is not an interval, we can therefore find a point $\xi_0 \in \mathcal{S}(\mu) \cap \Gamma$, $\xi_0 \notin \mathcal{L}$, such that $q(\xi_0) = 0$. But then $\xi_0 \in \Gamma$ is a zero of P_n and the preceeding argument (with z_0 replaced by ξ_0) shows that only finitely many points of $\mathcal{S}(\mu)$ can lie in an open half-plane bounded by a support line of $\mathrm{Co}(\mathcal{S}(\mu))$ through ξ_0. Thus $\mathcal{L}$ contains only finitely many points of $\mathcal{S}(\mu)$ and so $\mathcal{S}(\mu)$ is finite, which gives the desired contradiction.

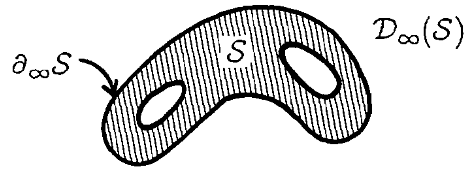


Figure 3

For example, if C is the unit circle $|z| = 1$ and $\mathcal{S}(\mu) \subseteq C$, then Theorem 2.2 asserts that all the zeros of $p_n(z;\mu)$ must lie in the open unit disk $|z| < 1$, which is a classical result of Szegő [Sz,§11.4].

How many zeros of $p_n(z;\mu)$ can lie outside $\mathcal{S}$? It is, of course, possible for all the zeros of $p_n(z;\mu)$ to lie off of $\mathcal{S}$. But, as we shall show, only a bounded number (independent of n) can lie on a fixed compact set exterior to $\mathcal{S}$. For this purpose we first introduce some notation.

With $\overline{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$, we let $\mathcal{D}_\infty(\mathcal{S})$ denote the component of $\overline{\mathbf{C}} \backslash \mathcal{S}$ containing ∞ (thus $\mathcal{D}_\infty(\mathcal{S})$ is an open, connected, unbounded set). The *outer boundary* of $\mathcal{S}$, denoted by $\partial_\infty \mathcal{S}$, is the boundary of $\mathcal{D}_\infty(\mathcal{S})$, i.e.

$$\partial_\infty \mathcal{S} := \partial \mathcal{D}_\infty(\mathcal{S})$$

(see Figure 3). Furthermore, we set

$$\mathrm{Pc}(\mathcal{S}) := \mathbf{C} \backslash \mathcal{D}_\infty(\mathcal{S}),$$

which is called the *polynomial convex hull* of $\mathcal{S}$. Roughly speaking, $\mathrm{Pc}(\mathcal{S})$ is obtained from $\mathcal{S}$ by filling in all of its "holes". Since $\mathrm{Pc}(\mathcal{S})$ is a compact set that does not separate the plane, Runge's theorem (cf. [G, p.76]) asserts that any function f analytic on $\mathrm{Pc}(\mathcal{S})$ can be uniformly approximated (as closely as desired) on $\mathrm{Pc}(\mathcal{S})$ by polynomials. Notice also that

$$\mathcal{S} \subseteq \mathrm{Pc}(\mathcal{S}) \subseteq \mathrm{Co}(\mathcal{S}).$$

The following lemma, implicit in the paper by Widom [Wi], will be used to examine the zeros of $p_n(z;\mu)$ that lie in $\mathcal{D}_\infty(\mathcal{S})$.

Lemma 2.3. *If E is a compact set such that $E \cap \mathrm{Pc}(\mathcal{S}) = \emptyset$ (i.e. $E \subset \mathcal{D}_\infty(\mathcal{S})$), then there exist an integer m and an α, $0 < \alpha < 1$, with the following property. For any m (not necessarily distinct) points $z_1, \ldots, z_m \in E$, there exist $w_1, \ldots, w_m \in \mathbf{C}$ such that*

$$\prod_{k=1}^{m} \left| \frac{z - w_k}{z - z_k} \right| < \alpha < 1, \qquad \forall \quad z \in \mathcal{S}. \tag{2.1}$$

Proof. We give a simple argument that is due to V. Totik.

Assume first that E consists of a single point, say $E = \{0\}$ with $0 \notin \mathrm{Pc}(\mathcal{S})$. Then proving (2.1) is equivalent to showing that there exists a monic polynomial $Q(z) = z^m + \cdots \in \mathcal{P}_m$ such that

$$|Q(z)/z^m| < \alpha < 1, \qquad \forall \quad z \in \mathcal{S}. \tag{2.2}$$

With the change of variable $\zeta = 1/z$, inequality (2.2) becomes

$$|\zeta^m Q(1/\zeta)| < \alpha < 1, \qquad \forall \quad \zeta \in \mathcal{S}^{-1} := \{\zeta | 1/\zeta \in \mathcal{S}\}. \tag{2.3}$$

Since $0 \notin \mathrm{Pc}(\mathcal{S})$ it is easy to see that $\mathcal{S}^{-1}$ is compact and that $0 \notin \mathrm{Pc}(\mathcal{S}^{-1})$. Now Q is monic and so (2.3) can be satisfied iff $\exists\, q_{m-1} \in \mathcal{P}_{m-1}$ such that

$$|1 - \zeta q_{m-1}(\zeta)| < \alpha < 1$$

i.e.,

$$|\zeta||1/\zeta - q_{m-1}(\zeta)| < \alpha < 1, \qquad \forall \quad \zeta \in \mathcal{S}^{-1}. \tag{2.4}$$

But the function $1/\zeta$ is analytic on $\mathrm{Pc}(\mathcal{S}^{-1})$, and so q_{m-1} exists by Runge's theorem (the factor $|\zeta|$ causes no difficulty since it is bounded on $\mathcal{S}^{-1}$).

Now we turn to the general case where E is compact and $E \cap \mathrm{Pc}(\mathcal{S}) = \emptyset$. By the first part of the proof, for each $z^* \in E$, there exist $m(z^*), \alpha(z^*)$, and $w_k(z^*), 1 \leq k \leq m(z^*)$, such that

$$\prod_{k=1}^{m(z^*)} \left| \frac{z - w_k(z^*)}{z - z^*} \right| < \alpha(z^*) < 1, \qquad \forall \quad z \in \mathcal{S}. \tag{2.5}$$

It then follows, by continuity, that there exists an ϵ–neighborhood $\mathcal{N}(z^*, \epsilon)$ of z^* such that whenever $\{z_1, \ldots, z_{m(z^*)}\} \subset \mathcal{N}(z^*, \epsilon)$, we have

$$\prod_{k=1}^{m(z^*)} \left| \frac{z - w_k(z^*)}{z - z_k} \right| < \alpha(z^*) < 1, \qquad \forall \quad z \in \mathcal{S}.$$

It is now possible to complete the proof by using a compactness argument. ■

With Lemma 2.3 in hand we can easily establish the following theorem of Widom [Wi].

Theorem 2.4. *If E is a closed set such that $E \cap \mathrm{Pc}(\mathcal{S}) = \emptyset$, then the number of zeros of $p_n(z;\mu)$ on E is uniformly bounded in n.*

Proof. By Theorem 2.1, we can assume that E is compact. Let m and α be as in Lemma 2.3 and suppose that $P_n = p_n/\gamma_n$ has $\geq m$ zeros in E, say $z_1, \ldots, z_m$. Then, by Lemma 2.3, $\exists\, w_1, \ldots, w_m$ such that (2.1) holds for all $z \in \mathcal{S}$. Let

$$Q_n(z) := P_n(z) \prod_{k=1}^{m} \left(\frac{z - w_k}{z - z_k} \right),$$

so that Q_n is a monic polynomial of degree n. From (2.1) we get

$$|Q_n(z)| < |P_n(z)|, \qquad \forall \quad z \in \mathcal{S}\backslash\{ \text{ zeros of } P_n\},$$

and so

$$\int |Q_n|^2 d\mu < \int |P_n|^2 d\mu.$$

As the last inequality contradicts Theorem 1.1, it follows that, for each n, the polynomial $p_n(z;\mu)$ has fewer than m zeros on E. ∎

Example 2.B. (*Szegő Polynomials*). We use the terminology "Szegő polynomials" to mean orthonormal polynomials $p_n(z;\mu)$ for which $\mathcal{S}=\text{supp}(\mu) \subseteq C:=\{z:|z|=1\}$. The zeros of these orthonormal polynomials play an important role in digital filter design. We consider separately the two cases $\mathcal{S} = C$, $\mathcal{S} \neq C$.

Case 1: $\mathcal{S} = C$. Then $\text{Pc}(\mathcal{S}) = \text{Co}(\mathcal{S}) = \{z : |z| \leq 1\}$ and, by Theorem 2.2, all the zeros of $p_n(z;\mu)$ lie in the open unit disk.

In Figure 4 we have plotted the zeros of $p_n(z;\mu)$ for $n = 15$ and $n = 25$ for

$$d\mu(e^{i\theta}) = |\sin(\theta/2)|^4 d\theta, \qquad 0 \leq \theta \leq 2\pi. \tag{2.6}$$

Notice that these zeros appear to be approaching the unit circle C and, except near $z = 1$, are close to being equally space in argument.

Figure 5 shows the zeros of $p_{15}(z;\mu)$ and $p_{25}(z;\mu)$ for

$$d\mu(e^{i\theta}) = (5/4 - \cos\theta)d\theta = |1 - e^{i\theta}/2|^2 d\theta, \qquad 0 \leq \theta \leq 2\pi. \tag{2.7}$$

Here the zeros seem to be approaching the circle $|z| = 1/2$, but again, the arguments of the zeros are nearly equally spaced.

We shall see in Section 5 (cf. Theorem 5.3) that the asymptotically uniform spacing of the arguments of the zeros of the $p_n(z;\mu)$ is a phenomenon that can be proved for a large class of measures μ whose support is C.

Case 2: $\mathcal{S} \neq C$. Again, all zeros of $p_n(z;\mu)$ lie in $|z| < 1$. But now $\text{Pc}(\mathcal{S}) = \mathcal{S}$ and so Theorem 2.4 implies that "most zeros" of $p_n(z;\mu)$ tend to $\mathcal{S}$ as $n \to \infty$.

In Figure 6 we have plotted the zeros of $p_6(z;\mu)$ and $p_{16}(z;\mu)$ for

$$d\mu(e^{i\theta}) = |\sin(\theta/2)|^4 d\theta, \quad 0 \leq \theta \leq \pi, \tag{2.8}$$

which is the restriction to the upper half-circle C^+ of the measure in (2.6). Notice that most (in fact, all) of the zeros are approaching $C^+ = \mathcal{S}$, as predicted by Theorem 2.4.

Figure 7 gives analogous plots for the measure that is the restriction of (2.7) to C^+. Again, the zeros are seen to be approaching C^+.

Notice further that the distribution of the zeros in Figures 6 and 7 look very much alike (there is some bunching near ± 1 and the zeros thin out near $z = i$). In Section 5 we shall show that both zero distributions are, in the limit, the equilibrium distribution for C^+. For this purpose we will utilize potential theory and norm comparisons with $L_\infty(\mathcal{S})$.

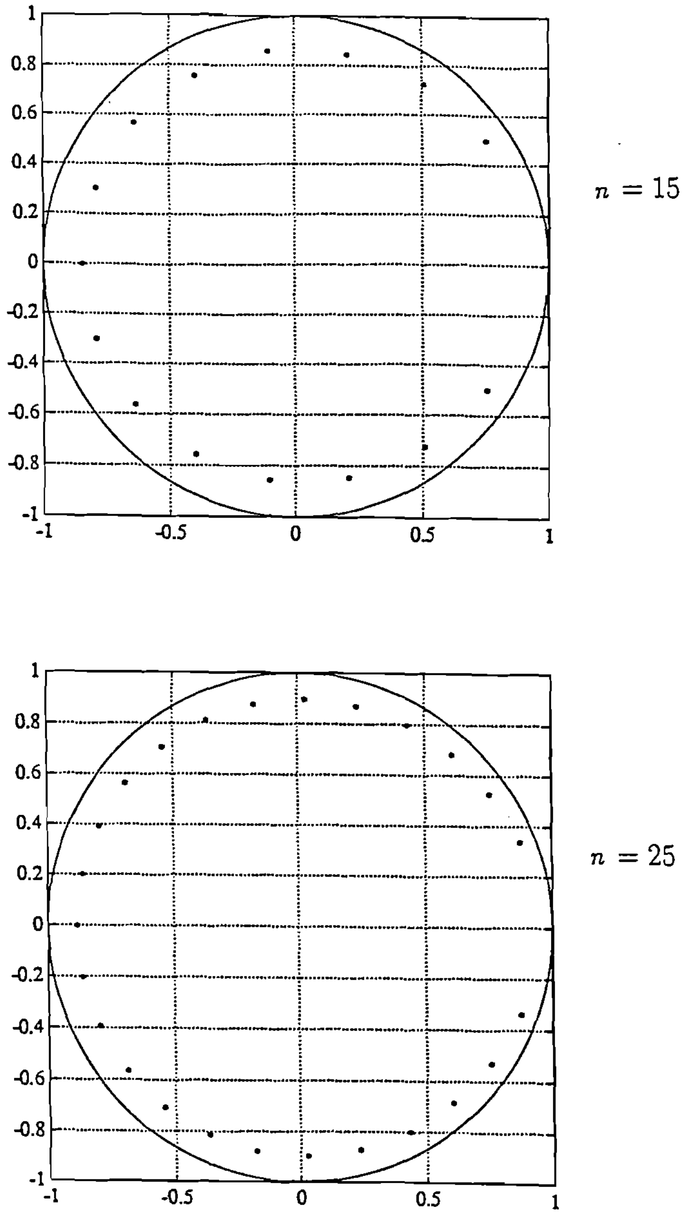
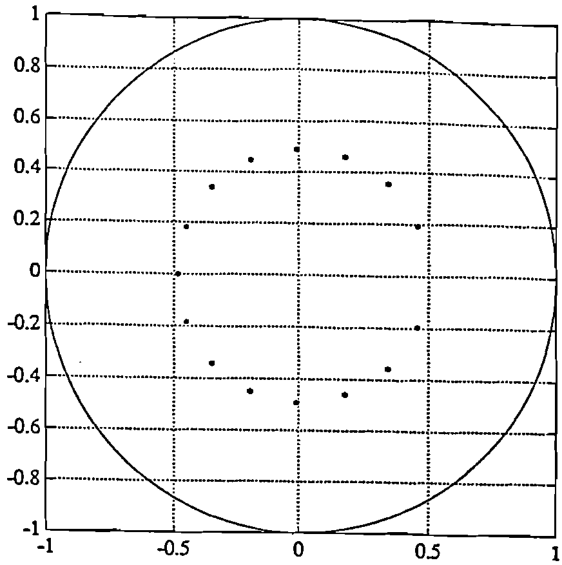
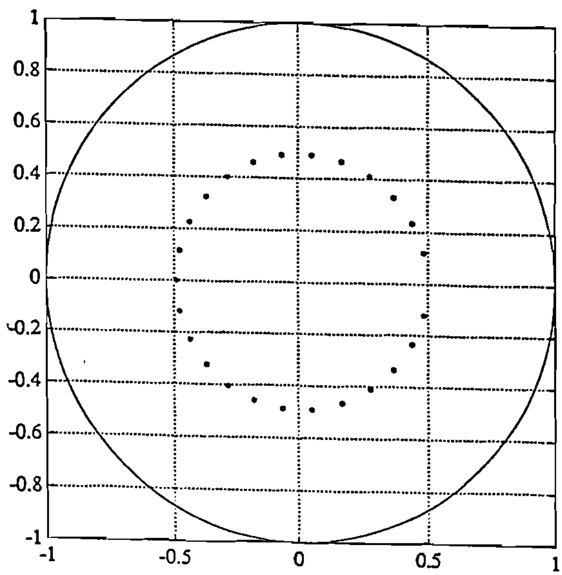


Figure 4: Zeros of $p_n(z;\mu)$ where $d\mu(e^{i\theta}) = |\sin(\theta/2)|^4 d\theta, \quad 0 \leq \theta \leq 2\pi$.

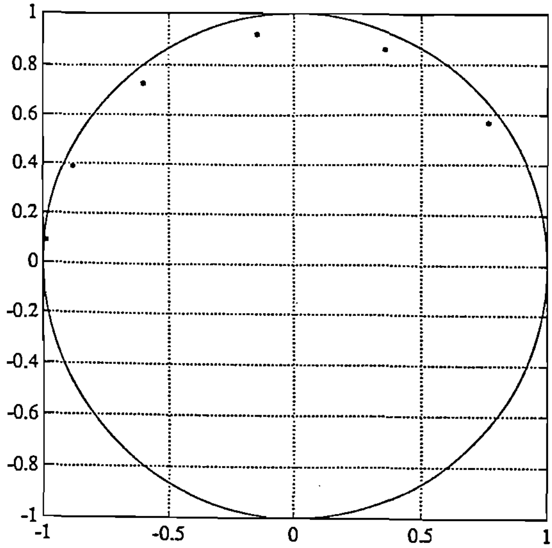


$n = 15$

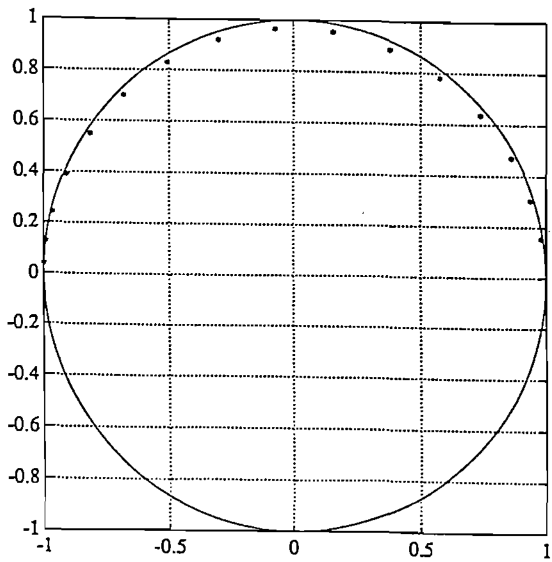


$n = 25$

Figure 5: Zeros of $p_n(z;\mu)$ where $d\mu(e^{i\theta}) = (5/4 - \cos\theta)d\theta, \quad 0 \le \theta \le 2\pi$.



$n = 6$



$n = 16$

Figure 6: Zeros of $p_n(z;\mu)$ where $d\mu(e^{i\theta}) = |\sin(\theta/2)|^4 d\theta, \quad 0 \le \theta \le \pi.$

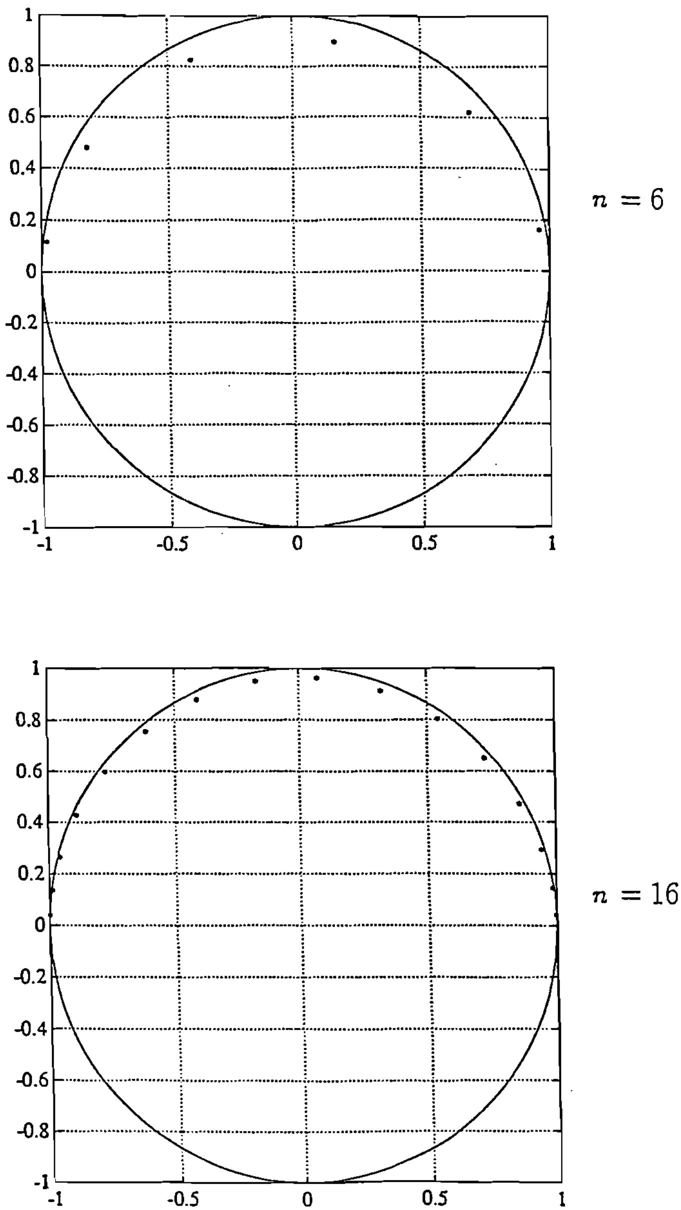


Figure 7: Zeros of $p_n(z;\mu)$ where $d\mu(e^{i\theta}) = (5/4 - \cos\theta)d\theta, \quad 0 \leq \theta \leq \pi$.

3 Completely Regular Measures

To obtain asymptotic results on the zeros of $p_n(z;\mu)$ we will first compare the monic orthogonal polynomials P_n with minimal sup norm polynomials. Let $\|\cdot\|_{\mathcal{S}}$ denote the sup norm over $\mathcal{S} = \mathcal{S}(\mu)$ and set

$$t_n(\mathcal{S}) := \min_{z^n+\cdots\in\mathcal{P}_n} \| z^n + \cdots \|_{\mathcal{S}}, \quad n = 0, 1, \ldots . \tag{3.1}$$

For each $n \geq 0$ there exists a unique monic polynomial $T_n(z) = z^n + \cdots \in \mathcal{P}_n$, called the *Chebyshev polynomial* of degree n, that satisfies

$$t_n(\mathcal{S}) = \| T_n \|_{\mathcal{S}} . \tag{3.2}$$

Recalling Theorem 1.1, we note that T_n is just an L_∞ analogue of P_n and that

$$\begin{aligned}\frac{1}{\gamma_n} = \left(\int |P_n(z)|^2 d\mu\right)^{1/2} &\leq \left(\int |T_n(z)|^2 d\mu\right)^{1/2} \\ &\leq \mu(\mathcal{S})^{1/2} t_n(\mathcal{S}),\end{aligned} \tag{3.3}$$

where γ_n is the leading coefficient of $p_n(z;\mu)$ (cf. (1.2)). We also remark that the zero results of Theorems 2.1 and 2.4 hold for the polynomials T_n (the same proofs apply).

One advantage of working with the sup norm instead of $L_2(\mu)$ is the fact that $\lim_{n\to\infty}[t_n(\mathcal{S})]^{1/n}$ always exists; this follows from the simple inequality

$$t_{m+n}(\mathcal{S}) = \| T_{m+n} \|_{\mathcal{S}} \leq \| T_m \cdot T_n \|_{\mathcal{S}} \leq t_m(\mathcal{S}) t_n(\mathcal{S}),$$

which shows that $\log t_n(\mathcal{S})$ is a subadditive function of n (cf. [T, §III.5]). In contrast, $\lim_{n\to\infty} \gamma_n^{1/n}$ need not exist.

We write

$$\text{cheb}(\mathcal{S}) := \lim_{n\to\infty} [t_n(\mathcal{S})]^{1/n}, \tag{3.4}$$

which is called the *Chebyshev constant* for $\mathcal{S}$. From (3.3) we immediately obtain

$$\liminf_{n\to\infty} \gamma_n^{1/n} \geq \frac{1}{\text{cheb}(\mathcal{S})}. \tag{3.5}$$

Definition 3.1. The measure μ is said to be *completely regular* if

$$\lim_{n\to\infty} \| p_n(z;\mu) \|_{\mathcal{S}(\mu)}^{1/n} = 1. \tag{3.6}$$

In other words, for a completely regular measure, there is no essential difference between $L_2(\mu)$ and $L_\infty(\mathcal{S})$ as far as n-th root asymptotics are concerned. Thus we might expect that P_n and T_n have some common limiting properties. For example, it is easy to see that $\lim_{n\to\infty} \gamma_n^{1/n}$ exists for such measures.

Proposition 3.2. *If μ is completely regular, then*

$$\lim_{n\to\infty} \gamma_n^{1/n} = \frac{1}{\text{cheb}(\mathcal{S}(\mu))}. \tag{3.7}$$

Proof. From the definition of $t_n(\mathcal{S})$ in (3.1) we have

$$t_n(\mathcal{S}) = \| T_n \|_{\mathcal{S}} \leq \| P_n \|_{\mathcal{S}} = \| p_n/\gamma_n \|_{\mathcal{S}},$$

and so, from (3.6),

$$\limsup_{n\to\infty} \gamma_n^{1/n} \leq \limsup_{n\to\infty} \frac{\| p_n \|_{\mathcal{S}}^{1/n}}{[t_n(\mathcal{S})]^{1/n}} = \frac{1}{\text{cheb}(\mathcal{S})}.$$

Together with (3.5), this proves (3.7). ■

We remark that in the paper of H. Stahl and V. Totik that appears in this Proceedings, property (3.7) is used to define a *regular* measure μ. Thus every completely regular measure is regular. Conversely, if μ is regular and $\mathcal{S}(\mu)$ is regular with respect to the Dirichlet problem for $\mathcal{D}_\infty(\mathcal{S}(\mu))$, then μ is completely regular.

Although the class of completely regular measures is quite restrictive, it does contain many important measures that arise in applications.

Example 3.A The following are examples of completely regular measures.

(i) $d\mu = w\,dx\,dy$ over a bounded Jordan region R, where the weight w (≥ 0) and some negative power of w are integrable with respect to area over R (cf. [Wa,§5.7]).

(ii) $d\mu = w\,ds$, where ds is arclength over a rectifiable curve Γ and the weight $w \geq 0$ and some negative power of w are integrable with respect to ds (cf. [Wa,§5.7]).

(iii) $\mathcal{S}(\mu) = [-1,1]$ and $\mu' > 0$ a.e. on $[-1,1]$ (cf. [EF],[ET]).

(iv) $\mathcal{S}(\mu) = C : |z| = 1$ and $\lim_{n\to\infty} \gamma_n^{1/n} = 1$ (cf. [LSS]); in particular, if μ belongs to the Szegő class, i.e.

$$\int_0^{2\pi} \log \mu'(e^{i\theta})d\theta > -\infty.$$

To deduce asymptotic properties of orthogonal polynomials with respect to a completely regular measure, we shall appeal to an alternate definition of the constant $\text{cheb}(\mathcal{S})$ that comes from potential theory.

4 Basics from Potential Theory

Introductions to potential theory can be found in [He], [Hi] and [T]; a more in depth treatment is given in [La]. Here we provide some basic facts.

Potential theory has its origin in the following

Electrostatics Problem. *Let $E \subset \mathbf{C}$ be compact. Place a unit positive charge on E so that equilibrium is reached in the sense that the energy with respect to the logarithmic potential is minimized.*

To create a mathematical framework for this problem, we let $\mathcal{M}(E)$ denote the collection of all positive, unit Borel measures ν supported on E (so that $\mathcal{M}(E)$ contains all possible distributions of charges placed on E). The *logarithmic potential* associated with $\nu \in \mathcal{M}(E)$ is

$$U^{\nu}(z) := \int \log |z - t|^{-1} d\nu(t), \tag{4.1}$$

which is a superharmonic, lower semi-continuous function on $\mathbf{C}$. The *energy* of such a potential is defined by

$$I[\nu] := \int U^{\nu} d\nu = \int \int \log |z - t|^{-1} d\nu(t) d\nu(z). \tag{4.2}$$

Thus, the electrostatics problem involves the determination of

$$V(E) := \inf_{\nu \in \mathcal{M}(E)} I[\nu], \tag{4.3}$$

which is called the *Robin's constant* for E. The *logarithmic capacity* of E, denoted by $\operatorname{cap}(E)$, is defined by

$$\operatorname{cap}(E) := e^{-V(E)}, \tag{4.4}$$

which is finite and nonnegative.

A fundamental theorem from potential theory asserts that if $\operatorname{cap}(E) > 0$, there exists a unique measure $\nu_E \in \mathcal{M}(E)$ such that

$$I[\nu_E] = V(E). \tag{4.5}$$

The extremal measure ν_E is called the *equilibrium distribution* for E and furnishes the solution to the electrostatics problem.

Some basic facts about $\operatorname{cap}(E)$ and ν_E are:

(i) $\mathcal{S}(\nu_E) = \operatorname{supp}(\nu_E) \subseteq \partial_\infty E$; moreover, the set $\partial_\infty E \backslash \mathcal{S}(\nu_E)$ has capacity zero.

(ii) The *conductor potential* $U^{\nu_E}(z)$ satisfies $U^{\nu_E}(z) \leq V(E)$ for all $z \in \mathbf{C}$, with equality holding on E except possibly for a set of capacity zero.

(iii) For any compact set E,

$$\operatorname{cap}(E) = \operatorname{cheb}(E). \tag{4.6}$$

Assertion (iii) provides the alternate interpretation of the Chebyshev constant that will be especially useful for our purposes. That cheb(E) has a plausible connection with potential theory can be seen from the fact that, for any monic polynomial $Q(z) = z^n + \cdots \in \mathcal{P}_n$, we can write

$$\frac{1}{n} \log \frac{1}{|Q(z)|} = U^{\nu(Q)}(z), \tag{4.7}$$

where $\nu(Q)$ is the discrete measure with mass $1/n$ at each zero of Q.

To gain some insight into the equilibrium distribution ν_E we turn to the (hopefully) more familiar concept of a Green function. If the outer boundary $\partial_\infty E$ consists of analytic curves, then the *Green function with pole at* ∞ for $\mathcal{D}_\infty(E)$ is denoted by $g_E(z, \infty)$ and is defined by the following three properties:

(a) $g_E(z,\infty)$ is harmonic in $\mathcal{D}_\infty(E)\backslash\{\infty\}$.

(b) $g_E(z,\infty) \to 0$ as $z \to \partial_\infty E, \quad z \in \mathcal{D}_\infty(E)$.

(c) $\exists$ a constant $\hat{V}$ such that

$$(g_E(z,\infty) - \log|z|) \to \hat{V} \quad \text{as} \quad z \to \infty.$$

Using Green's formula, we can derive the identity (cf. [Wa,§4.2])

$$\begin{aligned} \hat{V} - g_E(z,\infty) &= \frac{1}{2\pi} \int_{\partial_\infty E} \log|z-t|^{-1} \frac{\partial}{\partial n} g_E(t,\infty)|dt|, \\ &= \int_{\partial_\infty E} \log|z-t|^{-1} d\hat{\nu}, \end{aligned}$$

where n denotes the exterior normal for $\partial_\infty E$ and

$$d\hat{\nu} := \frac{1}{2\pi} \frac{\partial}{\partial n} g_E(t,\infty)|dt|. \tag{4.8}$$

The relationship between the Green function and the conductor potential is given in

Theorem 4.1. *If* $\partial_\infty E$ *consists of finitely many analytic curves, then* $\hat{V} = V(E)$, $\hat{\nu} = \nu_E$ *and*

$$\begin{aligned} U^{\nu_E}(z) &= V(E) - g_E(z,\infty) \\ &= \log \frac{1}{\operatorname{cap}(E)} - g_E(z,\infty). \end{aligned} \tag{4.9}$$

It is, of course, possible to define the Green function for the outer domain of more general compact sets. This is done by exhausting $\mathcal{D}_\infty(E)$ by a sequence of open sets $G_1 \subset G_2 \subset \cdots$ containing ∞ and having analytic boundaries, and then taking the limit of the associated Green functions. Provided this limit is not identically infinite, it defines the Green function $g_E(z,\infty)$. Moreover, equation (4.9) persists in this general setting.

It is helpful to keep in mind the following two simple examples.

Example 4.A. Let $E : |z| = R$. Then $\operatorname{cap}(E) = \operatorname{cheb}(E) = R$, $g_E(z,\infty) = \log|z/R|$ for $|z| \geq R$, and $d\nu_E = ds/2\pi R$, where ds is arclength on the circle $|z| = R$. Notice that the formula for $d\nu_E$ follows immediately from (4.8).

Example 4.B. Let $E = [-1,1]$. Then $\operatorname{cap}(E) = \operatorname{cheb}(E) = 1/2$, $g_E(z,\infty) = \log|z + \sqrt{z^2-1}|$ and

$$d\nu_E = \frac{1}{\pi}\frac{dx}{\sqrt{1-x^2}}, \quad x \in [-1,1],$$

which is the arcsine measure.

5 Asymptotic Behavior of Zeros

If Q is a polynomial of degree n with zeros $z_1, z_2, \ldots, z_n$, the *normalized zero distribution* associated with Q is defined by

$$\nu(Q) := \frac{1}{n}\sum_{k=1}^{n} \delta_{z_k}, \tag{5.1}$$

where δ_{z_k} denotes the unit point mass at z_k.

As we shall see, the following theorem due to Blatt, Saff and Simkani [BSS] not only leads to asymptotic results for the zeros of certain sequences of orthogonal polynomials, it is useful in many other contexts.

Theorem 5.1. *Let $\mathcal{S}$ be a compact set with positive capacity and suppose that the monic polynomials $Q_n(z) = z^n + \cdots \in \mathcal{P}_n$ satisfy the following two conditions:*

(a) $\limsup_{n\to\infty} \| Q_n \|_{\mathcal{S}}^{1/n} \leq \operatorname{cap}(\mathcal{S})$;

(b) $\lim_{n\to\infty} \nu(Q_n)(A) = 0$ *for all closed sets A contained in the (2-dimensional) interior of* $\operatorname{Pc}(\mathcal{S})$.

Then, in the weak-star sense,

$$\nu(Q_n) \longrightarrow \nu_{\mathcal{S}} \quad \textit{as} \quad n \to \infty, \tag{5.2}$$

where $\nu_{\mathcal{S}}$ is the equilibrium distribution for $\mathcal{S}$.

By (5.2) we mean that, for all continuous functions f on $\mathbf{C}$ having compact support,

$$\lim_{n\to\infty} \int f d\nu(Q_n) = \int f d\nu_{\mathcal{S}}.$$

Assumption (a) states that the L_∞ norms of the Q_n are asymptotically minimal (recall (3.4) and (4.6)), and (b) means that only $o(n)$ zeros of Q_n can lie on a compact subset of $\operatorname{int}(\operatorname{Pc}(\mathcal{S}))$.

The proof of Theorem 5.1 proceeds roughly along the following lines. First one shows that assumption (a) implies

$$\lim_{n\to\infty} \nu(Q_n)(B) = 0 \quad \forall \quad B \subset \mathcal{D}_\infty(\mathcal{S}), \quad B \quad \text{closed.}$$

This fact together with assumption (b) implies that any limit measure of the $\nu(Q_n)$'s is supported on $\partial_\infty \mathcal{S}$ (which is the case for the equilibrium measure $\nu_{\mathcal{S}}$). Next, if ν^* is any limit measure of $\{\nu(Q_n)\}_1^\infty$, one can use assumption (a), the representation (4.7), and the minimum principle for superharmonic functions to prove that

$$U^{\nu^*}(z) \le \log \frac{1}{\operatorname{cap}(\mathcal{S})} = V(\mathcal{S}) \quad \forall \quad z \in \partial_\infty \mathcal{S}. \tag{5.3}$$

Finally, on integrating this last inequality with respect to ν^* we obtain

$$I[\nu^*] = \int U^{\nu^*} d\nu^* \le \int V(\mathcal{S}) d\nu^* = V(\mathcal{S}) = I[\nu_{\mathcal{S}}],$$

that is, $\nu^* \in \mathcal{M}(\mathcal{S})$ has minimal energy. Consequently, by the uniqueness of the solution to the electrostatics problem, we get $\nu^* = \nu_{\mathcal{S}}$.

Notice that if μ is a completely regular measure, then the monic orthogonal polynomials $P_n(z) = p_n(z;\mu)/\gamma_n$ satisfy condition (a) of Theorem 5.1 for $\mathcal{S} = \mathcal{S}(\mu)$; indeed from (3.6) and (3.7),

$$\limsup_{n\to\infty} \| P_n \|_{\mathcal{S}(\mu)}^{1/n} \le \frac{\limsup_{n\to\infty} \| p_n \|_{\mathcal{S}(\mu)}^{1/n}}{\lim_{n\to\infty} \gamma_n^{1/n}} = \operatorname{cheb}(\mathcal{S}(\mu)) = \operatorname{cap}(\mathcal{S}(\mu)).$$

Moreover, if the interior of $\operatorname{Pc}(\mathcal{S}(\mu))$ is empty, then condition (b) of Theorem 5.1 is vacuously satisfied. Thus we obtain (compare [BSS, Cor. 2.1])

Theorem 5.2. *Let μ be a completely regular measure. If $\mathcal{S}(\mu) = \operatorname{supp}(\mu)$ has positive capacity and $\operatorname{Pc}(\mathcal{S}(\mu))$ has empty interior, then*

$$\nu(p_n(;\mu)) \longrightarrow \nu_{\mathcal{S}(\mu)} \quad \textit{as} \quad n \to \infty.$$

This result explains the behavior of the zeros plotted in Figures 6 and 7; they have limit distribution equal to ν_{C^+}. Moreover, it will follow from Theorem 5.1 that the limiting distribution for the zeros plotted in Figure 4 is $d\nu_C = d\theta/2\pi$, provided one can show that condition (b) is satisfied. That this indeed the case can be seen from the next result due to Mhaskar and Saff [MS4].

Theorem 5.3. *Let $\{p_n(z;\mu)\}_1^\infty$ be a sequence of Szegő polynomials (i.e. $\mathcal{S}(\mu) \subseteq C$: $|z| = 1$) so that $P_n(0) = p_n(0;\mu)/\gamma_n$ is the n-th 'reflection coefficient'. Set*

$$\limsup_{n\to\infty} |P_n(0)|^{1/n} =: \rho(\leq 1), \tag{5.4}$$

and let $\Lambda \subset \mathbf{N}$ satisfy

$$\lim_{\substack{n\to\infty \\ n\in\Lambda}} |P_n(0)|^{1/n} = \rho.$$

(a) *If $\rho < 1$, then $\nu(p_n(\cdot;\mu)) \to \nu_{C_\rho}$ as $n \to \infty$, $\quad n \in \Lambda$, where $d\nu_{C_\rho} = ds/(2\pi\rho)$, for $\rho > 0$, is the equilibrium distribution on the circle $C_\rho : |z| = \rho$ (cf. Example 4.A), and $\nu_{C_0} := \delta_0$.*

(b) *If $\rho = 1$ and*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} |P_k(0)| = 0, \tag{5.5}$$

then $\nu(p_n(\cdot;\mu)) \to \nu_{C_1}$ as $n \to \infty$, $\quad n \in \Lambda$.

The proof of this result follows by applying Theorem 5.1 to the polynomials $Q_n = P_n^*/\overline{P_n(0)}, n \in \Lambda$, on the set $\mathcal{S} = C_{1/\rho}$, where $P_n^*(z) := z^n\overline{P_n(1/\overline{z})}$ are the reverse polynomials.

For the Jacobi type weight $d\mu(e^{i\theta}) = |\sin(\theta/2)|^4 d\theta, \quad 0 \leq \theta \leq 2\pi$ (cf. (2.8)), the reflection coefficients are $P_n(0) = 2/(n+2)$ so that $\rho = 1, \quad \Lambda = \mathbf{N}$, and condition (5.5) is satisfied. Thus by part (b) of the above theorem, $\nu(p_n(\cdot;\mu)) \to \nu_{C_1} = d\theta/2\pi$ as $n \to \infty$, which confirms our expectations from the plots in Figure 4.

Concerning the plots in Figure 5 where

$$d\mu(e^{i\theta}) = (5/4 - \cos\theta)d\theta = |1 - e^{i\theta}/2|^2 d\theta, \quad 0 \leq \theta \leq 2\pi, \tag{5.6}$$

it turns out that $\rho = 1/2$ and $\Lambda = \mathbf{N}$, so that from part (a) of Theorem 5.3 we get $\nu(p_n(\cdot;\mu)) \to \nu_{C_{1/2}}$, the uniform distribution on the circle of radius 1/2.

We remark that it is not necessary to have an explicit form for the reflection coefficients in order to determine the constant ρ of (5.4). As observed in [NT], for measures μ belonging to the Szegő class, ρ can be deduced from the analytic properties of the *Szegő function* $D(\mu;z)$ (cf. [Sz, §10.2]). Recall that $D(\mu;z)$ is analytic and nonzero in $|z| < 1$, and satisfies

$$|D(\mu;z)|^2 = \mu'(e^{i\theta}) \quad \text{a.e. on} \quad [0,2\pi].$$

By considering the orthogonal expansion for the reciprocal $D(\mu;z)^{-1}$, it is easy to see that ρ in (5.4) is the smallest number such that $D(\mu;z)^{-1}$ has an analytic extension to the disk $|z| < 1/\rho$. For example, the Szegő function for the weight (5.6) is $D(\mu;z) = 1 - z/2$, from which we see that $|z| < 2$ is the largest disk for which $D(\mu;z)^{-1} = 1/(1-z/2)$ is analytic; hence $\rho = 1/2$.

6 Bounds for Polynomials

Another important application of potential theory is in determining bounds for the growth of polynomials in the complex plane; that is, for attacking the following

Problem. *Let $E \subset \mathbf{C}$ be compact with* $\operatorname{cap}(E) > 0$. *Given that $q_n \in \mathcal{P}_n$ and*

$$|q_n(z)| \le M, \quad \forall\, z \in E, \tag{6.1}$$

estimate $|q_n(z)|$ for $z \notin E$.

In analyzing this problem we shall make use of the Green function $g_E(z,\infty)$ with pole at ∞ for $\mathcal{D}_\infty(E)$ (cf. Section 4). The level curves of this function shall be denoted by Γ_ρ, that is,

$$\Gamma_\rho := \{z \in \mathbf{C} \,|\, g_E(z,\infty) = \log\rho\}, \qquad \rho > 1. \tag{6.2}$$

Example 6.A. If $E = [-1,1]$, then $\Gamma_\rho(\rho > 1)$ is the ellipse with foci at ± 1 and semi-major axis equal to $(\rho + \rho^{-1})/2$ (cf. Example 4.B).

The following result, known as the *Bernstein-Walsh lemma* [Wa, p.77, 87], is a simple application of the maximum principle for subharmonic functions that provides an answer to the above problem.

Lemma 6.1. *Let $E \subset \mathbf{C}$ be compact and have positive capacity. If $q_n \in \mathcal{P}_n$ satisfies* (6.1), *then*

$$|q_n(z)| \le M \exp(n g_E(z,\infty)), \quad \forall \quad z \in \mathcal{D}_\infty(E).$$

In particular,

$$|q_n(z)| \le M\rho^n, \qquad \forall \quad z \in \Gamma_\rho.$$

Proof. From (6.1) we have

$$\frac{1}{n}\log|q_n(z)| \leq \frac{1}{n}\log M, \quad \forall \quad z \in E,$$

and so

$$\frac{1}{n}\log|q_n(z)| - g_E(z,\infty) \leq \frac{1}{n}\log M \tag{6.3}$$

holds for $z \in \partial_\infty E$ except possibly for a set of capacity zero (recall property (ii) of $U^{\nu_E}(z)$ in Section 4 and the representation (4.9)). But the left-hand side of (6.3) is *subharmonic* in $\mathcal{D}_\infty(E)$ (including ∞), and so the maximum principle (cf. [T, p.77]) implies that (6.3) holds for all $z \in \mathcal{D}_\infty(E)$. ∎

We now show how the above lemma can be used to establish the convergence of certain Fourier expansions. Let $f \in L_2(\mu)$, i.e.

$$\| f \|_{L_2(\mu)} := \left(\int |f|^2 d\mu\right)^{1/2} < \infty.$$

Then the Fourier expansion of f is given by

$$f \sim \sum_{k=0}^{\infty} a_k p_k(z;\mu), \quad a_k = a_k(f) := \int f\overline{p_k} d\mu, \tag{6.4}$$

and its partial sums

$$s_n(z) = s_n(z;f) := \sum_{k=0}^{n} a_k p_k(z;\mu) \tag{6.5}$$

are best polynomial approximants to f; more precisely, we have the following well-known result:

Theorem 6.2. *The partial sum s_n is the best $L_2(\mu)$ approximant to f out of $\mathcal{P}_n$ in the sense that*

$$\| f - s_n \|_{L_2(\mu)} \leq \| f - q_n \|_{L_2(\mu)}, \quad \forall \quad q_n \in \mathcal{P}_n.$$

To simplify our discussion (yet still convey the general spirit) we assume throughout the remainder of this section that $\mathrm{supp}(\mu) = [-1,1]$ and that μ is completely regular; we denote this by writing $\mu \in \mathrm{CR}[-1,1]$.

The next theorem describes the convergence of the Fourier expansion (6.4) for analytic functions f.

Theorem 6.3. *Let $\mu \in \mathrm{CR}[-1,1]$ and assume f is analytic inside the ellipse $\Gamma_\rho (\rho > 1)$ of Example 6.A. Then*

$$s_n(z;f) = \sum_0^n a_k p_k(z;\mu) \to f(z) \quad as \quad n \to \infty$$

locally uniformly inside Γ_ρ.

Proof. Let Q_{n-1} denote the unique polynomial in $\mathcal{P}_{n-1}$ that interpolates f in the n points that are the extrema on [-1,1] of the classical Chebyshev polynomial $\cos((n-1)\arccos x)$. Using the analyticity of f and the Hermite error formula (cf. [Wa,§3.1]) for $f - Q_{n-1}$, it is easy to verify the following fact:

$$\limsup_{n\to\infty} \| f - Q_{n-1} \|_{[-1,1]}^{1/n} \leq 1/\rho < 1, \tag{6.6}$$

where $\| \cdot \|_{[-1,1]}$ denotes the sup norm over $[-1,1]$. Since

$$|a_n| = \left| \int_{-1}^1 f p_n d\mu \right| = \left| \int_{-1}^1 (f - Q_{n-1}) p_n d\mu \right|,$$

we deduce from (6.6) that

$$\limsup_{n\to\infty} |a_n|^{1/n} \leq 1/\rho. \tag{6.7}$$

From the assumption that μ is completely regular we also have

$$\lim_{n\to\infty} \| p_n \|_{[-1,1]}^{1/n} = 1,$$

and so, for the sup norm over any ellipse (level curve) Γ_σ, Lemma 6.1 yields

$$\limsup_{n\to\infty} \| p_n \|_{\Gamma_\sigma}^{1/n} \leq \sigma.$$

Hence, for $1 < \sigma < \rho$, we get from (6.7) that

$$\limsup_{n\to\infty} \| a_n p_n \|_{\Gamma_\sigma}^{1/n} \leq \sigma/\rho < 1,$$

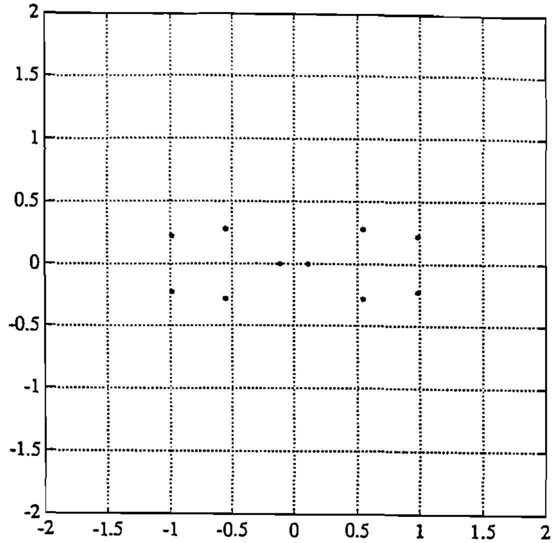
which implies that $\sum_0^\infty a_n p_n(z)$ converges uniformly on Γ_σ for each $\sigma < \rho$. Letting $\sigma \to \rho$, the theorem follows. ∎

For the case of Jacobi series, the above result is given in [Sz,Chap. 9].

What can be said about the behavior of the partial sums s_n in the complex plane when f is *not* analytic on $[-1,1]$? To gain some insight, let's consider the Chebyshev and Legendre expansions of

$$f(x) = |x| \quad \text{on} \quad [-1,1].$$

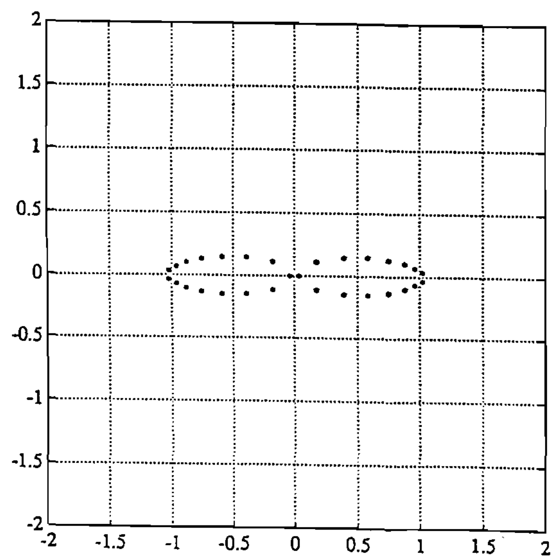
In Figure 8 we have plotted the zeros of the partial sums $s_{10}(z;f)$ and $s_{20}(z;f)$ for the Chebyshev expansion and, in Figure 9, the analogous plots are given for the



$n = 10$

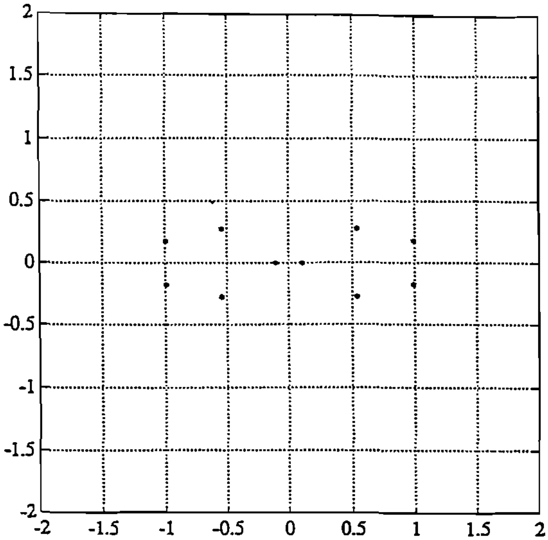
Zeros of n–th partial sum of **Chebyshev** expansion

$$|x| \sim \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k T_{2k}(x)}{4k^2 - 1}$$



$n = 30$

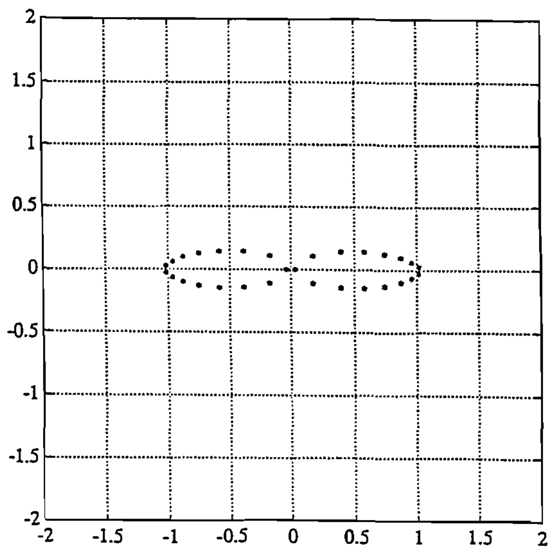
Figure 8



$n = 10$

Zeros of n−th partial sum of **Legendre** expansion

$$|x| \sim \frac{1}{2} - \sum_{k=1}^{\infty} (-1)^k \binom{2k}{k-1} \frac{4k+1}{2^{2k+1} k(2k-1)} P_{2k}(x)$$



$n = 30$

Legendre case

Figure 9

Legendre expansion. We observe that these zeros seem to surround and approach the orthogonality interval $[-1,1]$ as n increases. Moreover, the distributions of the zeros in Figures 8 and 9 are very much alike. This phenomenon is explained in the following theorem due to Li, Saff and Sha [LSS].

Theorem 6.4. *Let $\mu \in \mathrm{CR}[-1,1]$. If $f \in L_2(\mu)$ is not equal $d\mu$-a.e. to a function analytic on an open set containing $[-1,1]$, then there exists a subsequence $\Lambda(f) \subseteq \mathbf{N}$ for which the zero measures $\nu(s_n)$ of the partial sums $s_n(z;f)$ satisfy*

$$\nu(s_n) \to \frac{1}{\pi}\frac{dx}{\sqrt{1-x^2}} \quad \text{as} \quad n \to \infty, \quad n \in \Lambda(f). \tag{6.8}$$

Proof. We first claim that the Fourier coefficients a_n of f satisfy

$$\limsup_{n\to\infty} |a_n|^{1/n} = 1. \tag{6.9}$$

If not, then $\limsup_{n\to\infty} |a_n|^{1/n} = 1/\rho < 1$ and the argument used in the proof of Theorem 6.3 shows that $\sum_0^\infty a_n p_n$ converges inside the ellipse Γ_ρ to an analytic function. As this contradicts the assumption on f, equation (6.9) follows.

Now choose $\Lambda \subseteq \mathbf{N}$ so that

$$\lim_{\substack{n\to\infty\\ n\in\Lambda}} |a_n|^{1/n} = 1, \tag{6.10}$$

and let

$$Q_n(z) := s_n(z)/(a_n\gamma_n) = z^n + \cdots.$$

Since $\mu \in \mathrm{CR}[-1,1]$ and the polynomials s_n have uniformly bounded $L_2(\mu)$-norm, it is easy to see (cf. (3.6)) that

$$\limsup_{n\to\infty} \| s_n \|_{[-1,1]}^{1/n} \leq 1.$$

Hence, from (3.7) and (6.10) we get

$$\begin{aligned} \limsup_{\substack{n\to\infty\\ n\in\Lambda}} \| Q_n \|_{[-1,1]}^{1/n} &= \limsup_{\substack{n\to\infty\\ n\in\Lambda}} \frac{\| s_n \|_{[-1,1]}^{1/n}}{|a_n|^{1/n}\, \gamma_n^{1/n}} \\ &\leq \mathrm{cap}([-1,1]) = \frac{1}{2}. \end{aligned}$$

But now we are in position to apply Theorem 5.1: we have just verified condition

(a) and condition (b) holds vacuously. Thus

$$\nu(s_n) = \nu(Q_n) \to \nu_{[-1,1]} = \frac{1}{\pi}\frac{dx}{\sqrt{1-x^2}} \quad \text{as} \quad n \to \infty, \quad n \in \Lambda. \qquad \blacksquare$$

From Theorem 6.4 we see that the zeros plotted in Figures 8 and 9 have limiting distribution equal to the arcsine measure. We also obtain the following

Corollary 6.5. *If* $\mu \in \mathrm{CR}[-1,1]$ *and* $f \in L_2(\mu)$ *is not equal* $d\mu$*-a.e. to a function analytic on an open set containing* $[-1,1]$*, then every point of* $[-1,1]$ *is a limit point of the zeros of the partial sums* $\{s_n(z;f)\}_1^\infty$.

Consequently, $\{s_n(z;f)\}_1^\infty$ *does not converge uniformly in any (2-dimensional) neighborhood of a point of* $[-1,1]$.

This result illustrates a shortcoming of the partial sums s_n. Although, by Theorem 6.2, they are *globally* best approximants to f on $[-1,1]$, *locally* (in the 2-dimensional sense) the sequence $\{s_n\}_1^\infty$ cannot imitate f; it is useless for the purpose of analytic continuation. We therefore have an example of the following

MORAL: WHAT'S BEST GLOBALLY IS LOCALLY NOT SO GOOD.

To lend further support to this moral we mention a recent result of Li, Saff and Sha dealing with the rate of convergence of the Fourier series on subintervals of $[-1,1]$. Let $s_n(z;f)$ denote, as above, the partial sums of the Fourier expansion for f over $[-1,1]$ and set

$$\| f - s_n \|_{L_2(\mu)} = \left(\int_{-1}^{1} |f - s_n|^2 d\mu\right)^{1/2},$$

$$\| f - s_n \|_{L_2(\mu,[a,b])} := \left(\int_{a}^{b} |f - s_n|^2 d\mu\right)^{1/2}.$$

Then we have (cf. [LSS])

Theorem 6.6. *If* $\mu' > 0$ *a.e. on* $[-1,1]$ *and* f *is not* ($d\mu$*- a.e.) a polynomial, then*

$$\sum_{n=0}^{\infty} \left(\frac{\| f - s_n \|_{L_2(\mu,[a,b])}}{\| f - s_n \|_{L_2(\mu)}}\right)^2 = \infty. \tag{6.11}$$

for every subinterval $[a,b] \subseteq [-1,1]$ $(a \neq b)$.

Moreover, (6.11) *is sharp in the sense that the exponent* 2 *cannot, in general, be replaced by any larger constant.*

The proof of this theorem is based upon a lemma of Máté, Nevai and Totik [MNT].

Notice that the divergence of the series in (6.11) implies that infinitely many of its terms must exceed $1/n^{1+\epsilon}$, $\epsilon > 0$. Indeed we have (cf. [LSS])

Corollary 6.7. *With the assumptions of Theorem 6.6, for each $\epsilon > 0$, there exists a subsequence $\Lambda \subseteq \mathbf{N}$ such that for any $[a,b] \subseteq [-1,1]$ $(a \neq b)$,*

$$\| f - s_n \|_{L_2(\mu,[a,b])} \geq \frac{C}{n^{\frac{1}{2}+\epsilon}} \| f - s_n \|_{L_2(\mu)}, \quad n \in \Lambda, \tag{6.12}$$

where the constant $C > 0$ depends only on $b - a$.

Returning to the moral mentioned above, we see from (6.12) that, with reference to the partial sums s_n, what is globally best cannot locally be much better (only improvements of order $1/\sqrt{n}$ are possible for the rate of convergence on subintervals).

7 Weighted Polynomials over Unbounded Sets

Thus far we have restricted our discussion to orthogonal polynomials over a compact subset of $\mathbf{C}$. In this section we describe an approach for analyzing orthogonal polynomials with respect to unbounded sets; for example, when $\text{supp}(\mu) = \mathbf{R}$.

Let $E \subseteq \mathbf{C}$ be a closed (but not necessarily bounded) set having positive capacity and let $W(z)$ be a nonnegative weight function on E. We now wish to attack the following generalized version of the problem stated at the beginning of Section 6.

Problem. *Given that $q_n \in \mathcal{P}_n$ and*

$$|W(z)q_n(z)| \leq M, \quad \forall \quad z \in E,$$

estimate $|q_n(z)|$ for $z \in \mathbf{C}$.

For example, if

$$|e^{-x^2} q_n(x)| \leq M, \quad \forall \quad x \in \mathbf{R} = (-\infty, \infty), \tag{7.1}$$

then what can be said about $|q_n(z)|$ for $z \in \mathbf{C}$? The first point to observe is that (7.1) contains superfluous information. That is, since $|e^{-x^2} q_n(x)| \to 0$ as $|x| \to \infty$, inequality (7.1) is only needed over some *finite* interval. Indeed, if $q_n \not\equiv 0$ and $\xi \in \mathbf{R}$ is a point for which

$$|e^{-\xi^2} q_n(\xi)| = \max_{\mathbf{R}} |e^{-x^2} q_n(x)|$$

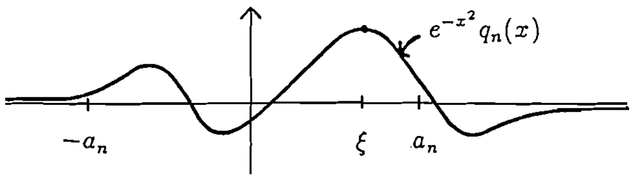


Figure 10

(see Figure 10), then ξ must belong to a finite interval $[-a_n, a_n]$. Moreover, such an a_n can be found that is *independent* of $q_n \in \mathcal{P}_n$. As shown in [MS1], $a_n = \sqrt{n}$ is an asymptotically sharp (smallest) choice. Consequently (7.1) should be replaced by

$$|e^{-x^2}q_n(x)| \leq M, \quad \forall \quad x \in [-\sqrt{n}, \sqrt{n}].$$

What we wish to emphasize is that the above problem is intimately related to the following question: *Where does the sup norm of a weighted polynomial live*?

In answering this question it is convenient to fix n and write $W = w^n$. Our goal is to imitate the Bernstein-Walsh lemma (Lemma 6.1) and in order to carry over the proof we shall assume that the weight w is of a special form that allows us to apply the maximum principle. Namely, we assume that

$$w(z) = \exp(U^\sigma(z) - F_w), \quad z \in E, \tag{7.2}$$

where σ is some probability measure with compact support $\mathcal{S}(\sigma) \subseteq E$, $U^\sigma(z)$ is continuous on C, and F_w is a constant. Then the generalization of Lemma 6.1 becomes

Lemma 7.1. *Let $w : E \to [0, \infty)$ be of the form (7.2). If $q_n \in \mathcal{P}_n$ satisfies*

$$|w(z)^n q_n(z)| \leq M, \quad \forall \quad z \in \mathcal{S}(\sigma), \tag{7.3}$$

then $|w(z)^n q_n(z)| \leq M$ for all $z \in E$. Consequently,

$$\| w^n q_n \|_E = \| w^n q_n \|_{\mathcal{S}(\sigma)} . \tag{7.4}$$

Furthermore,

$$|q_n(z)| \leq M \exp\{-n(U^\sigma(z) - F_w)\}, \quad \forall \quad z \in \mathbf{C}. \tag{7.5}$$

Proof. Inequality (7.3) and the representation (7.2) yield

$$U^\sigma(z) - F_w + \frac{1}{n}\log|q_n(z)| \leq \frac{1}{n}\log M, \quad \forall \quad z \in \mathcal{S}(\sigma). \tag{7.6}$$

But the left-hand side of (7.6) is subharmonic in $\overline{\mathbf{C}}\backslash\mathcal{S}(\sigma)$, even at infinity. Thus, since U^σ is continuous, we can apply the maximum principle to deduce that (7.6) holds for all $z \in \overline{\mathbf{C}}\backslash\mathcal{S}(\sigma)$, which completes the proof. ■

Analyzing the above proof we see that it is not necessary to assume that w is of the form (7.2) for all $z \in E$. Indeed, Lemma 7.1 remains valid provided

$$w(z) = \exp(U^\sigma(z) - F_w), \quad \forall \quad z \in \mathcal{S}(\sigma) \tag{7.7}$$

and

$$w(z) \leq \exp(U^\sigma(z) - F_w), \quad \forall \quad z \in E\backslash\mathcal{S}(\sigma). \tag{7.8}$$

In fact, it's enough to assume that, on the indicated sets, (7.7) and (7.8) hold *quasi-everywhere* (q.e); that is, with the possible exception of a set having capacity zero. Thus to handle general weight functions w we are lead to the following

Question. *Given* $w : E \to [0,\infty)$, *how do we find a probability measure* σ *and a constant* F_w *so that* (7.7) *and* (7.8) *hold?*

Readers familiar with the Szegő theory for orthogonal polynomials on the unit circle will recognize that what we seek is essentially a generalized version of the *Szegő function* $D(\mu; z)$.

It turns out that the above question is related to the following

Generalized Electrostatics Problem. *Let* $E \subseteq \mathbf{C}$ *be closed. Place a unit charge on* E *so that equilibrium is reached in the presence of an external field due to* w (see Figure 11).

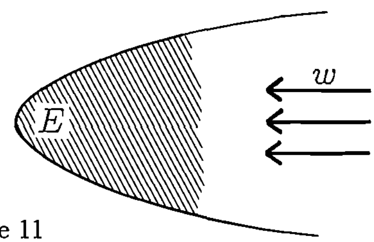


Figure 11

As before, we let $\mathcal{M}(E)$ denote the collection of all positive, unit Borel measures supported on E. Then the energy integral that takes into account the field due to w is

$$I_w[\sigma] := \int\int \log[|z-t|w(z)w(t)]^{-1} d\sigma(z)d\sigma(t)$$

and the generalized Robin's constant is

$$V_w(E) := \inf_{\sigma\in\mathcal{M}(E)} I_w[\sigma]$$

(cf. (4.2) and (4.3)). What we therefore seek is a measure $\sigma_w \in \mathcal{M}(E)$ such that

$$I_w[\sigma_w] = V_w(E). \tag{7.9}$$

For such a measure to exist we need to make some mild assumptions on w.

Definition 7.2. A weight $w : E \to [0, \infty)$ is said to be *admissible* if the following three conditions hold:
(i) w is upper semi-continuous;
(ii) The set $\{z \in E : w(z) > 0\}$ has positive (inner) capacity;
(iii) If E is unbounded, then $|z|w(z) \to 0$ as $|z| \to \infty$, $z \in E$.

Since the external field due to w has a strong repelling effect near points where $w = 0$, the assumption (iii) physically means that this repelling effect is sufficient to prevent charges placed on E from rushing to ∞.

The theory of weighted polynomials and potentials is developed in [GR1],[MS2], [MS3],[GR2] and [STM]. The basis for this theory is the following result (cf. [MS2] for the case when $E \subseteq \mathbf{R}$).

Theorem 7.3. *Let $E \subseteq \mathbf{C}$ be closed and $w : E \to [0, \infty)$ be admissible. Then*
(a) $\exists$ *a unique* $\sigma_w \in \mathcal{M}(E)$ *such that* (7.9) *holds.*
(b) $\mathcal{S}(\sigma_w) = \mathrm{supp}(\sigma_w)$ *is compact.*
(c) $w(z) = \exp(U^{\sigma_w}(z) - F_w)$ *q.e. on* $\mathcal{S}(\sigma_w)$
and

$$w(z) \le \exp(U^{\sigma_w}(z) - F_w) \qquad \text{q.e. on} \quad E,$$

where the constant F_w is given by

$$F_w := V_w(E) + \int \log w \, d\sigma_w.$$

Notice that if E is compact, $\mathrm{cap}(E) > 0$ and $w \equiv 1$ on E, then $\sigma_w = \nu_E$, the equilibrium distribution for E.

From Lemma 7.1 and Theorem 7.3 we get

Corollary 7.4. *If $w : E \to [0, \infty)$ is admissible and $q_n \in \mathcal{P}_n$, then*

$$|q_n(z)| \le \| w^n q_n \|_E \exp\{-n(U^{\sigma_w}(z) - F_w)\}, \quad \forall \quad z \in \mathbf{C}.$$

We remark that in the above inequality, $\| w^n q_n \|_E$ can be replaced by $\| w^n q_n \|_{\mathcal{S}(\sigma_w)}$.

Corollary 7.4 (with $W = w^n$) gives an answer to the problem stated at the beginning of this section. Thus it provides a starting point for the analysis of orthogonal polynomials on unbounded sets. For example, together with some "hard analysis", it leads to the solution of the Freud conjecture dealing with exponential weights on $\mathbf{R}$ (cf. [LMS]).

References

[BSS] H.–P. Blatt, E.B. Saff and M. Simkani (1988), *Jentzsch–Szegö type theorems for the zeros of best approximants*, J. London Math. Soc., 38: 307–316.

[D] P.J. Davis (1963), *Interpolation and Approximation*, Blaisdell Pub. Co, New York.

[EF] P. Erdös and G. Freud (1974), *On orthogonal polynomials with regularly distributed zeros*, Proc. London Math. Soc., 29: 521–537.

[ET] P. Erdös and P. Turán (1940), *On interpolation III*, Ann. of Math., 41: 510–555.

[F] G. Freud (1971), *Orthogonal Polynomials*, Pergamon Press, London.

[G] D. Gaier (1987), *Lectures on Complex Approximation*, Birkhauser, Boston.

[GR1] A.A. Gonchar and E.A. Rakhmanov (1984), *The equilibrium measure and the distribution of zeros of extremal polynomials*, Math. Sb. 125(167): 117–127. Math. USSR. Sbornik 53(1986), 119–130.

[GR2] A.A. Gonchar and E.A. Rakhmanov (1987), *Equilibrium distributions and the rate of rational approximation to analytic functions*, Math. Sb. 134(176): 305–352. (Russian).

[He] L.L. Helms (1969), *Introduction to Potential Theory*, Wiley–Interscience (Pure and Applied Mathematics, vol. XXII), New York.

[Hi] E. Hille (1962), *Analytic Function Theory* (Introduction to Higher Mathematics, vol. II), Ginn and Co., Boston.

[La] N.S. Landkof (1972), *Foundations of Modern Potential Theory*, Springer–Verlag, Berlin.

[LMS] D.S. Lubinsky, H.N. Mhaskar and E.B. Saff (1986), *Freud's conjecture for exponential weights*, Bull. Amer. Math. Soc., 15: 217–221.

[LS] D.S. Lubinsky and E.B. Saff (1988), *Strong Asymptotics for Extremal Polynomials Associated with Weights on* $\mathbb{R}$, Lecture Notes in Math,Vol. 1305, Springer–Verlag, Berlin.

[LSS] X. Li, E.B. Saff and Z. Sha (to appear), *Behavior of best* L_p *polynomial approximants on the unit interval and on the unit circle*, J. Approx. Theory.

[MNT] A. Máté, P. Nevai and V. Totik (1987), *Strong and weak convergence of orthogonal polynomials*, Amer. J. Math., 109: 239–282.

[MS1] H.N. Mhaskar and E.B. Saff (1984), *Extremal problems for polynomials with exponential weights*, Trans. Amer. Math. Soc. **285**: 203–234.

[MS2] H.N. Mhaskar and E.B. Saff (1985), *Where does the sup norm of a weighted polynomial live?*, Constr. Approx. 1: 71–91.

[MS3] H.N. Mhaskar and E.B. Saff (1987), *Where does the L_p–norm of a weighted polynomial live?*, Trans. Amer. Math. Soc., **303**: 109–124. *Errata* (1988), **303**:431.

[MS4] H.N. Mhaskar and E.B. Saff (to appear), *On the distribution of zeros of polynomials orthogonal on the unit circle*, J. Approx. Theory.

[NT] P. Nevai and V. Totik (to appear), *Orthogonal polynomials and their zeros.*

[R] E.A.· Rakhmanov (1984), *On asymptotic properties of polynomials orthogonal on the real axis*, Math. USSR. Sb. **47**: 155–193.

[STM] E.B. Saff, V. Totik and H. Mhaskar (to appear), *Weighted Polynomials and Potentials in the Complex Plane.*

[Sz] G. Szegő (1967), *Orthogonal Polynomials*, 3rd. ed., Amer. Math. Soc. Colloq. Pub, Vol. **23**, Amer. Math. Soc., Providence, R.I.

[T] M. Tsuji (1959), *Potential Theory in Modern Function Theory*, Dover, New York.

[Wa] J.L. Walsh (1960), *Interpolation and Approximation by Rational Functions in the Complex Domain.* 3rd ed., Amer. Math. Soc. Colloq. Publ., Vol. 20, Amer. Math. Soc., Providence, R.I.

[Wi] H. Widom (1967), *Polynomials associated with measures in the complex plane*, J. Math. Mech., **16**: 997–1013.