Seminar in Analysis:
Operator Algebras and
Acylindrically Hyperbolic Groups

Vanderbilt University
Spring 2021

Lebesgue 1902 - PhD defense

How do we measure subsets of $\mathbb{R}^n$?

He developed Lebesgue measure.

Vitali: 1905 Problem: There exist nonmeasurable sets.

Solutions:

1. Insist on countably additive measures restrict the sets that can be measured.
2. Allow finitely additive measures Try to measure all possible sets.

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1. A general theory developed.

Haar 1933 showed that every loc. cpt. group G has a left-invariant regular measure s.t. compact sets have finite measure & open sets have positive measure.

von Neumann 1934 Haar measure is unique up to a constant multiple.

- If $H < G$ a closed subgroup then G/H has a ! measure class G-invariant. Moreover, this class contains an invariant measure iff $\Delta_G|_H = \Delta_H$.

Thm (Vitali 1905) G loc. cpt. non-discrete Then $\nexists$ an extension of Haar measure which is countably additive, G-invariant, and defined on all sets.

Proof:

Let $K \subset G$ be a compact subset with positive measure (e.g. $K = [0,1]$ if $G = \mathbb{R}$)

Let $\Lambda < G$ be a group generated by a countably infinite subset of $K \cdot K^{-1}$ (eg $\Lambda = \mathbb{Q}$ if $G = \mathbb{R}$)

Define an ER on G by the orbits of Λ.

Let $V \subset K$ be any set that intersects in exactly one element for each equivalence class that intersects K.

If $\lambda_1 \lambda_2 \in \Lambda$ then $\lambda_1 V \cap \lambda_2 V = \phi$ if $\lambda_1 \neq \lambda_2$.

If $k \in K$ then $\exists$ $t \in \Lambda$ st $tk \in V \subset K$

$\therefore$ $t = (tk)k^{-1} \in K \cdot K^{-1}$

$\therefore$

$$K \subset (\Lambda \cap K \cdot K^{-1}) \cdot V \subset K K^{-1} K$$

$\therefore$

$$\lambda((\Lambda \cap K \cdot K^{-1}) V) = \sum_{t \in \Lambda \cap K K^{-1}} \lambda(tV) = \sum_{t \in \Lambda \cap K K^{-1}} \lambda(V) \in \{0, \infty\}$$

but

$$0 < \lambda(K) \leq \lambda((\Lambda \cap K K^{-1}) V) \leq \lambda(K \cdot K^{-1} K) < \infty$$

Giving a contradiction.

② Finitely additive measures

Thm (Banach 1923) on both $\mathbb{R}$ and $\mathbb{R}^2$ there do exist finitely additive extensions of Lebesgue measure, that are invariant under rigid motions.

Thm (Banach–Tarski 1926, Hausdorff 1914) If $d \geq 3$ then $\mathbb{R}^d$ does not have a finitely additive extension of Lebesgue measure, that is invariant under rigid motions.

Def: (von Neumann 1929) Γ a group. X a set & suppose $\Gamma \curvearrowright X$. This action is amenable (in the sense of von Neumann) if there exists a Γ-inv. finitely additive probability measure on X. Γ is amenable if $\Gamma \curvearrowright \Gamma$ by left translation is amenable.

$$\mathbb{R}^d \rtimes O(d) \text{ is amenable iff } d = 1, 2.$$

Rmk: finitely additive measures are in bijective correspondence to states on $\ell^\infty(X)$ $\quad (t \cdot f)(x) = f(t^{-1}x).$

$$\ell^\infty X \simeq C(\beta X)$$

states on $C(\beta X) \xleftrightarrow[\text{Riez rep}]{}$ Prob. measures on βX.

Ex: $\mathbb{Z}^d$ is amenable

we have $F_n = [-n, n]^d \cap \mathbb{Z}^d$ s.t.

$$\frac{|aF_n \triangle F_n|}{|F_n|} \xrightarrow[n\to\infty]{} 0 \quad \forall\, a \in \mathbb{Z}^d$$

Define $\varphi_n \in \ell^\infty(\mathbb{Z}^d)^*$ states by

$$\varphi_n(f) = \frac{1}{|F_n|} \sum_{t \in \mathbb{Z}^d} f(t).$$

Let φ be some wk*-accumulation pt.

$$|\varphi_n(f) - \varphi_n(t\cdot f)|$$

$$= \frac{1}{|F_n|} \Big| \sum_{s\in F_n} f(s) - f(t^{-1}s) \Big|$$

$$= \frac{1}{|F_n|} \Big| \sum_{s \in F_n \setminus tF_n} f(s) - \sum_{s \in tF_n \setminus F_n} f(s) \Big|$$

$$\leq \frac{|F_n \triangle tF_n|}{|F_n|} \|f\|_\infty \xrightarrow[n\to\infty]{} 0.$$

$$\therefore\ \varphi(f) = \varphi(t\cdot f) \qquad \square.$$

Γ a group. A Følner sequence (or net) is a sequence of finite subsets $\{F_n\}$ s.t.

$$\frac{|tF_n \triangle F_n|}{|F_n|} \xrightarrow[n\to\infty]{} 0 \quad \forall\, t \in \Gamma.$$

Remark: If every fg subgroup of Γ is amenable then Γ is amenable

proof: $\forall$ fg subgroup Γ_0 we define a state φ_{Γ_0} on $\ell^\infty\Gamma$ by $\varphi_{\Gamma_0}(f) = \Psi_{\Gamma_0}(f|_{\ell^\infty\Gamma_0})$ where $\Psi_{\Gamma_0} \in (\ell^\infty\Gamma_0)^*$ is an inv state.

If φ is any wk^*-accumulation pt of $\{\varphi_{\Gamma_0} : \Gamma_0 < \Gamma \text{ fg}\}$ then φ is a Γ-inv state.

Prop: If Γ is amenable and $\Sigma \trianglelefteq \Gamma$ then Γ/Σ is amenable.

Proof:

$$\ell^\infty(\Gamma/\Sigma) \xrightarrow[\Theta]{\sim} (\ell^\infty\Gamma)^\Sigma = \{f \in \ell^\infty\Gamma \text{ st } f(t\sigma) = f(t) \ \forall t \in \Gamma \ \sigma \in \Sigma\}$$

If φ is a Γ-inv state on $\ell^\infty\Gamma$ then $\varphi \circ \Theta$ is Γ/Σ-invariant on $\ell^\infty(\Gamma/\Sigma)$

Cor: All abelian groups are amenable.

Prop: If Γ is amenable and $\Sigma < \Gamma$ then Σ is amenable.

Proof:

Let T be a set of coset representatives for $\Sigma \backslash \Gamma$

ie $\Gamma = \Sigma \cdot T$

we define $\phi : \ell^\infty\Sigma \to \ell^\infty\Gamma$ by

$$\phi(f)(\sigma \cdot t) = f(\sigma).$$

ϕ is Σ-equivariant

If $\varphi \in (\ell^\infty\Gamma)^*$ is a Γ-inv state then $\varphi \circ \phi$ is a Σ-inv state on $\ell^\infty\Sigma$.

Ex: All finite groups are amenable.

Thm: Γ a group. TFAE

1. Γ is amenable
2. whenever Γ acts by homeomorphisms on a compact Hausdorff space there exists an invariant probability measure.
3. If V is a lctvs and $\mathcal{K} \subset V$ is a non-empty compact convex subset and Γ acts on $\mathcal{K}$ by continuous affine transformations, then Γ has a fixed point in $\mathcal{K}$.

Proof: $\beta\Gamma$ is cpt Hausdorff.

$(2 \Rightarrow 1)$ $\beta\Gamma$ is cpt Hausdorff.

$(1 \Rightarrow 2)$ suppose $\Gamma \curvearrowright K$ K cpt Hausdorff

Fix $\kappa \in K$ consider the map $\Gamma \to K$ defined by $\Gamma \ni t \mapsto t \cdot \kappa \in K$

$\exists$ $\pi : \beta\Gamma \to K$ cont st $\pi(t) = t\kappa$ for $t \in \Gamma$. This will be equivariant.

The push-forward of a Γ-inv prob measure on $\beta\Gamma$ gives a Γ-inv prob measure on K.

$(2 \Rightarrow 3)$

If $\mu \in Prob(\mathcal{K})$ is Γ-invariant

$bar(\mu) \in \mathcal{K}$ is Γ-invariant.

$(3 \Rightarrow 2)$ $\underline{Prob(K)} \subset C(K)^*$ is wk^* compact convex.

Prop: If Γ is a group and $\Sigma \trianglelefteq \Gamma$.
If Σ and Γ/Σ are amenable
then Γ is amenable.

Proof:
Suppose $\Gamma \curvearrowright K$ a compact Hausdorff space.

Consider $\text{Prob}(K)^\Sigma$ the space of Σ-invariant prob measures

then $\Gamma/\Sigma \curvearrowright \text{Prob}(K)^\Sigma$ a compact convex non-empty subset of $C(K)^*$

Since Γ/Σ is amenable there then exists a Γ-fixed point in $\text{Prob}(K)^\Sigma$. $\square$

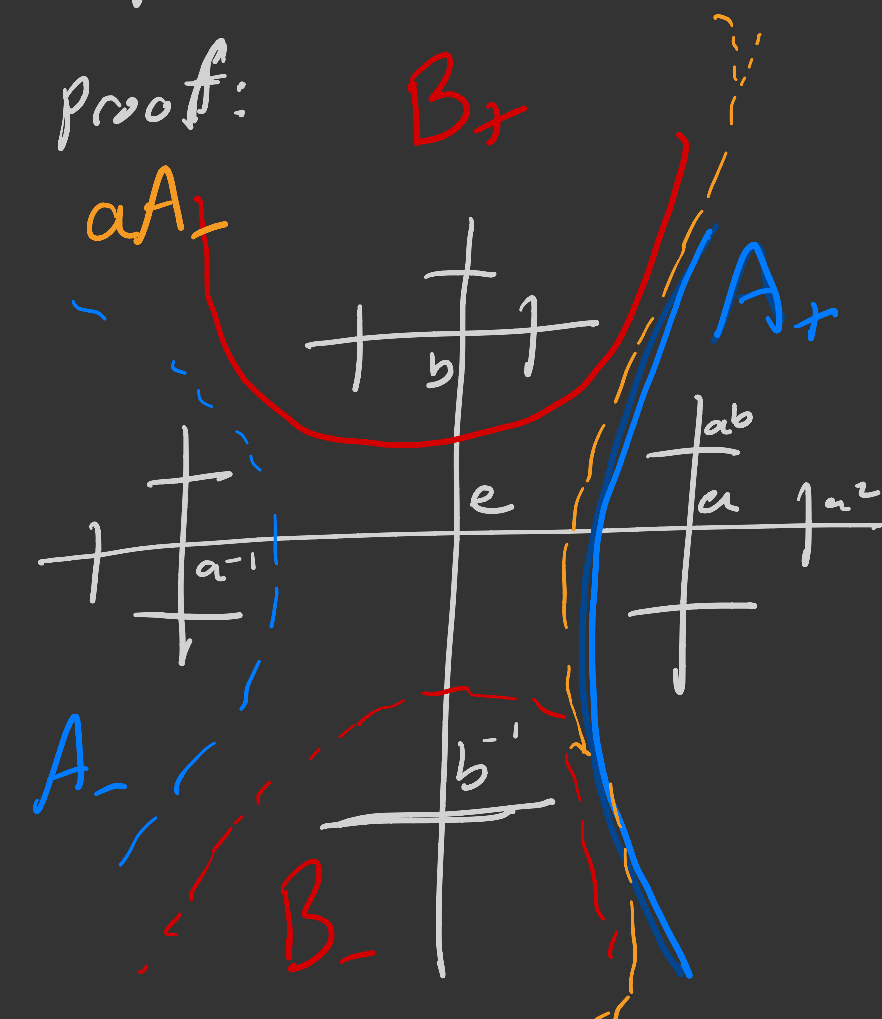
Abelian gps, finite groups, inductive unions, extensions, subgroups, quotients $\rightarrow$ elementary amenable

Amenable

Grigorchuk groups

Prop: $\mathbb{F}_2 = \langle a, b \rangle$ is not amenable.

proof:



$\mathbb{F}_2 = A_+ \sqcup A_- \sqcup B_+ \sqcup B_- \sqcup \{e\}$

$aA_- \cup A_+ = \mathbb{F}_2$

$bB_- \cup B_+ = \mathbb{F}_2$

$\rightarrow$ Paradoxical decomposition

If $\varphi \in (\ell^\infty \mathbb{F}_2)^*$ is a left invariant pos. linear functional then

$$\varphi(1) = \varphi(1_{A_+} + 1_{A_-} + 1_{B_+} + 1_{B_-} + 1_{\{e\}})$$

$$= \varphi(1_{A_+} + a \cdot 1_{A_-})$$

$$+ \varphi(1_{B_+} + b \cdot 1_{B_-}) + \underbrace{\varphi(1_{\{e\}})}$$

$$\geq \varphi(1) + \varphi(1) = \varphi(2)$$

$$\Longrightarrow \varphi(1) = 0, \text{ ie } \varphi \equiv 0. \quad \square$$

$\therefore$ If a group Γ contains $\mathbb{F}_2$ then Γ is not amenable.

Olshanski : Tarski monsters.

Thompsons group F

Open: Is F amenable.

Def: If $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation, then π has almost invariant vectors if there exists a sequence (net) of unit vectors $\{\xi_n\}_n$ st

$$\| \pi(t)\xi_n - \xi_n \| \xrightarrow[n \to \infty]{} 0 \quad \forall t \in \Gamma.$$

Thm: TFAE

① Γ is amenable

② $\exists\ \mu_n \in \mathrm{Prob}(\Gamma) \subset \ell^1\Gamma$ st (net)

$$\|\mu_n - t\mu_n\|_1 \xrightarrow[n\to\infty]{} 0 \quad t \in \Gamma.$$

③ The left-regular rep $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ has almost invariant vector.

④ There exists a Følner sequence (net)

Proof:

④ ⟹ ②: If $\{F_n\}$ is a Følner net then define $\mu_n = \frac{1}{|F_n|} 1_{F_n}$

$$\|\mu_n - t\mu_n\|_1 = \frac{|F_n \triangle tF_n|}{|F_n|}$$

② ⟹ ① $\mathrm{Prob}(\Gamma) \subset \ell^1\Gamma = (\ell^\infty\Gamma)_*$

Any accumulation pt of $\{\mu_n\}$ in the state space of $\ell^\infty\Gamma$ will be an invariant mean.

② ⟺ ③: If $\mu \in (\ell^1\Gamma)_+$ consider $\xi \in \ell^2\Gamma$ given by $\xi(t) = \sqrt{\mu(t)}$

$$\|\xi - \lambda_t \xi\|_2^2 = \sum_{s\in\Gamma} \left|\sqrt{\mu(s)} - \sqrt{\mu(t^{-1}s)}\right|^2$$

$$\leq 2 \sum_{s\in\Gamma} |\mu(s) - \mu(t^{-1}s)|$$

$$= 2\|\mu - t\mu\|_1$$

$$\left(|a-b|^2 \leq 2(a^2 - b^2)\right)$$

If $\xi \in \ell^2\Gamma$ unit then consider $\mu \in \mathrm{Prob}(\Gamma)$ by $\mu(s) = |\xi(s)|^2$

$$\|\mu - t\mu\|_1 = \sum_{s\in\Gamma} |\mu(s) - \mu(t^{-1}s)|$$

$$\leq 2 \sum_{s\in\Gamma} |\xi(s) - \xi(t^{-1}s)|^2$$

(1) $\Rightarrow$ (2) (Day's trick)

suppose $\ell^\infty\Gamma$ has a left invariant state.

Fix $t_1, \ldots, t_n \in \Gamma$

consider
$$\mathcal{C} = \{(\mu - t_1\mu, \mu - t_2\mu, \ldots, \mu - t_n\mu) : \mu \in Prob(\Gamma)\}$$
$$\in (\ell^1\Gamma)^{\oplus n}$$

Note $\mathcal{C}$ is convex

(Goldstein: $(\ell^1\Gamma)_1 \hookrightarrow ((\ell^\infty\Gamma)^*)_1$ is wk*-dense)

$\therefore$ $Prob(\Gamma)$ is wk*-dense in the state space of $\ell^\infty\Gamma$.

$$\therefore\ 0 \in \overline{\mathcal{C}}^{wk^*} \subset (\ell^\infty\Gamma)^*$$

$$\therefore\ 0 \in \overline{\mathcal{C}}^{wk} \subset \ell^1\Gamma$$

By Hahn-Banach $\quad 0 \in \overline{\mathcal{C}}^{\|\cdot\|_1} \subset \ell^1\Gamma$

$\therefore$ If $\varepsilon > 0$ $\exists$ $\mu \in Prob(\Gamma)$ st
$$\sum_{i=1}^n \|\mu - t_i\mu\|_1 < \varepsilon.$$

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(2) $\Longrightarrow$ (4): (Namioka's trick)

Fix $t_1, \ldots, t_n \in \Gamma$, $\varepsilon > 0$.

Take $\mu \in Prob(\Gamma)$ st
$$(*)\quad \sum_{i=1}^n \|\mu - t_i\mu\|_1 < \varepsilon\|\mu\|_1$$

For each $r > 0$ let $1_{(r,\infty)}$ be the characteristic function on the interval $(0,r)$.

If $\alpha, \beta \ge 0$ then
$$|\alpha - \beta| = \int_0^\infty |1_{(r,\infty)}(\alpha) - 1_{(r,\infty)}(\beta)|\, dr$$

$$\therefore\ \sum_{i=1}^n \sum_{s\in\Gamma} \int_0^\infty \left| 1_{(r,\infty)}(\mu(s)) - 1_{(r,\infty)}(\mu(t_i^{-1}s)) \right| dr$$
$$< \varepsilon \sum_{s\in\Gamma} \int_0^\infty 1_{(r,\infty)}(\mu(s))\, dr$$

$$\sum_{i=1}^{n} \sum_{s\in\Gamma} \int_0^\infty \left| 1_{(r,\infty)}(\mu(s)) - 1_{(r,\infty)}(\mu(t_i^{-1}s)) \right| dr$$

$$< \varepsilon \sum_{s\in\Gamma} \int_0^\infty 1_{(r,\infty)}(\mu(s))\, dr$$

$\therefore$ there exists $r \geq 0$ st

$$\sum_{i=1}^{n} \sum_{s\in\Gamma} \left| \underbrace{1_{(r,\infty)}(\mu(s))} - \underbrace{1_{(r,\infty)}(\mu(t_i^{-1}s))} \right|$$

$$< \varepsilon \sum_{s\in\Gamma} \underbrace{1_{(r,\infty)}(\mu(s))\, dr}$$

$\therefore$ If $F = \{ s \in \Gamma \mid \mu(s) > r \}$

then F is finite and

$$\sum_{i=1}^{n} |F \triangle t_i F| < \varepsilon |F| \qquad \square$$

If Γ is a group then

$C^*_r\Gamma$ is the C^*-algebra generated by the left-regular representation

$\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$.

The full group C^*-algebra is the C^*-alg generated by the universal representation

$$\pi_u := \bigoplus_{\text{rep}} \underline{\pi}$$

we have the universal property that if $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is any unitary representation then there is a unique $*$-hom $\pi : C^*\Gamma \to \mathcal{B}(\mathcal{H})$ extending this representation.

$$C^*\Gamma \xrightarrow{\lambda} C^*_r\Gamma$$

Thm: Γ is amenable iff the $*$-hom $C^*\Gamma \to C^*_r\Gamma$ is a $*$-isomorphism.

Proof:

($\Leftarrow$) Then $C^*_r\Gamma$ has a non-trivial 1-dimensional $*$-representation π.

$C^*_r\Gamma \subset B(\ell^2\Gamma)$.

By the Hahn-Banach theorem there exists a state φ on $\underline{B(\ell^2\Gamma)}$ st $\varphi(\lambda_\gamma) = \pi(\lambda_\gamma)$

$$\ell^\infty\Gamma \xrightarrow[*\text{-hom}]{M} B(\ell^2\Gamma)$$

by $M_f\delta_t = f(t)\delta_t$

Then $\lambda_t M_f \lambda_t^* = M_{t\cdot f}$, since

$$(\lambda_t M_f \lambda_t^*)\delta_s = \lambda_t M_f \delta_{t^{-1}s}$$
$$= \lambda_t \underline{f(t^{-1}s)}\,\delta_{t^{-1}s}$$
$$= f(t^{-1}s)\,\delta_s$$
$$= M_{t\cdot f}\,\delta_s$$

$$\varphi(M_{t\cdot f}) = \varphi(\lambda_t \underline{M_f} \lambda_t^*)$$
$$\boxed{=}\ \pi(\lambda_t)\,\varphi(M_f)\,\pi(\lambda_t)^*$$
$$= \varphi(M_f).$$

Fell's absorption lemma:

If $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary rep then $\underline{\pi \otimes \lambda} \sim \underline{1 \otimes \lambda}$

λ the left-regular rep.

Proof:

Define $U : \mathcal{H} \bar{\otimes} \ell^2\Gamma \to \mathcal{H} \bar{\otimes} \ell^2\Gamma$

$\overset{\shortparallel}{\bigoplus_{\Gamma} \mathcal{H}}$

by $U(\xi \otimes \delta_t) = \pi(t^{-1})\xi \otimes \delta_t$

$$\underbrace{U^*(1 \otimes \lambda)_t U}(\xi \otimes \delta_s)$$
$$= U^*(1 \otimes \lambda)_t \, \pi(s^{-1})\xi \otimes \delta_s$$
$$= U^* \pi(s^{-1})\xi \otimes \delta_{ts}$$
$$= \pi(t)\xi \otimes \delta_{ts}$$
$$= \underbrace{(\pi \otimes \lambda)_t}\, \xi \otimes \delta_s$$

$\therefore \; 1 \otimes \lambda \sim \pi \otimes \lambda$

$1 \prec \lambda$ from amenability

$\pi \sim 1 \otimes \pi \prec \lambda \otimes \pi \sim \lambda \otimes 1$

$C^*\Gamma \to C^*_r\Gamma$ is a $*$-isomorphism.

Lem: A a C^*-alg with unit

Fact: If φ is a state on A

& $x \in A$ st $\varphi(xx^*) = \varphi(x^*x) = |\varphi(x)|^2$

then $\varphi(xT) = \varphi(x)\varphi(T) \;\; \forall\, T \in A$

$\varphi(Tx) = \varphi(T)\varphi(x)$

Proof:

GNS construction the state φ corresponds to a representation $\pi : A \to \mathcal{B}(\mathcal{H})$ and a unit vector $\xi \in \mathcal{H}$ st

$$\varphi(a) = \langle \pi(a)\xi, \xi \rangle \quad \forall a \in A.$$

$$\varphi(x^*x) = \langle \pi(x^*x)\xi, \xi\rangle = \|\pi(x)\xi\|^2$$

$$\varphi(x^*)\varphi(x) = |\langle \pi(x)\xi, \xi\rangle|^2$$

$$\therefore \|\pi(x)\xi\| = |\langle \pi(x)\xi, \xi\rangle| \le \|\pi(x)\xi\|$$

$$\therefore \pi(x)\xi = \alpha\xi \quad \text{for } \alpha \in \mathbb{T}$$

$$\varphi(x^*a) = \langle \pi(x^*a)\xi, \xi\rangle$$

$$= \langle \pi(a)\xi, \alpha\xi\rangle = \bar{\alpha}\varphi(a)$$

$$= \overline{\varphi(x)}\varphi(a) \quad \square$$

$(\Longrightarrow)$ (amenability $\Longrightarrow$ $C^*\Gamma \to C^*_r\Gamma$ is an isomorphism)

Suppose $T \in C^*\Gamma$ is in the kernel of this homomorphism.

$\therefore \exists\ T_n \in \mathbb{C}\Gamma$ st $\|T - T_n\| \to 0$

Fix a rep $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$,

Fix $\xi, \eta \in \mathcal{H}$

$$|\langle \pi(T)\xi, \eta\rangle| = \lim_{n\to\infty} |\langle \pi(T_n)\xi, \eta\rangle|$$

$$= \lim_{n\to\infty} \lim_{m\to\infty} |\langle (\pi \otimes \lambda)(T_n)\xi \otimes \zeta_m, \eta \otimes \zeta_m\rangle|$$

where $\zeta_m \in \ell^2\Gamma$ is an almost invariant sequence.

$$= \lim_{n\to\infty} \lim_{m\to\infty} |\langle (1 \otimes \lambda)(T_n)\tilde{\zeta}_m, \tilde{\tilde{\zeta}}_m\rangle|$$

$$\le \lim_{n\to\infty} \lim_{m\to\infty} \|(1\otimes\lambda)(T_n)\|\,\|\xi\|\,\|\eta\|$$

$$\longrightarrow 0$$

Def: A tracial von Neumann algebra (M, τ) is amenable if there exists a <u>hypertrace</u> φ on $\mathcal{B}(L^2(M,\tau))$, i.e. φ is a state on $\mathcal{B}(L^2(M,\tau))$ s.t.

① $\varphi|_M = \tau$

② $\varphi(xT) = \varphi(Tx)$ $\forall$ $\underline{x \in M}$, $T \in \mathcal{B}(L^2(M,\tau))$.

$$L\Gamma := \lambda(\Gamma)''$$

$$\tau(x) = \langle x\,\delta_e, \delta_e \rangle.$$

Thm: A group Γ is amenable iff $L\Gamma := \lambda(\Gamma)''$ is amenable.

Proof:

$(\Longrightarrow)$

Consider $\mathcal{S}$ the set of states on $\mathcal{B}(\ell^2\Gamma)$ that extend the canonical trace on $L\Gamma$.

a wk*-cpt set, convex.

$\Gamma \curvearrowright \mathcal{S}$ by conjugation with the left-regular representation.

Since Γ is amenable there is a fixed point φ. Note that $\underline{\varphi|_{L\Gamma} = \tau}$

$$\varphi(\lambda(\gamma) T \lambda(\gamma^{-1})) = \varphi(T) \quad \forall\, T \in \mathcal{B}(\ell^2\Gamma),\ \gamma \in \Gamma$$

$$\therefore\ \varphi(\lambda(\gamma)T) = \varphi(T\lambda(\gamma))$$

$$\therefore\ \varphi(xT) = \varphi(Tx) \quad \forall\, T \in \mathcal{B}(\ell^2\Gamma),\ \underline{x \in \mathbb{C}\Gamma}$$

If $x \in L\Gamma$, take $x_n \in \mathbb{C}\Gamma$

$x_n \to x$ in the SOT $\|x_n\|$ unif bdd

$$|\varphi(xT) - \varphi(x_nT)|$$

$$\leq \underbrace{\varphi((x-x_n)(x-x_n)^*)^{1/2}}_{\longrightarrow 0} \varphi(T^*T)^{1/2}$$

similarly $|\varphi(Tx) - \varphi(Tx_n)| \to 0$

$\therefore$ $\varphi(xT) - \varphi(Tx)$

$$= \lim_{n\to\infty} \varphi(x_nT) - \varphi(Tx_n) = 0.$$

($\Longleftarrow$) If φ is a hypertrace for $L\Gamma$, then $\varphi|_{\ell^\infty\Gamma}$ is a state

and if $s \in \Gamma$ $f \in \ell^\infty\Gamma$

then $\varphi(s \cdot f) = \varphi(\lambda(s) f \lambda(s)^*)$

$= \varphi(f)$. $\square$

Property (T)

Def: (Kazdan, Margulis): If $\Sigma < \Gamma$ then (Γ, Σ) has relative property (T) if whenever $\pi: \Gamma \to \mathcal{U}(\mathcal{H})$ has almost invariant vectors, then there is a non-zero Σ-invariant vector.

Γ has property (T) if (Γ, Γ) has relative property (T).

Note: If Γ is amenable and has (T) then Γ is finite.

Lem: If $\Gamma \twoheadrightarrow \Lambda$ and Γ has (T) then Λ has (T).

Lem: If Γ has (T) then Γ is finitely generated.

Proof:

$\forall$ finite subset $F \subset \Gamma$ consider $\Gamma \curvearrowright \ell^2 \Gamma/\langle F \rangle$. $\delta_{\langle F \rangle}$ is $\langle F \rangle$ invariant.

Set $\mathcal{H} = \underbrace{\bigoplus_{F \subset \Gamma} \ell^2 \Gamma/\langle F \rangle}$

Prop (T) $\Rightarrow$ $\mathcal{H}$ has a non-zero Γ-inv. vector. Projecting then gives a Γ-inv vector in some $\ell^2 \Gamma/\langle F \rangle$

$\Rightarrow$ $\langle F \rangle$ has finite index

$\Rightarrow$ Γ is f.g.

$SL_n\mathbb{Z}$ has (T) for $n\geq 3$

1st step: $(SL_n\mathbb{Z}\ltimes\mathbb{Z}^n, \mathbb{Z}^n)$ has relative (T) for $n\geq 2$.

If A is an abelian discrete group

& π is a rep, then

then we get a representation of

$\underline{C^*_r(A)\simeq C(\hat{A})}$ $\hat{A}$ compact.

$\therefore \exists$ a $\underline{\text{spectral measure}}$ on $\hat{A}$

giving this representation, i.e. $\forall$

unit vector $\xi\in\mathcal{H}$ there is

a probability measure $\mu_\xi\in Prob(\hat{A})$

s.t. $\langle\pi(a)\xi,\xi\rangle=\int\langle a,x\rangle\, d\mu_\xi(x)$

$\boxed{\text{Prop:}}$ A abelian. If $\Lambda\curvearrowright A$ by automorphisms

$\Gamma=\Lambda\ltimes A$. If (Γ, A) does not have ~~has~~

rel (T) then $\exists$ a net $\nu_i\in Prob(\hat{A})$

s.t.

① $\underline{\nu_i(\{e\})=0}$

② $\underline{\nu_i\longrightarrow\delta_{\{e\}}}$ $\underline{\text{wk}^*}$

③ $\|\underline{t\nu_i-\nu_i}\|\longrightarrow 0$ $\forall\, t\in\Lambda$.

Proof:

If π has a.i. vectors $\underline{\xi_n}$ then π has no A-inv vectors.

$\underline{\mu_{\xi_n}}$ satisfies

$\langle\pi(a)\xi_n,\xi_n\rangle=\underline{\int\langle a,x\rangle\, d\mu_{\xi_n}(x)}$

$\Longrightarrow$ $\nu_i\longrightarrow\delta_{\{e\}}$ wk*

$\|t\mu_{\xi_n}-\mu_{\xi_m}\|=\sup_{f\in C(\hat{A})}\left|\int f\, d(t\mu_{\xi_n}-\mu_{\xi_n})\right|$

cor: If (Γ, A) does not have relative (T) then $\exists$ a finitely additive probability measure m on $\mathrm{Borel}(\hat{A})$ st $m(\{e\}) = 0$ and is Λ-invariant, and st $m(E) = 1$ for every open nbhd of $\{e\}$.

Proof:
Let m be an wk*-accumulation point of $\{\nu_i\} \subset \mathrm{Prob}(\hat{A})$
$\subset$ states on $\mathrm{Borel}^{\infty}(\hat{A})$

$\nu_i \to \delta_{\{e\}}$ wk*.

$\Lambda \curvearrowright A$

$t \cdot a \cdot t^{-1}$ $\quad t \in \Lambda$

$$\langle \pi(t a t^{-1}) \xi_n, \xi_n \rangle = \int \langle t a t^{-1}, x \rangle \, d\mu_{\xi_n}(x) = \int \langle a, x \rangle \, d t_* \mu_{\xi_n}(x)$$

$$\approx \langle \pi(a) \xi_n, \xi_n \rangle = \int \langle a, x \rangle \, d\mu_{\xi_n}(x)$$

$\|\pi(t)\xi_n - \xi_n\|$

$$\left| \int \hat{f}(x) \, d(\mu_{\xi_n} - t_* \mu_{\xi_n})(x) \right| \le C_n \|f\|_1$$

$$\Longrightarrow \|\mu_{\xi_n} - t_* \mu_{\xi}\| \xrightarrow[n \to \infty]{} 0.$$

Thm: $(SL_2\mathbb{Z} \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ has rel. (T).

Proof:

$SL_2\mathbb{Z} \curvearrowright \mathbb{Z}^2$ by matrix mult.

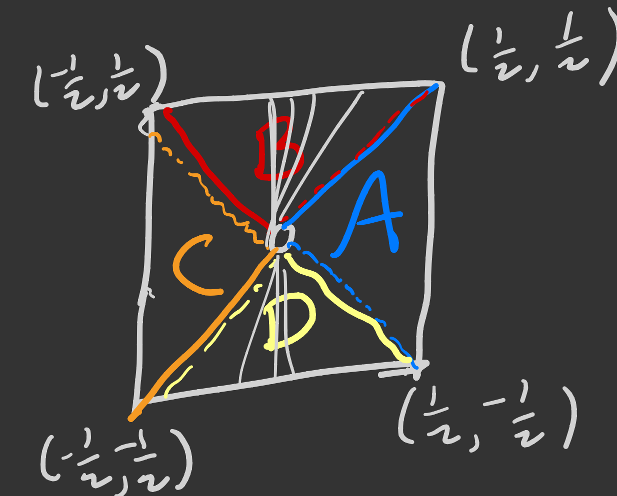
$$\widehat{\mathbb{Z}^2} = (\mathbb{R}/\mathbb{Z})^2$$

$$\langle v_1, v_2 \rangle = e^{2\pi i (v_1 \cdot v_2)}$$

$SL_2\mathbb{Z} \curvearrowright (\mathbb{R}/\mathbb{Z})^2$ is matrix mult. by inverse transpose

If we do not have rel (T) then $\exists$ a finitely additive prob measure m on Borel$(\mathbb{R}/\mathbb{Z})^2$ st $m(\{e\})=0$

m is $SL_2\mathbb{Z}$-invariant and $m(O)=1$ for any nbhd of $\{e\}$.



consider the following sets:

$$A = \left\{\begin{pmatrix}x\\y\end{pmatrix} \,\middle|\, x>0,\ -x<y\leq x\right\}$$

$$B = \left\{\begin{pmatrix}x\\y\end{pmatrix} \,\middle|\, y>0,\ -y\leq x<y\right\}$$

$$C = \left\{\begin{pmatrix}x\\y\end{pmatrix} \,\middle|\, x<0,\ x\leq y<-x\right\}$$

$$D = \left\{\begin{pmatrix}x\\y\end{pmatrix} \,\middle|\, y<0,\ y<x\leq -y\right\}.$$

$$\underline{\begin{pmatrix}1&0\\2k&1\end{pmatrix} \cdot A} = \left\{\begin{pmatrix}x\\y\end{pmatrix} \,\middle|\, x>0\ \ (2k-1)x<y\leq(2k+1)x\right\}$$

Pairwise disjoint for $k\in\mathbb{N}$

$\Longrightarrow m(A)=0.$

$\begin{pmatrix}1&2k\\0&1\end{pmatrix} B$ pairwise disjoint

$\begin{pmatrix}1&2k\\0&1\end{pmatrix} D$ pairwise disjoint

$\begin{pmatrix}1&0\\2k&1\end{pmatrix} C$ pairwise disjoint $m(C)=0$

A contradiction.

Thm (Kassabov): If R is any f.g. ring then $(EL_2(R) \ltimes R^2, R^2)$ has rel (T).

Thm: $SL_3\mathbb{Z}$ has (T).

If S is a set then a map $\varphi: S\times S \to \mathbb{C}$ is of positive type if the matrix $(\varphi(s,t))_{s,t}$ is non-negative definite, i.e. $\forall \alpha_1, \dots, \alpha_n \in \mathbb{C}$ $s_1, \dots, s_n \in S$

$$\sum_{i,j=1}^n \overline{\alpha_j}\alpha_i \varphi(s_i, s_j) \geq 0. \quad (*)$$

Prop: (GNS-type construction) If $\varphi: S\times S \to \mathbb{C}$ then φ is of pos. type iff $\exists$ $\xi: S \to \mathcal{H}$ a Hilbert space s.t.

$$\varphi(s,t) = \langle \xi_s, \xi_t \rangle.$$

Pf: ($\Leftarrow$) $\sum_{i,j=1}^n \overline{\alpha_j}\alpha_i \langle \xi_{s_i}, \xi_{s_j} \rangle$

$$= \left\| \sum_{i=1}^n \alpha_i \xi_{s_i} \right\|^2 \geq 0$$

($\Rightarrow$) Define $\langle \cdot, \cdot \rangle_\varphi$ on $\mathbb{C}S$ by

$$\left\langle \sum_{i=1}^n \alpha_i s_i, \sum_{j=1}^m \beta_j t_j \right\rangle = \sum_{i,j=1}^{n,m} \overline{\beta_j}\alpha_i \varphi(s_i, t_j)$$

$$\mathcal{H} = \overline{\mathbb{C}S/\ker(\langle \cdot, \cdot \rangle_\varphi)}^{\langle \cdot, \cdot \rangle_\varphi}$$

$\xi: S \to \mathcal{H}$ is given by $\xi_s = [s] \in \mathcal{H}$

$$\langle \xi_s, \xi_t \rangle = \langle [s], [t] \rangle = \varphi(s,t) \quad \square$$

If Γ is a group, $\varphi : \Gamma \to \mathbb{C}$ is of positive type if the map $\Gamma \times \Gamma \ni (s,t) \mapsto \varphi(t^{-1}s)$ is of positive type.

GNS: φ is pos type iff $\exists$ a unitary rep $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ $\xi \in \mathcal{H}$ st $\varphi(t) = \langle \pi(t)\xi, \xi \rangle$.

$\psi(s,t) = \psi(t,s) \quad \psi(s,s) = 0$

S a set. A map $\psi : S \times S \to \mathbb{R}$ is of conditionally negative type if $\forall$ $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ st. $\boxed{\sum_{i=1}^{n} \alpha_i = 0}$ $\forall$ $s_1, \dots, s_n \in S$ we have

$$\sum_{i,j=1}^{n} \alpha_j \alpha_i \psi(s_i, s_j) \leq 0.$$

cor: If φ_1, φ_2 are of pos. type then $\varphi_1 \cdot \varphi_2$ is of pos. type.

proof: $\varphi_i(s,t) = \langle \xi^i_s, \xi^i_t \rangle$

$$\varphi_1(s,t)\varphi_2(s,t) = \langle \xi^1_s \otimes \xi^2_s, \xi^1_t \otimes \xi^2_t \rangle. \ \square$$

Prop: If $\psi : S \times S \to \mathbb{R}$ then ψ is cond. of neg. type iff $\exists$ a Hilbert space $\mathcal{H}$ and a map $\eta : S \to \mathcal{H}$ st

$$\psi(s,t) = \| \eta_s - \eta_t \|^2.$$

Proof: ($\Longleftarrow$): $\sum_{i,j=1}^{n} \alpha_j \alpha_i \| \eta_{s_i} - \eta_{s_j} \|^2$

$$= \underbrace{\sum_{i,j=1}^{n}} \alpha_j \alpha_i \left(\underbrace{\|\eta_{s_i}\|^2} - 2\mathrm{Re}\langle \eta_{s_i}, \eta_{s_j} \rangle + \underbrace{\|\eta_{s_j}\|^2} \right)$$

$$= \sum_{i,j=1}^{n} -2\mathrm{Re}\langle \alpha_i \eta_{s_i}, \alpha_j \eta_{s_j} \rangle = -2 \left\| \sum_i \alpha_i \eta_{s_i} \right\|^2 \leq 0.$$

$(\Longrightarrow)$ $\mathbb{R}_0 S := \{ \sum_{i=1}^{n} \alpha_i s_i \mid \sum_{i=1}^{n} \alpha_i = 0 \}$

on $\mathbb{R}_0 S$ we define

$$\left\langle \sum_{i=1}^{n} \alpha_i s_i , \sum_{j=1}^{m} \beta_j t_j \right\rangle_\psi = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j \psi(s_i, t_j)$$

$$\mathcal{H} = \overline{\mathbb{R}_0 S}^{\langle \cdot, \cdot \rangle_\psi}$$

Fix $s_0 \in S$.

define $\eta : S \to \mathcal{H}$ by $\eta(s) = [s - s_0] \in \mathcal{H}$

$$\begin{aligned}
\|\eta(s) - \eta(t)\|^2 &= \|[s - s_0] - [t - t_0]\|^2 \\
&= \|[s - t]\|^2 \\
&= \underbrace{-\frac{1}{2}\psi(s,s)}_{0} + \frac{1}{2}\psi(s,t) + \frac{1}{2}\psi(t,s) - \underbrace{\frac{1}{2}\psi(t,t)}_{0} \\
&= \psi(s,t)
\end{aligned}$$

Γ a group. $\psi : \Gamma \to \mathbb{R}$ is of cond. neg type if the kernel $\Gamma \times \Gamma \ni (s,t) \longmapsto \psi(t^{-1}s)$ is cond. neg. type.

If $\psi : \Gamma \to \mathbb{R}$ is of cond. neg. type then $\exists$ a rep. $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ and a map $c : \Gamma \to \mathcal{H}$ a cocycle

$$\boxed{c(st) = c(s) + \pi(s)\, c(t)}$$

(*) cocycle identity

s.t. $\psi(t) = \| c(t) \|^2$.

Pf: $c(t) = [t - e]$ is a cocycle and $c(e) = 0$

$$\psi(t) = \psi(e^{-1}t) = \| c(t) - c(e) \|^2 = \| c(t) \|^2.$$

conversely if $c : \Gamma \to \mathcal{H}$ is a cocycle for π then

$$\|c(t^{-1}s)\|^2$$
$$= \|c(t^{-1}) + \pi(t^{-1})c(s)\|^2$$
$$= \|c(s) + \underbrace{\pi(t)c(t^{-1})}\|^2$$
$$= \|c(s) - c(t)\|^2 \quad \text{cond. neg. type}$$

Thm (Schoenberg) S a set $\psi : S \times S \to \mathbb{R}$ $\psi(s,s) = 0$, $\psi(s,t) = \psi(t,s)$. Then ψ is cond. of neg. type iff $\forall\, a \geq 0$ the map $\exp(-a\psi)$ is of pos. type.

Lemma: If $\varphi : S \times S \to \mathbb{C}$ is of pos. type then

$$(s,t) \mapsto \varphi(s,s) - 2\mathrm{Re}(\varphi(s,t)) + \varphi(t,t)$$

is of cond. neg. type.

Proof: $\varphi(s,t) = \langle \xi_s, \xi_t \rangle$ where $\xi : S \to \mathcal{H}$

then $\varphi(s,s) - 2\mathrm{Re}\,\varphi(s,t) + \varphi(t,t)$

$$= \|\xi_s\|^2 - 2\mathrm{Re}\langle \xi_s, \xi_t \rangle + \|\xi_t\|^2$$
$$= \|\xi_s - \xi_t\|^2$$

Lem: If $\psi : S \times S \to \mathbb{R}$ is of cond. neg. type. and if $s_0 \in S$ is fixed. Then

$$(s,t) \mapsto \underbrace{\psi(s_0, s) - \psi(s,t) + \psi(t, s_0)}$$

is of pos. type.

Proof,

$$\|\eta_{s_0} - \eta_s\|^2 - \|\eta_s - \eta_t\|^2 + \|\eta_t - \eta_{s_0}\|^2$$
$$\equiv 2 \quad \langle \eta_s - \eta_{s_0}, \eta_t - \eta_{t_0} \rangle$$

Proof of Schoenberg's thm:

If $\exp(-a\psi)$ is of pos type.

$\therefore (s,t) \mapsto e^{-a\psi(s,s)} - 2e^{-a\psi(s,t)} + e^{-a\psi(t,t)}$

is of cond. neg type.

$$2\left(\frac{1-e^{-a\psi(s,t)}}{a}\right)$$

$\hookrightarrow$ cond neg type

$\therefore \psi = \lim_{a\to 0} 2\left(\frac{1-e^{-a\psi}}{a}\right)$ is of cond. neg type.

If ψ is of cond. neg type

Fix $s_0 \in S$

$\varphi_{(s,t)} = \psi(s_0,s) - \psi(s,t) + \psi(t,s_0)$

is of pos type.

$\therefore$ $\exp(\varphi)$ is of pos type.

$S^2 \ni (s,t) \mapsto \tilde{\varphi}\ \underbrace{\exp(-\psi(r_0,s))\ \exp(-\psi(t,r_0))}_{= \langle \eta_s, \eta_t \rangle}$

is of pos. type.

$\langle \eta_s, \eta_t \rangle$ when $\eta_s = \exp(-\psi(r_0,s)) \in \mathbb{R} \subseteq \mathcal{H}$

$\therefore \exp(-\psi(s,t))$

$= \exp(\varphi(s,t))\tilde{\varphi}(s,t)$ is of pos. type.

$\therefore \forall\ a \geq 0$

$\exp(-a\psi(s,t))$ is of pos. type. $\square$

Thm: If Γ is a group and $\Sigma < \Gamma$

then TFAE

① (Γ, Σ) has rel (T)

② If $\pi: \Gamma \to U(\mathcal{H})$ is a rep. with $(\xi_n)_n$ almost invariant then

$\| \boxed{\mathrm{Proj}_{\mathcal{H}^\Sigma}}(\xi_n) - \xi_n \| \xrightarrow[n\to\infty]{} 0.$

orthogonal projection onto the space of Σ-invariant vectors.

③ $\varphi_n : \Gamma \to \mathbb{C}$ pos type $\varphi_n \to 1$ pointwise

then $\| \varphi_n|_\Sigma - 1 \|_\infty \xrightarrow[n\to\infty]{} 0$

Thm: If Γ is a countable group and $\Sigma < \Gamma$ then TFAE

1. (Γ, Σ) has rel (T)
2. If $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a rep. with $(\xi_n)_n$ almost invariant then $$\|\boxed{\mathrm{Proj}_{\mathcal{H}^\Sigma}}(\xi_n) - \xi_n\| \xrightarrow[n\to\infty]{} 0.$$
3. $\varphi_n : \Gamma \to \mathbb{C}$ pos type $\varphi_n \to 1$ pointwise then $\|\varphi_n|_\Sigma - 1\|_\infty \xrightarrow[n\to\infty]{} 0$
4. Every cond. neg. type function on Γ is bounded when restricted to Σ
5. Every cocycle (for every rep) of Γ is bounded when restricted to Σ

Pf: $(4 \Leftrightarrow 5)$: ψ cnt. $\longleftrightarrow$ c cocycle

$$\psi(t) = \|c(t)\|^2$$

$(2 \Rightarrow 3)$: φ_n pos type functions $\varphi_n \to 1$ ptwise

$\overset{\text{GNS}}{\rightsquigarrow}$ reps $\pi_n : \Gamma \to \mathcal{U}(\mathcal{H}_n)$ $\xi_n \in \mathcal{H}_n$ st

$$\varphi_n(t) = \langle \pi_n(t)\xi_n, \xi_n\rangle$$

$$\|\xi_n\|^2 = \varphi_n(e) \to 1$$

$$\|\xi_n - \pi_n(t)\xi_n\|^2 = 2(\|\xi_n\|^2 - \mathrm{Re}\langle \pi_n(t)\xi_n, \xi_n\rangle) = 2(\varphi_n(e) - \mathrm{Re}(\varphi_n(t))) \xrightarrow[n\to\infty]{} 0 \quad \forall\, t \in \Gamma.$$

$$\mathcal{H}_n \subset \bigoplus_n \mathcal{H}_n$$

If $\exists\ \eta_n = \mathrm{Proj}_{\mathcal{H}^\Sigma}(\xi_n)$ st

$$\|\eta_n - \xi_n\| \to 0$$

then

$$(\varphi_n(e) - \varphi_n(t)) = \langle \xi_n - \pi_n(t)\xi_n, \xi_n\rangle \le \underbrace{2\|\xi_n\|\,\|\xi_n - \eta_n\|}_{\xrightarrow[n\to\infty]{} 0} + \underbrace{\langle \eta_n - \pi_n(t)\eta_n, \xi_n\rangle}_{=0 \text{ for } t \in \Sigma.}$$

$3 \Rightarrow 2)$: suppose π a rep with

$\{\xi_n\}_n$ almost Γ-invariant.

set $\varphi_n(t) = \langle \pi(t)\xi_n, \xi_n \rangle$ pos type.

$$1 - \varphi_n(t) = \langle \xi_n - \pi(t)\xi_n, \xi_n \rangle$$
$$\leq \|\xi_n - \pi(t)\xi_n\| \xrightarrow[n\to\infty]{} 0$$

$\therefore \sup_{t\in\Sigma} |1 - \varphi_n(t)| \xrightarrow[n\to\infty]{} 0$

For each $n \geq 1$ set $\sup_{t\in\Sigma} |1-\varphi_n(t)| = \varepsilon_n$

consider $K = \overline{co}\{\pi(t)\xi_n \mid t \in \Sigma\} \subset \mathcal{H}$

let η be the ! element of minimal norm in K.

$\pi(t)\eta$ is the ! element of minimal norm in $\pi(t)K = K$ for $t \in \Sigma$.

$\therefore \pi(t)\eta = \eta$ for $t \in \Sigma$

Note for $t \in \Sigma$ we have

$$\|\xi_n - \pi(t)\xi_n\| = 2(1 - \mathrm{Re}(\varphi_n(t)))^{1/2}$$
$$\leq \varepsilon_n^{1/2}$$

$\therefore$ taking convex combinations we have

$$\|\xi_n - \eta\| \leq \varepsilon_n^{1/2} \longrightarrow 0$$

$\boxed{5 \Rightarrow 2)}$ $\boxed{\text{when } \Gamma \text{ is countable}}$ By contraposition Suppose π is a rep $\{\xi_n\}_n \subset \mathcal{H}$ almost invariant,

$$\|\mathrm{Proj}_{\mathcal{H}^\Sigma}(\xi_n) - \xi_n\| \geq c_0 > 0.$$

Enumerate Γ as $\{t_k\}_{k=1}^\infty$ For each n assume $\|\xi_n - \pi(t_k)\xi_n\|^2 < \frac{1}{4^n}$

for $1 \leq k \leq n$.

Define $c : \Gamma \to \bigoplus_{\mathbb{N}} \mathcal{H}$ by

$$c(t) = \bigoplus 2^n(\xi_n - \pi(t)\xi_n)$$

Note that

$$\|C(t_n)\|^2 \le \sum_{k=1}^{n-1} 2^k \|\xi_k - \pi(t_n)\xi_k\|^2 + \frac{1}{2^{n-1}} < \infty$$

$\therefore$ C is well defined.

claim: C is unbounded on Σ

If $\|C(t)\|^2 \le M$

otherwise we have

$$\boxed{\sup_{t\in\Sigma} \|\underline{\xi_n} - \pi(t)\xi_n\|^2} \le \sup_{t\in\Sigma} \frac{\|C(t)\|^2}{2^n} \le \frac{M}{2^n} \to 0$$

by the previous part ~~of~~ $\boxed{(3\Rightarrow 2)}$ the proof

we showed that this would

imply $\|P_{\mathcal{H}^\Sigma}(\xi_n) - \xi_n\| \longrightarrow 0$

a contradiction.

$(1 \Rightarrow 5)$ By contraposition there exists a cocycle $c: \Gamma \to \mathcal{H}$ for some rep π

st $c|_\Sigma$ is unbounded.

Choose $\sigma_n \in \Sigma$ st $\|c(\sigma_n)\| \to \infty$

Note $\|c(t\sigma_n s)\|$

$$= \|\underbrace{c(t)}_{} + \pi(t)\underbrace{c(\sigma_n)}_{} + \underbrace{\pi(t\sigma_n)c(s)}_{}\| \xrightarrow[n\to\infty]{} \infty \quad \text{for each fixed } t, s \in \Gamma$$

By Schoenbergs theorem we have representation $\pi_a: \Gamma \to U(\mathcal{H}_a)$ and ξ_a unit cyclic vectors st

$$\underbrace{\langle \pi_a(t)\xi_a, \xi_a\rangle}_{} = \exp\left(-a\|c(t)\|^2\right)$$

Note: $\langle \underline{\pi_a(\sigma_n)}\, \underline{\pi(s)\xi_a},\, \underline{\pi(t)\xi_a} \rangle \xrightarrow[n\to\infty]{} 0$ $\forall\, a>0$ $t, s \in \Gamma$ fixed

taking spans and closures then shows that

$\forall\, \xi, \eta \in \mathcal{H}_a$ $\langle \pi_a(\sigma_n)\xi, \eta \rangle \longrightarrow 0.$

$\therefore \bigoplus_a \pi_a$ has no non-zero Σ-invariant vector.

But $\|\pi_a(t)\xi_a - \xi_a\|^2$

$$= 2(1 - \langle \pi_a(t)\xi_a, \xi_a \rangle)$$

$$= 2(1 - \exp(-a\|c(t)\|^2))$$

$$\xrightarrow[a\to\infty]{} 0 \quad \text{for each fixed } t$$

$\therefore \{\xi_a\}_{a>0}^{C\oplus\mathcal{H}_a}$ is almost invariant.

Let Γ be a finitely gen group w/ gen set S.

Assume $S = S^{-1}$ and $e \in S$.

If $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a rep. we define the gradient of π to be the operator

$$\nabla_S : \mathcal{H} \to \bigoplus_S \mathcal{H}. \text{ by}$$

$$\nabla_S \xi = \frac{1}{|S|^{1/2}} \bigoplus (\xi - \pi(s)\xi).$$

$$\langle \nabla_S \xi, \bigoplus_{s\in S} \eta_s \rangle$$

$$= \frac{1}{|S|^{1/2}} \sum_{s\in S} \langle \xi - \pi(s)\xi, \eta_s \rangle$$

$$= \frac{1}{|S|^{1/2}} \sum_{s\in S} \langle \xi, \eta_s - \pi(s^{-1})\eta_s \rangle$$

the divergence is the operator

$$\nabla_S^*\left(\bigoplus_{s\in S} \eta_s\right) = \frac{1}{|S|^{1/2}} \sum_{s\in S} \eta_s - \pi(s^{-1})\eta_s.$$

The Laplacian is defined as

$$\nabla_S^* \nabla_S = \Delta_S.$$

$$\Delta_S(\xi) = \frac{1}{|S|^{1/2}} \nabla_S^* \Big(\bigoplus_{s \in S} \xi - \pi(s)\xi \Big)$$

$$= \frac{1}{|S|} \sum_{s \in S} (\xi - \pi(s)\xi) - \pi(s^{-1})(\xi - \pi(s)\xi)$$

$$= 2\Big(\xi - \underbrace{\frac{1}{|S|} \sum_{s \in S} \pi(s)\xi}\Big)$$

Note $\sigma(\Delta_S) \subset [0, \infty)$

Thm: Γ has (T) iff $\exists$ $c > 0$ st $\forall$ reps π

$$\sigma(\Delta_S) \cap (0, c) = \phi.$$

Proof:

Note $\operatorname{Ker}(\Delta_S) \supseteq \mathcal{H}^\Gamma$.

In fact $\operatorname{Ker}(\Delta_S) = \mathcal{H}^\Gamma$

since if $\frac{1}{|S|} \sum_{s \in S} \pi(s)\xi = \xi$

$\Longrightarrow$ $\pi(s)\xi = \xi$ $\forall$ $s \in S$

$\hookrightarrow$ strict convexity of Hilbert space

Note: If $\alpha, \beta \geq 0$ $\alpha + \beta = 1$, then

$$\|\alpha\xi + \beta\eta\|^2 = \alpha^2\|\xi\|^2 + 2\alpha\beta \operatorname{Re}\langle \xi, \eta \rangle + \beta^2 \|\eta\|^2$$

$$= \alpha^2 \|\xi\|^2 + 2\alpha\beta \|\xi\|\|\eta\| + \beta^2\|\eta\|^2$$

$$= (\alpha\|\xi\| + \beta\|\xi\|)^2$$

iff $\xi = \lambda\eta$ for $\lambda > 0$.

Similarly: Δ_S has an approximate kernel iff π has almost invariant vectors. iff we do not have spectral gap.

Recall: If $\pi: \Gamma \to \mathcal{U}(\mathcal{H})$ is a rep,
a 1-cocycle is a map $c: \Gamma \to \mathcal{H}$ st

$$c(st) = c(s) + \pi(s)c(t).$$

We let $Z^1(\Gamma, \pi)$ be the space of cocycles.

A cocycle c is inner if $\exists\ \xi \in \mathcal{H}$ st $c(t) = \xi - \pi(t)\xi \quad \forall\ t \in \Gamma$

We let $B^1(\Gamma, \pi)$ be the space of inner cocycles.

$$H^1(\Gamma, \pi) = Z^1(\Gamma, \pi)/B^1(\Gamma, \pi).$$

Thm: (Delorme-Guichardet) Γ has (T) iff $H^1(\Gamma, \pi) = \{0\}$ for all reps π.

Lem: A cocycle is bounded iff it is inner.

Proof:

($\Leftarrow$) obvious.

($\Rightarrow$) set $X = \{c(t)\}_{t \in \Gamma}$ a bounded set

Let ξ_0 be the Chebyshev center of X,

i.e. ξ_0 is the ! element realizing the infimum of the function

$$\xi \longmapsto \sup_{\eta \in X} \|\xi - \eta\|.$$

Fix $s \in \Gamma$

Note since $c(st) = c(s) + \pi(s)c(t)$

$$\therefore \quad \xi_0 = c(s) + \pi(s)\xi_0 \quad \forall s \in \Gamma$$

$$\therefore \quad c(s) = \xi_0 - \pi(s)\xi_0. \quad \square$$

Lem: $\mathcal{H}$ Hilbert space $X \overset{\neq\phi}{\subset} \mathcal{H}$ bounded

then $\exists!\ \xi_0 \in \mathcal{H}$ st this realizes the infemum of

$$\mathcal{H} \ni \xi \longmapsto \sup_{\eta \in X} \|\xi - \eta\|$$

Proof:

Let $d \geq 0$ denote the infemum.

choose $\xi_n \in \mathcal{H}$ st

$$d_n = \sup_{\eta \in X} \|\xi_n - \eta\| \longrightarrow d.$$

Then

$$\left\| \frac{\xi_n + \xi_m}{2} - \eta \right\|^2 = \frac{1}{2}\|\xi_n - \eta\|^2 + \frac{1}{2}\|\xi_m - \eta\|^2 - \left\|\frac{\xi_n - \xi_m}{2}\right\|^2$$

Taking sup over $\eta \in X$

$$d^2 \leq \sup = \frac{1}{2} d_n^2 + \frac{1}{2} d_m^2 - \left\|\frac{\xi_n - \xi_m}{2}\right\|^2$$

$$\therefore \left\|\frac{\xi_n - \xi_m}{2}\right\|^2 = \frac{1}{2} d_n^2 + \frac{1}{2} d_m^2 - d^2 \xrightarrow[n,m \to \infty]{} 0$$

$\therefore \{\xi_n\}_{n=1}^{\infty}$ is Cauchy and so converges to ξ_0.

This shows existence, but also uniqueness $\square$

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on $Z^1(\Gamma, \pi)$ we consider the top. of pointwise convergence, ie $c_i \longrightarrow c$ iff $c_i(t) \longrightarrow c(t)$ $\forall\ t \in \Gamma$.

we let $\overline{H^1}(\Gamma, \pi) = Z^1(\Gamma, \pi) / \overline{B^1(\Gamma, \pi)}$.

☆ Note: If $\Gamma = \langle S \rangle$ S finite symmetric, then If $\Delta_S = \nabla_S^* \nabla_S$ does not have spectral gap, then take $c_n \in \sigma(\Delta_S)$ st $c_n \longrightarrow 0$ then $\exists$ unit vectors ξ_n st

$$\Delta_S \xi_n \approx c_n \xi_n$$

$\{\|\Delta_S \xi_n - c_n \xi_n\| < c_n / n\}$

ultrafilters on $\mathbb{N}$

An ultrafilter on $\mathbb{N}$ is a point

$$\omega \in \beta\mathbb{N} = \mathrm{Hom}(\ell^\infty\mathbb{N}, \mathbb{C})$$

$$\ell^\infty\mathbb{N} \simeq C(\beta\mathbb{N})$$

Note: If K is cpt Hausdorff

$a : \mathbb{N} \to K$ (continuous) then

$\exists!$ cont extension $\tilde{a} : \beta\mathbb{N} \to K$

we define $\operatorname{Lim}_{n\to\omega} a_n := \tilde{a}(\omega)$

ω is free if it is not in $\mathbb{N}$.

Properties:

1. If $\lim_{n\to\infty} a_n$ exists then

$$\operatorname{Lim}_{n\to\omega} a_n = \lim_{n\to\infty} a_n.$$

2. If $b, a \in \ell^\infty\mathbb{N}$ then

$$\operatorname{Lim}_{n\to\omega} (a_n + b_n) = \operatorname{Lim}_{n\to\infty} a_n + \operatorname{Lim}_{n\to\infty} b_n$$

$$\operatorname{Lim}_{n\to\omega} (a_n b_n) = \left(\operatorname{Lim}_{n\to\omega} a_n\right)\left(\operatorname{Lim}_{n\to\omega} b_n\right).$$

Suppose $\mathcal{H}_n$ are Hilbert spaces, $\omega \in \beta\mathbb{N}$ free ultrafilter.

Consider $\ell^\infty(\mathbb{N}, \mathcal{H}_n)$

$$= \left\{ \xi : \mathbb{N} \to \bigcup_n \mathcal{H}_n \;\middle|\; \xi_n \in \mathcal{H}_n \text{ and } \sup_n \|\xi_n\| < \infty \right\}$$

This is a Banach space.

If $\xi, \eta \in \ell^\infty(\mathbb{N}, \mathcal{H}_n)$

define $\langle \xi, \eta \rangle_\omega = \operatorname{Lim}_{n\to\omega} \langle \xi_n, \eta_n \rangle$

This is a non-neg definite sesquilinear form.

$\ker(\langle , \rangle_\omega) = \{ \xi \in \ell^\infty(\mathbb{N}, \mathcal{H}_n)$ st $\lim_{n\to\omega} \|\xi_n\|^2 = 0 \}$.

This is a closed subspace of $\ell^\infty(\mathbb{N}, \mathcal{H}_n)$.

$$\prod_{n\to\omega} \mathcal{H}_n := \ell^\infty(\mathbb{N}, \mathcal{H}_n) / \ker(\langle , \rangle_\omega)$$

a Hilbert space.

If we have representations $\pi_n : \Gamma \to U(\mathcal{H}_n)$ then we get a rep $\pi^\omega : \Gamma \to U(\prod_{n\to\omega} \mathcal{H}_n)$ by

$$\pi^\omega(t)(\xi_n)_n = (\pi_n(t)\xi_n)_n$$

Γ is a f.g. gp with symmetric gen set S.

Note If $c_i \in Z^1(\Gamma, \pi)$ then $c_i \to c$ iff $c_i(s) \to c(s)$ for $s \in S$.

We consider the embedding $Z^1(\Gamma, \pi) \hookrightarrow \mathcal{H}^{\oplus S}$ by

$$c \longmapsto \bigoplus_{s\in S} c(s)$$

$\therefore Z^1(\Gamma, \pi)$ is a closed subspace of $\mathcal{H}^{\oplus S}$.

$\therefore \overline{H^1}(\Gamma, \pi) \neq \{0\}$ iff $\exists$ some cocycle $c \neq 0$ st $c \perp B^1(\Gamma, \pi)$.

I.e. $\forall\ \xi \in \mathcal{H}$ we have

$$0 = \langle c, \xi - \pi(\cdot)\xi \rangle$$

$$= \sum_{s\in S} \langle c(s), \xi - \pi(s)\xi \rangle$$

$$= \sum_{s\in S} \langle c(s) - \underbrace{\pi(s^{-1})c(s)}_{-c(s^{-1})}, \xi \rangle$$

$$= \langle 2 \sum_{s\in S} c(s), \xi \rangle$$

A cocycle is S-harmonic if $\sum_{s\in S} c(s) = 0$.

$\therefore \overline{H}^1(\Gamma, \pi) \neq \{0\}$ iff $\exists$ a non-zero harmonic cocycle.

Note: If $\Gamma = \langle S \rangle$ S finite symmetric, then If $\Delta_S = \nabla_S^* \nabla_S$ does not have spectral gap, then take $c_n \in \sigma(\Delta_S)$ st $c_n \longrightarrow 0$ then $\exists$ unit vectors ξ_n st

$$\Delta_S \xi_n \approx c_n \xi_n$$

set $\eta_n = \frac{1}{\sqrt{c_n}} \xi_n$

then $\nabla_S \eta_n = \frac{1}{|S|^{1/2}} \bigoplus \eta_n - \pi(s)\eta_n$

$$\|\nabla_S \eta_n\|^2 = \langle \nabla_S \eta_n, \nabla_S \eta_n \rangle$$
$$= \langle \Delta_S \eta_n, \eta_n \rangle$$
$$= \frac{1}{c_n} \langle \Delta \xi_n, \xi_n \rangle$$

$$\|\xi_n\|^2 = 1$$

but $\|\Delta_S \eta_n\|^2 \approx c_n \|\xi_n\|^2 = c_n \longrightarrow 0$

Define $c_n : \Gamma \to \mathcal{H}_n$ by

$$c_n(t) = \eta_n - \pi(t)\eta_n$$

$$\sum_{s \in S} \|c_n(s)\|^2 = \|\nabla \eta_n\|^2 \simeq 1$$

$\therefore$ $\forall\, t \in \Gamma$ $\sup_n \|c_n(t)\| < \infty$

Specifically $\|c_n(t)\| \leq \ell_S(t)$

$$\left\|\frac{1}{|S|}\sum_{s\in S} c_n(s)\right\|^2$$

$$= \left\|\frac{1}{|S|}\sum_{s \in S} \eta_n - \pi(s)\eta_n\right\|^2$$

$$= \|\Delta \eta_n\|^2 \longrightarrow 0.$$

define $c_\omega : \Gamma \to \prod_{n\to\omega} \mathcal{H}_n$ by

$$c_\omega(t) = (c_n(t))_n$$

This is a cocycle.

$$\left\|\frac{1}{|S|}\sum_{s\in S} c_\omega(s)\right\| := \lim_{n\to\omega} \left\|\frac{1}{|S|}\sum_{s\in S} c_n(s)\right\| = 0$$

$\therefore$ c_ω is harmonic.

$$\frac{1}{|S|}\sum_{s\in S} \|c_\omega(s)\|^2 = 1.$$

$\therefore$ c_ω is non-zero.

Thm: If Γ is finitely generated then Γ has (T) iff $\overline{H}^1(\Gamma, \pi) = \{0\}$ for all representation.

T has an approximate kernel if

$\exists\ \xi_n \in \mathcal{H}$ $\|\xi_n\| = 1$ s.t.

$T\xi_n \longrightarrow 0$

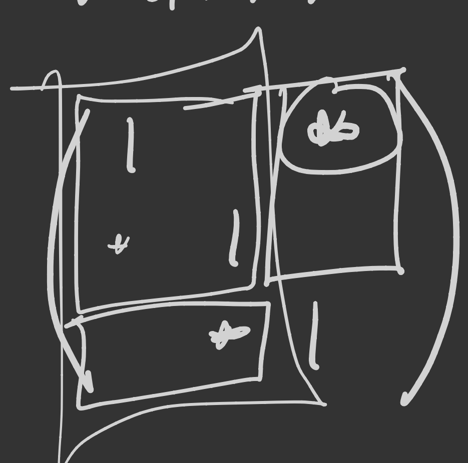
If $\underline{\ker(T) = \{0\}}$ $T = T^*$ then

T has an approximate kernel

iff $0 \in \sigma(T)$

iff 0 is not isolated in $\sigma(T)$.

Elementary matrix:



$\longrightarrow$ one more term that's not zero.

Thm: $SL_3\mathbb{Z}$ has (T)

1st proof:

Fact (Carter–Keller '83) every element of $SL_3\mathbb{Z}$ is a product of at most 48 elementary matrices.

$\therefore$ $SL_3\mathbb{Z}$ is boundedly generated by 6 subgroups each having relative property (T).

If $c : \Gamma \to \mathcal{H}$ is any cocycle then on each subgroup it is bounded say by M

$\therefore$ If $t = t_1 t_2 \cdots t_{48}$ with t_i in these subgroups

then $\|c(t)\| \leq 48M$

$\therefore$ $H^1(\Gamma, \pi) = \{0\} \Longrightarrow \Gamma = SL_3\mathbb{Z}$ has (T).

Set $H = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} < SL_3\mathbb{Z}$

$$\mathbb{Z}^2 \simeq \Sigma_1 = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\} < SL_3\mathbb{Z}$$

$$\Sigma_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & 1 \end{pmatrix} \right\} \simeq \mathbb{Z}^2$$

$\langle H, \Sigma_i \rangle \simeq SL_2\mathbb{Z} \ltimes \mathbb{Z}^2$

& hence Σ_i has rel Property (T) in $SL_3\mathbb{Z}$.

Also $SL_3\mathbb{Z} = \langle \Sigma_1, \Sigma_2 \rangle$

Lemma: $SL_3\mathbb{Z} = \Sigma_1 \Sigma_2 \Sigma_1 \Sigma_2 H$

Proof:

Fix $\underbrace{\gamma = \begin{pmatrix} * & * & * \\ * & * & * \\ x & y & z \end{pmatrix}}_{\gcd(x,y,z)=1} \in SL_3\mathbb{Z}$

The Chinese Remainder Theorem $\Longrightarrow$

$\exists\ m \in \mathbb{Z}$ st. $\overbrace{x + mz} \equiv 1 \pmod{p}$ for any prime $p \mid y$ st $p \nmid z$.

$\therefore \gcd(\underline{x+mz}, y) = 1$

$$\begin{pmatrix} * & * & * \\ * & * & * \\ x & y & z \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & 0 & 1 \end{pmatrix}}_{\in \Sigma_2} = \begin{pmatrix} * & * & * \\ * & * & * \\ x' & y & z \end{pmatrix}$$

$\gcd(x', y) = 1$

$\therefore \exists\ s, t \in \mathbb{Z}$ st $sx' + ty + z = 1$

$$\therefore \begin{pmatrix} * & * & * \\ * & * & * \\ x' & y & z \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}}_{\in \Sigma_1} = \begin{pmatrix} * & * & u \\ * & * & v \\ x' & y & 1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & -u \\ 0 & 1 & -v \\ 0 & 0 & 1 \end{pmatrix}}_{\Sigma_1} \begin{pmatrix} * & * & u \\ * & * & v \\ x' & y & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x' & -y & 1 \end{pmatrix}}_{\Sigma_2} = \underbrace{\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{} \in H$$

$\Sigma_1 \cdot \gamma \cdot \Sigma_2 \Sigma_1 \Sigma_2 \cap H = \phi$

$\therefore \gamma \in \Sigma_1 H \Sigma_2 \Sigma_1 \Sigma_2 = \Sigma_1 \Sigma_2 \Sigma_1 \Sigma_2 H$

$\square$

2nd proof that $SL_3\mathbb{Z}$ has (T): (Shalom).

Thm: If Γ is a fg gp $H, \Sigma_1, \Sigma_2 < \Gamma$ st.

① (Γ, Σ_i) has rel (T) $i=1,2$

② H normalizes Σ_1 and Σ_2

③ $\Gamma = \langle \Sigma_1, \Sigma_2 \rangle$

④ Γ is boundedly generated by $\{H, \Sigma_1, \Sigma_2\}$

then Γ has (T).

Proof:

Fix a rep. $\pi: \Gamma \to \mathcal{U}(\mathcal{H})$ and we will show $\overline{H^1}(\Gamma, \pi) = \{0\}$.

Note if $c: \Gamma \to \mathcal{H}$ is a cocycle & P is the projection onto $\mathcal{H}^\Gamma$ then Pc is a homomorphism into $P(\mathcal{H})$.

Since Σ_i have rel (T) we have that

$$Pc|_{\Sigma_i} \equiv 0 \Longrightarrow Pc|_{\underbrace{\langle \Sigma_1, \Sigma_2 \rangle}_{=\Gamma}} \equiv 0$$

$\therefore$ we suppose $\mathcal{H}^\Gamma \neq \{0\}$.

Fix $c: \Gamma \to \mathcal{H}$ 1-cocycle

Since Σ_i has rel (T) in Γ

$\exists \; \xi_i^c \in \mathcal{H}$ st $c(g) = \xi_i^c - \pi(g)\xi_i^c$

$\forall \; g \in \Sigma_i$ (replace ξ_i^c with $P_i^\perp \xi_i^c$)

Moreover, there is a unique ξ_i^c st for $P_i = \mathrm{Proj}_{\mathcal{H}^{\Sigma_i}}$ we have $P_i \xi_i^c = 0$.

we embedd $Z^1(\Gamma, \pi) \hookrightarrow \mathcal{H} \oplus \mathcal{H}$ by

$$c \longmapsto \xi_1^c \oplus \xi_2^c$$

This gives the topology on $Z^1(\Gamma, \pi)$

Fix $c \in Z^1(\Gamma, \pi) \ominus B^1(\Gamma, \pi)$.

Note: If $\xi \in \mathcal{H}$ and $c_\xi(g) = \xi - \pi(g)\xi$

$$\therefore \quad \xi_1 = P_1^\perp \xi \qquad \xi_2 = P_2^\perp \xi$$

Since $c \perp B^1(\Gamma, \pi)$ we have

$$0 = \langle c, c_\xi \rangle = \langle \xi_1^c, \cancel{P_1^\perp} \xi \rangle + \langle \xi_2^c, \cancel{P_2^\perp} \xi \rangle$$

$$= \langle \xi_1^c + \xi_2^c, \xi \rangle$$

$$\therefore \quad \xi_1^c + \xi_2^c = 0$$

If $h \in H$, $t \in \Sigma_i$, Then

$$\xi_i^c - \pi(hth^{-1})\xi_i^c = c(hth^{-1})$$

$$= c(h) + \pi(h)c(t) + \pi(ht)c(h^{-1})$$

$$= c(h) + \pi(h)(\xi_i^c - \pi(t)\xi_i^c) - \pi(hth^{-1})c(h)$$

$$= (c(h) + \pi(h)\xi_i^c) - \pi(hth^{-1})(\pi(h)\xi_i^c + c(h))$$

$$\therefore \quad \xi_i^c - c(h) - \pi(h)\xi_i^c = \pi(hth^{-1})(\xi_i^c - c(h) - \pi(h)\xi_i^c) \quad \forall t \in \Sigma_i$$

$$\therefore \quad P_1^\perp(c(h)) = P_1^\perp(\xi_1^c - \pi(h)\xi_1^c)$$

$$= \xi_1^c - \pi(h)\xi_1^c$$

$$= -(\xi_2^c - \pi(h)\xi_2^c)$$

$$= -P_2^\perp(c(h))$$

$$\therefore \quad (P_1^\perp + P_2^\perp)(c(h)) = 0$$

since $P_1^\perp + P_2^\perp \geq P_i^\perp \geq 0$

$$\Longrightarrow \quad P_i^\perp(c(h)) = 0$$

$$\therefore \quad c(h) \in \mathcal{H}^{\Sigma_1} \cap \mathcal{H}^{\Sigma_2} = \mathcal{H}^{\langle \Sigma_1, \Sigma_2 \rangle} = \{0\}$$

$\forall h \in H$

Since Γ is boundedly gen by $\{H, \Sigma_1, \Sigma_2\}$ it follows that c is bounded & hence inner, hence 0 $\square$

$c : \Gamma \to \mathcal{H}$ cocycle

$\rightsquigarrow$ affine isometric action on $\mathcal{H}$

by $\alpha_t \xi = \pi(t)\xi + c(t)$.

If $c_i \to c$

$\Sigma < \Gamma$ with rel (T)

$\therefore$ $c_i|_\Sigma = \xi_i - \pi(\cdot)\xi_i$

$P_{\mathcal{H}^\Sigma} \xi_i = 0$

claim: $\|\xi - \xi_i\| \to 0$

pf: we may assume $c \equiv 0$

$\xi_i^{\Sigma_1}$ $\xi_i^{\Sigma_2}$

$c_i : \Gamma \to \mathcal{H}$ st $c_i|_{\Sigma_i} = \xi_i^{\Sigma_i} - \pi(\cdot)\xi_i^{\Sigma_i}$

Suppose $\underbrace{\xi_i^{\Sigma_1} \to \xi^{\Sigma_1}}$ $\underbrace{\xi_i^{\Sigma_2} \to \xi^{\Sigma_2}}$

claim: $\exists$ $c : \Gamma \to \mathcal{H}$ cocycle st

$\underbrace{c|_{\Sigma_{K_i}}} = \xi^{\Sigma_K} - \pi(\cdot)\xi^{\Sigma_K}$.

Check: For each $t \in \Gamma$ the sequence

$c_i(t)$ is Cauchy

Since $\Gamma = \langle \Sigma_1, \Sigma_2 \rangle$.

$\therefore$ $\exists$ a cocycle $\tilde{c}$ st $\tilde{c}(t) = \lim_{i\to\infty} c_i(t)$.

$\underline{\tilde{c}(\sigma_1)} = \lim \underline{c_i(\sigma_1)} = \lim_i \xi_i^{\Sigma_1} - \pi(\sigma_1)\xi_i^{\Sigma_1}$

$= \underline{\xi^{\Sigma_1} - \pi(\sigma_1)\xi^{\Sigma_1}}$

$\boxed{\text{the top on } \mathcal{H} \oplus \mathcal{H} \Longrightarrow \text{pointwise convergence in } Z^1(\Gamma, \pi).}$

Open problem: Do some infinite Burnside groups have property (T)?

Haagerup's property

Γ an infinite group.

Def: A unitary rep $\pi:\Gamma\to\mathcal{U}(\mathcal{H})$ is **mixing** if $\forall\ \xi,\eta\in\mathcal{H}$ the map $\Gamma\ni t\longmapsto\langle\pi(t)\xi,\eta\rangle$ is in $c_0(\Gamma)$ $\left(=\{f:\Gamma\to\mathbb{C} \text{ st } \forall\ \varepsilon>0\ \{t\in\Gamma \mid |f(t)|>\varepsilon\} \text{ is finite}\}\right)$

Ex: If Γ is infinite then $\lambda:\Gamma\to\mathcal{U}(\ell^2\Gamma)$ is mixing.

Ex: If $\varphi:\Gamma\to\mathbb{C}$ is of pos. type and if $\varphi\in c_0(\Gamma)$ then the GNS rep π,ξ is mixing.

Note $\forall\ x,y\in\Gamma$

$$t\longmapsto\langle\pi(t)\pi(x)\xi,\pi(y)\xi\rangle = \varphi(y^{-1}tx)\xrightarrow[t\to\infty]{}0$$

Thm/Def: Γ countably infinite has the Haagerup property if the following equivalent conditions hold:

1. There is a mixing rep with a.i. vectors.
2. There are c_0 pos. type functions φ_i st $\varphi_i\longrightarrow 1$ pointwise.
3. There is some rep. π and a **proper** cocycle c. $\left(\forall\ C>0\ \{t\in\Gamma \mid \|c(t)\|<C\} \text{ is finite}\right)$
4. Same as 3 but the rep π is mixing.
5. There is a proper cond. neg. type function.

Proof: "Same as for property (T)."

Note ψ proper cond. neg. type

$\Rightarrow \exp(-t\psi)$ c_0 and pos. type.

If $\pi:\Gamma\to\mathcal{U}(\mathcal{H})$ mixing rep with a.i. vectors

take ξ_n a.i., Enumerate Γ as $\{t_k\}$ choose a subsequence of $\{\xi_n\}$

st $\|\pi(t_k)\xi_n-\xi_n\|<\frac{1}{4^n}$ for $k\le n$.

consider $c:\Gamma\to\mathcal{H}^{\oplus\infty}$ by

$$c(t)=\bigoplus 2^n(\xi_n-\pi(t)\xi_n)$$

$$\lim_{t\to\infty}\|c(t)\|^2 \ge \lim_{t\to\infty}(2^n)^2\underbrace{\|\xi_n-\pi(t)\xi_n\|^2}_{(2\|\xi_n\|^2-2\mathrm{Re}\langle\pi(t)\xi_n,\xi_n\rangle)}$$

$$=2^{2n}\cdot 2$$

$\therefore$ c is a proper cocycle.

Amenable $\Longrightarrow$ Haagerup

Haagerup + infinite $\Longrightarrow$ not property (T).

$\Longrightarrow$ $\boxed{\text{no infinite subgroup has relative (T)}}$

If π is mixing and $\Sigma<\Gamma$ has an invariant vector then $|\Sigma|<\infty$.

eg $SL_3\mathbb{Z}\longrightarrow(T)$

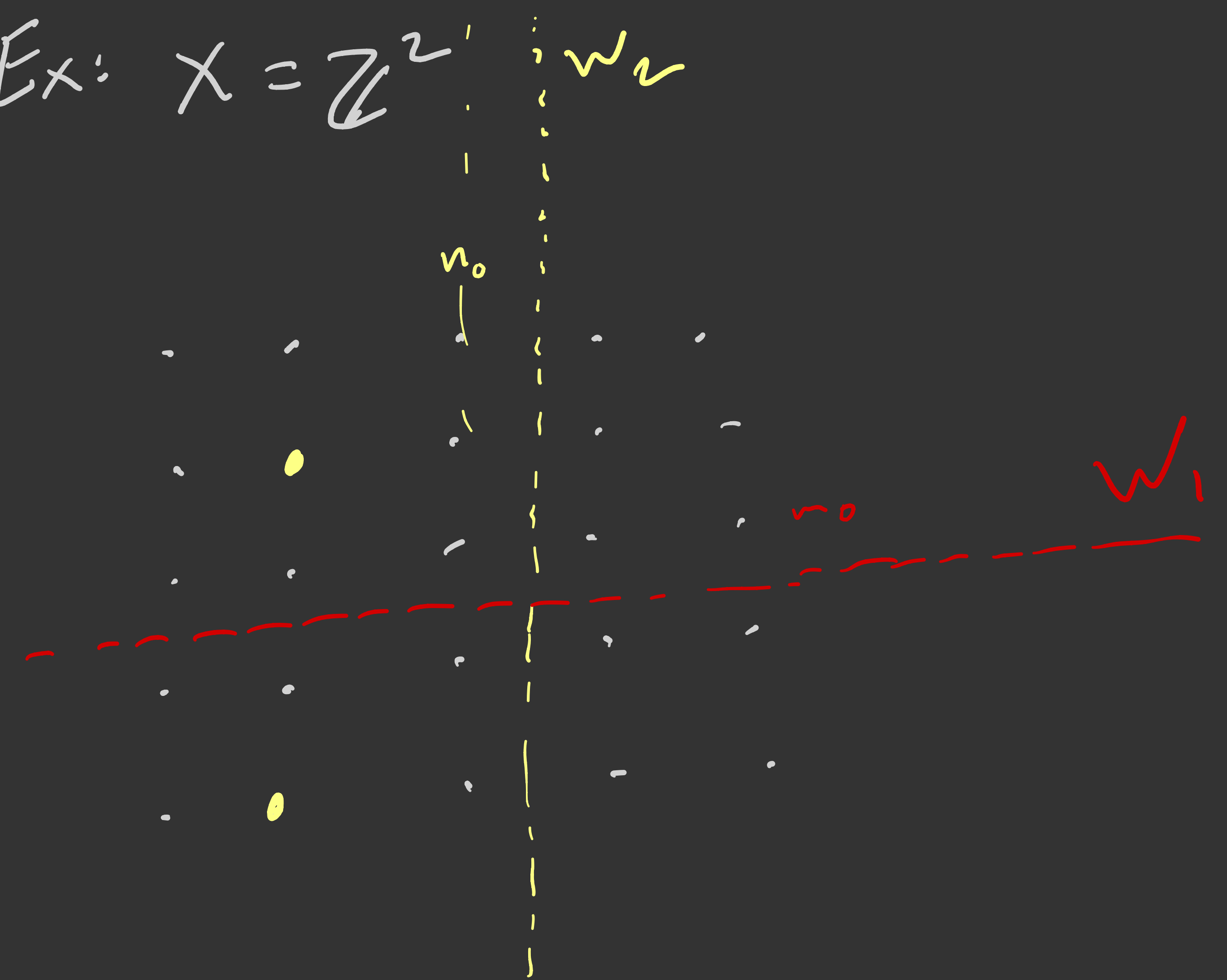
$SL_2\mathbb{Z}\ltimes\mathbb{Z}^2\longrightarrow$ not Haagerup, not (T)

Def: A space X with walls is a set together with partitions into two disjoint non-empty subsets $\mathcal{W}=\{\underbrace{\{H,H^c\}}_{\text{walls}}$ Half-spaces$\}$

st. For each $x\ne y\in X$ there are finitely many walls $\{H,H^c\}$ in $\mathcal{W}$ that separate x and y,

ie st $x\in H$, $y\in H^c$ or $x\in H^c$, $y\in H$.

Ex: $X = \mathbb{Z}^2$



A half space will be

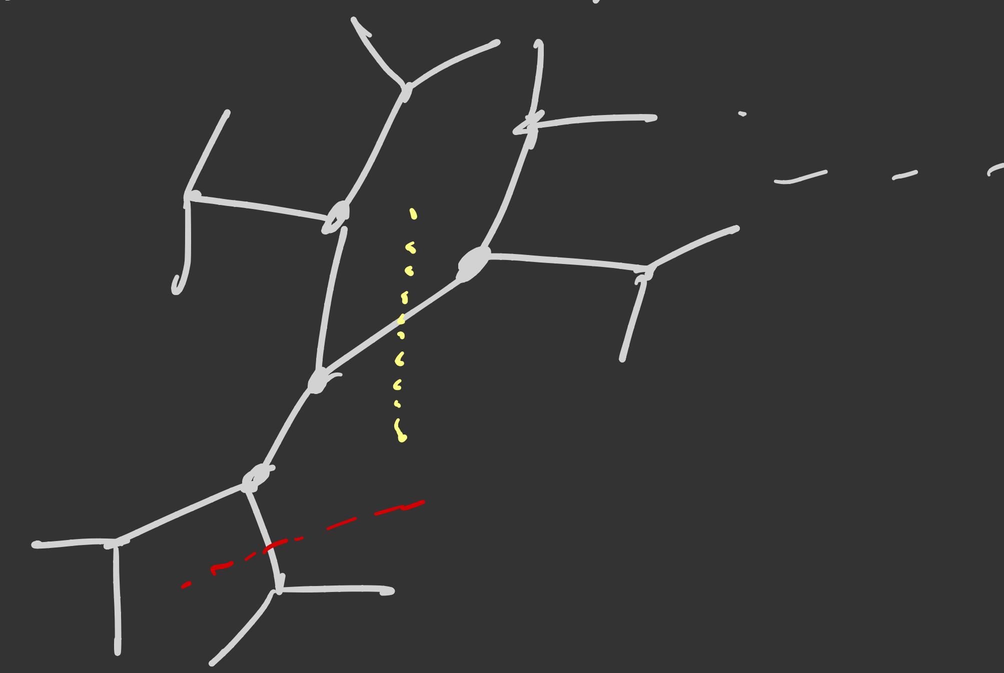
$$H = \{(n,m) \mid n \le n_0\}$$

or $H = \{(n,m) \mid m \le m_0\}$

or their complements.

Wall will be the corresponding partitions.

Ex: T a simplicial tree. $\boxed{\text{graph with edges}}$



For each edge e we let the two connected components of T after removing e be a wall.

Thm: If $(X, \mathcal{W})$ is a space with walls and if $d_{\mathcal{W}}(x,y)$ denotes the number of walls separating x and y, then $d_{\mathcal{W}}$ is a kernel of negative type.

Proof: let $\mathcal{H}$ denote the collection of Half-spaces.

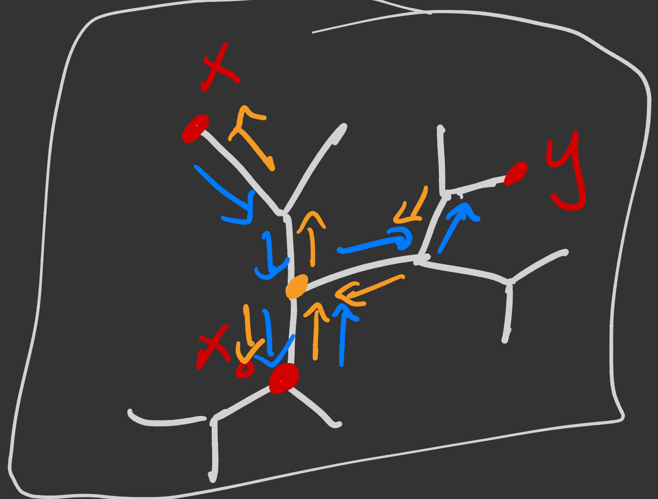
we define $\xi : X \to \ell^2(\mathcal{H})$

by Fixing $x_0 \in X$ and setting

$$\xi_x = 1_{\{H \in \mathcal{H} \mid x \in H \;\; x_0 \notin H\}}$$
$$- 1_{\{H \in \mathcal{H} \mid x \notin H \;\; x_0 \in H\}}.$$

$$(\xi_x - \xi_y) = 1_{\{H \in \mathcal{H} \mid x \in H \;\; y \notin H\}}$$
$$- 1_{\{H \in \mathcal{H} \mid y \in H \;\; x \notin H\}}.$$

$$\|\xi_x - \xi_y\|^2 = 2\, d_{\mathcal{W}}(x, y).$$



$$\|\xi_x - \xi_y\|^2 = 4 + 4$$

$\forall N > 0$
$\forall x \in X$ $\{t \in \Gamma \mid d_{\mathcal{W}}(tx, x) \le N\}$ is finite

Cor: If Γ acts Properly on a space with walls,

$\lim_{t \to \infty} d_{\mathcal{W}}(tx, x) = \infty$

Γ has the Haagerup property.

specifically, we get a cond. neg. type function on Γ by

$$\Gamma \ni t \mapsto d_{\mathcal{W}}(t \cdot x_0, x_0)$$

for any fixed x_0.

Ex: $\mathbb{F}_n$ $n \ge 1$ have Haagerup's property.

they act on trees.

$\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. [In fact. any free product of Haagerup groups has the Haagerup prop.

Ex: $\underbrace{(\mathbb{Z}/4\mathbb{Z}) *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z})}$ has Haagerup's property.

$\mathbb{Z} \curvearrowright SL_2(\mathbb{Z})$

claim: $\underline{PSL_2(\mathbb{Z})} \simeq \underline{\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}}$.

$$\| \quad \langle \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{= s}, \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}}_{= t} \rangle$$

Note: $st = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad ts = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

$\underline{s^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}$ $\quad \underline{t^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}$ $\quad t^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$

claim: There are no relations between s and t.

Suppose w is some non-trivial word in s, t.

By conjugating by t or perhaps t^2 we may assume that this word begins and ends with a power of t.

$PSL_2\mathbb{Z} \curvearrowright \mathbb{R} \cup \{\infty\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}$$

$s \cdot x = \frac{-1}{x}$ $\quad t \cdot x = \frac{-1}{x+1}$ $\quad t^2 \cdot x = -x-1$

$s \cdot (-\infty, 0) = (0, \infty)$

$t \cdot (0, \infty) \subset (-\infty, 0)$

$t^2 \cdot (0, \infty) \subset (-\infty, 0)$

$\therefore$ $w \cdot \underbrace{(0, \infty)} \subset (-\infty, 0)$

$\therefore$ $w \neq e$.

Def: A group Γ has a proper wall structure, if there is a family of walls $\mathcal{W}$ on Γ st $(\Gamma, \mathcal{W})$ is a space with walls st. the action of Γ on itself by left mult. is a proper action on a space with walls.

Def: If Λ and Σ are groups, the wreath product of Σ with Λ is the group

$$\Gamma = \Lambda \ltimes \bigoplus_{\Lambda} \Sigma =: \Sigma \wr \Lambda$$

where Λ acts on $\boxed{\bigoplus_{\Lambda} \Sigma}$ by permuting the entries via left mult.

Thm: If Σ is finite and Λ has a proper wall structure, then $\Sigma \wr \Lambda$ has a proper wall structure. In particular, this will have Haagerup's property.

Thm: (Cornulier, Tessera, Valette): If Σ and Λ have Haagerup's property then so does $\Sigma \wr \Lambda$.

Thm: If Σ is finite and Λ has a proper wall structure, then $\Gamma = \Sigma \wr \Lambda$ has a proper wall structure. In particular, this will have Haagerup's property.

Proof:

Let $\mathcal{W}_\Lambda$ be a proper wall structure for Λ. Let $\mathcal{H}$ be the collection of half-spaces. For each $H \in \mathcal{H}$ and $\mu : H^c \to \Sigma$ finitely supported, set

$$E(H,\mu) := \{ \underbrace{x\,s}_{} \in \Gamma \mid \underbrace{s \in H}_{} \text{ and } \underbrace{x|_{H^c} = \mu}_{} \}$$

set

$$\mathcal{W} = \left\{ \{E(H,\mu), E(H,\mu)^c\} \;\middle|\; \begin{array}{l} H \in \mathcal{H} \\ \mu : H^c \to \Sigma \\ \text{finitely supported} \end{array} \right\}.$$

If $xs, yt \in \Gamma$

$xs \in \underline{E(H,\mu)}$ means $s \in H$ and $\underline{x|_{H^c} = \mu}$.

$yt \notin \underline{E(H,\mu)}$ means either $t \notin H$ or $\underline{y|_{H^c} \neq \mu}$.

$$\therefore H^c \cap \left(\{t\} \cup \mathrm{supp}(x^{-1}y) \right) \neq \emptyset$$

$\therefore \{H, H^c\}$ separate s from some element $\underline{\{t\} \cup \mathrm{supp}(x^{-1}y)} \to$ finite

$\therefore$ there are only finitely many H's in $\mathcal{H}$ st $E(H,\mu)$ separates $\underline{xs}$ from yt.

Since μ is determined by $x|_{H^c}$

$\therefore$ only finitely many $E(H,\mu)$'s separate xs from yt.

$\therefore \underline{(\Gamma, \mathcal{W})}$ is a space with walls.

If $t \in \Lambda$ then

$$t E(H, \mu) = E(tH, t\mu)$$

If $x \in \underbrace{\oplus \sum}_{\Lambda}$ then

$$x E(H, \mu) = E(H, x 1_{H^c} \cdot \mu)$$

we need to show that the action is proper, i.e.,

$$\lim_{\gamma \to \infty} d_{\mathcal{Y}}(\gamma \cdot 1, 1) = \infty.$$

Fix $N \geq 1$ suppose $d_{\mathcal{Y}}(xs, 1) \leq N$,

then $\{s\} \cup \operatorname{supp}(x) \subset \underbrace{\{t \in \Lambda \mid d_{\mathcal{Y}_\Lambda}(t, 1) \leq N\}}_{\text{finite}}$

$\therefore \{xs \in \Gamma \mid d_{\mathcal{Y}}(xs, e) \leq N\}$ is finite $\square$

Def: If A and B are C^*-algebras and $\phi: A \to B$ is linear then ϕ is **positive** if $\phi(x^* x) \geq 0 \ \forall \ x \in A$.

ϕ is **completely positive** if $\phi^{(n)}: M_n(A) \longrightarrow M_n(B)$ is positive $\forall n \geq 1$

where $\phi^{(n)}((a_{ij})_{ij}) = (\phi(a_{ij}))_{ij}$.

Ex: • If $\pi: A \to B$ is a $*$-homomorphism, then π is c.p. (completely positive)

• If $x \in A$ then $\phi_x(a) = x^* a x$ is c.p.

Thm: (Stinespring dilation theorem)

If A is a unital C^*-alg.

$\phi : A \to \underline{B(\mathcal{H})}$ is UCP. (unital CP)

then $\exists$ a Hilbert space $\mathcal{K}$, an

isometry $V: \mathcal{H} \to \mathcal{K}$ and a

$*$-representation $\pi : A \to B(\mathcal{K})$ st

$$\phi(a) = V^* \pi(a) V \quad \forall\, a \in A.$$

proof:

on $\underbrace{A \otimes_{alg} \mathcal{H}}$, we define an

inner-product by

$$\langle a \otimes \xi, b \otimes \eta \rangle = \langle \phi(b^* a)\xi, \eta \rangle$$

Then $\langle \sum a_i \otimes \xi_i, \sum a_j \otimes \xi_j \rangle$

$$= \sum_{i,j=1}^{n} \langle \phi(a_j^* a_i)\xi_i, \xi_j \rangle$$

$\in M_{1,n}(A)$

$$= \left\langle \phi^{(n)}\Big(\underbrace{(a_1\ a_2 \cdots a_n)^* (a_1\ a_2 \cdots a_n)}_{\geq 0} \Big) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \right\rangle \geq 0$$

$$\mathcal{K} = \overline{A \otimes_{alg} \mathcal{H} / \ker \langle\, , \rangle} \quad \text{Hilbert space}$$

$V: \mathcal{H} \to \mathcal{K} \qquad V\xi = 1 \otimes \xi$

$\pi(a)(b \otimes \xi) = ab \otimes \xi$

$$\phi(a) = V^* \pi(a) V. \qquad \square$$

Remark: (Arveson) A a unital C^*-alg,

$\phi: A \to \mathcal{B}(\mathcal{H})$ u.c.p., then $\exists$ Hilbert space $\mathcal{K}$

isometry $V: \mathcal{H} \to \mathcal{K}$, a $*$-rep

$\pi: A \to \mathcal{B}(\mathcal{K})$, and a normal $*$-rep

$\rho: \phi(A)' \to \pi(A)' \cap \mathcal{B}(\mathcal{K})$ s.t.

$\phi(a) = V^* \pi(a) V \quad \forall \; a \in A$,

and $\rho(T)V = VT \quad \forall \; T \in \phi(A)'$.

---

we define $\rho: \phi(A)' \to \pi(A)' \cap \mathcal{B}(\mathcal{K})$

by $\rho(T)(a \otimes \xi) = a \otimes T\xi$

$\langle \rho(T)\pi(x)(a \otimes \xi), b \otimes \eta \rangle$

$= \langle \phi(b^* x a) T\xi, \eta \rangle$

$= \langle \phi(b^* x a)\xi, T^* \eta \rangle$

$= \langle a \otimes \xi, \rho(T^*) x^* b \otimes \xi \rangle$

$= \langle \pi(x) \rho(T)(a \otimes \xi), b \otimes \xi \rangle$

$\therefore \rho(T) \in \pi(A)' \cap \mathcal{B}(\mathcal{K})$

$\rho(T) V \xi = \rho(T)(1 \otimes \xi) = 1 \otimes T\xi = VT\xi$.

$\therefore \rho(T) V = VT$.

If S is a set and $K: S \times S \to \mathbb{C}$,

then the Schur multiplier, is

$m_K: \mathcal{B}(\ell^2 S) \to \mathcal{B}(\ell^2 S)$

given by $m_K([x_{s,t}]_{s,t})$

$$= \underbrace{\left[K(s,t) \, x_{s,t} \right]_{s,t}}$$

if this is well defined.

S a set $\quad K : S \times S \to \mathbb{C}$, $\overline{K(s,s) = 1 \quad \forall s \in S}$.

Thm: K is of positive type iff

$m_K : B(\ell^2 S) \to B(\ell^2 S)$ is ucp.

Proof:

$(\Longrightarrow)$ K pos. type $\Rightarrow \exists$ a Hilbert space $\mathcal{H}$ and $\xi : S \to \mathcal{H}$ st

$$K(s,t) = \langle \xi_s , \xi_t \rangle .$$

Define $V_\xi : \ell^2 S \to \ell^2 S \bar{\otimes} \mathcal{H}$ by

$$V_\xi(\delta_s) = \underline{\delta_s \otimes \xi_s}$$

V_ξ is then an isometry

claim:

$$m_K(x) = \underline{V_\xi^* (x \otimes 1) V_\xi} \quad \text{ucp}$$

indeed $\langle \underline{V_\xi^*(x \otimes 1) V_\xi} \delta_t , \delta_s \rangle$

$$= \langle (x \otimes 1)\, \delta_t \otimes \xi_t , \delta_s \otimes \xi_s \rangle$$

$$= \langle \xi_t , \xi_s \rangle \langle x \delta_t , \delta_s \rangle$$

$$= \langle \underline{K(t,s) x} \, \delta_t , \delta_s \rangle$$

$$= \langle m_K(x) \delta_t , \delta_s \rangle .$$

$(\Longleftarrow)$ If $E \subset S$ is a finite subset

$$\underline{[K(t,s)]_{t,s}} = [\underline{m_K(1_{E \times E})}] \geq 0$$

$\therefore$ K is of positive type. $\square$

Note:

$$\begin{array}{c} \\ E\{ \end{array} \overset{T^*}{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}} \overset{T}{\underline{\begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}}} = 1_{E \times E} \geq 0 .$$

$\in M_{E,\{\cdot\}}(\mathbb{C})$

$\cong B(\mathbb{C}, \ell^2 E)$

If Γ is a group and $\varphi:\Gamma\to\mathbb{C}$ is bounded, then the multiplier is $m_\varphi: C_r^*\Gamma\to C_r^*\Gamma$ s.t.

$$m_\varphi(\lambda_t)=\varphi(t)\lambda_t \quad \forall\ t\in\Gamma.$$

Thm: $\varphi:\Gamma\to\mathbb{C}$ bounded $\varphi(e)=1$. TFAE

1. φ is of positive type
2. $m_\varphi: \underline{C_r^*\Gamma}\to C_r^*\Gamma$ is ucp.
3. $m_\varphi: L\Gamma\to L\Gamma$ is ucp, normal

Recall: $L\Gamma := \lambda(\Gamma)''\subset B(\ell^2\Gamma)$

$$\lambda(\Gamma)'' = \overline{\mathrm{sp}\,\lambda(\Gamma)}^{WOT} = \overline{\mathrm{sp}\,\lambda(\Gamma)}^{SOT}$$

$R\Gamma := \rho(\Gamma)''$

Lemma: $L\Gamma=\rho(\Gamma)'$ and $R\Gamma=\lambda(\Gamma)'$

Proof:

Note that $L\Gamma\subset\rho(\Gamma)'$ is obvious since $\lambda_t\in\underline{\rho(\Gamma)'}$ and

$$L\Gamma=\overline{\mathrm{sp}\,\lambda(\Gamma)}^{WOT}.$$

we need to show that if $T\in\rho(\Gamma)'$ and if $S\in\lambda(\Gamma)'$ then $ST=TS$.

Suppose $T\delta_e=\sum_{t\in\Gamma}\underline{\alpha_t\delta_t}$

then $\langle\delta_s,\underline{T^*\delta_e}\rangle=\langle T\underline{\rho_{s^{-1}}}\delta_e,\delta_e\rangle$

$$=\langle T\delta_e,\delta_{s^{-1}}\rangle=\underline{\alpha_{s^{-1}}}$$

$$\therefore\ T^*\delta_e=\sum_{t\in\Gamma}\overline{\alpha_{t^{-1}}}\,\delta_t.$$

Similarly. If $S\delta_e=\sum_{s\in\Gamma}\beta_s\delta_s$ then $S^*\delta_e=\sum_{s\in\Gamma}\overline{\beta_{s^{-1}}}\delta_s$.

$$\therefore \langle TS\,\delta_e, \delta_e\rangle = \langle S\,\delta_e, T^*\delta_e\rangle$$

$$= \sum_{t\in\Gamma} \beta_t \alpha_{t^{-1}} = \sum_{t\in\Gamma} \beta_{t^{-1}}\alpha_t$$

$$= \langle T\,\delta_e, S^*\delta_e\rangle = \langle ST\,\delta_e, \delta_e\rangle$$

If $x, y \in \Gamma$

$$\langle TS\,\delta_x, \delta_y\rangle = \langle TS\lambda_x\,\delta_e, \rho_{y^{-1}}\delta_e\rangle$$

$$= \langle (T\lambda_x)(\rho_y S)\,\delta_e, \delta_e\rangle$$

$$= \langle \rho_y S T \lambda_x\,\delta_e, \delta_e\rangle$$

$$= \langle ST\,\delta_x, \delta_y\rangle$$

$\therefore\ TS = ST$

Hence $\rho(\Gamma)' \subset \lambda(\Gamma)'' = L\Gamma$. $\square$

pf of thm:

(1) $\Rightarrow$ (2) $\varphi : \Gamma \to \mathbb{C}$ is of pos. type

$\therefore$ the kernel $\Gamma\times\Gamma \ni (s,t) \longmapsto \varphi(s t^{-1})$

is of positive type.

$\therefore$ the Schur multiplier $m_\varphi : B(\ell^2\Gamma) \to B(\ell^2\Gamma)$

is ucp.

$$\langle m_\varphi(\lambda_t)\,\delta_x, \delta_y\rangle$$

$$= \varphi(y x^{-1}) \langle \lambda_t\,\delta_x, \delta_y\rangle$$

$$= \langle \varphi(t)\lambda_t\,\delta_x, \delta_y\rangle$$

$$\therefore\ m_\varphi(\lambda_t) = \varphi(t)\lambda_t.$$

$\therefore\ m_\varphi(C^*_r\Gamma) \subset C^*_r\Gamma$ and m_φ is ucp.

Note: If $T \in \rho(\Gamma)'$ then

$$\langle (m_\varphi(T)\rho_s - \rho_s m_\varphi(T))\delta_x, \delta_y \rangle$$

$$= \langle m_\varphi(T)\delta_{xs^{-1}}, \delta_y \rangle - \langle m_\varphi(T)\delta_x, \delta_{ys} \rangle$$

$$= \varphi(ysx^{-1})\big(\langle T\delta_{xs^{-1}}, \delta_y \rangle$$

$$- \langle T\delta_x, \delta_{ys} \rangle\big)$$

$= 0$ since $T \in \rho(\Gamma)'$

$\therefore$ $m_\varphi : L\Gamma \longrightarrow L\Gamma$ ucp

Conversely if $m_\varphi : C^*_r\Gamma \to C^*_r\Gamma \subset B(\ell^2\Gamma)$ is ucp

By Stinespring $\exists$ $\mathcal{H}$ a Hilbert space

$V : \ell^2\Gamma \to \mathcal{H}$ isometry

$\pi : C^*_r\Gamma \longrightarrow B(\mathcal{H})$ $*$-rep

$\tilde{\rho} : R\Gamma \longrightarrow B(\mathcal{H})$ st

$m_\varphi(a) = V^*\pi(a)V$ and

$\underbrace{\tilde{\rho}(x)V = Vx}$, $x \in R\Gamma$

we consider the representation of Γ on $\mathcal{H}$ given by

$$t \longmapsto \pi(\lambda_t)\tilde{\rho}(\rho_t)$$

$$\langle \pi(\lambda_t)\underbrace{\tilde{\rho}(\rho_t)V}\delta_e, \underbrace{V\delta_e} \rangle$$

$$= \langle \pi(\lambda_t)V\delta_{t^{-1}}, V\delta_e \rangle$$

$$= \langle m_\varphi(\lambda_t)\delta_{t^{-1}}, \delta_e \rangle$$

$$= \langle \varphi(t)\lambda_t\delta_{t^{-1}}, \delta_e \rangle$$

$$= \underbrace{\varphi(t)}$$

hence φ is of positive type.

$\square$

Preliminary results in von Neumann algebras

if $M \subset \mathcal{B}(\mathcal{H})$ is a $*$-subalgebra st $1 \in M$ then M is a **von Neumann algebra** iff it is closed in the SOT $\left(T_i \to T \text{ iff } T_i\xi \to T\xi \text{ in } \mathcal{H}\ \forall \xi \in \mathcal{H} \right)$

iff it is closed in the WOT

$\left(T_i \to T \text{ iff } T_i\xi \to T\xi \text{ weakly } \forall \xi \in \mathcal{H} \right)$

iff $M = M''$ (von Neumann '29).

$$S' := \{ T \in \mathcal{B}(\mathcal{H}) \mid TS = ST\ \forall S \in S \}$$

$t\mu$ = push forward of μ wrt t

ie $t\mu(E) := \mu(tE)$.

Ex: (X, μ) is a standard measure space.

$\underline{L^\infty(X,\mu)} \subset \mathcal{B}(L^2(X,\mu))$ by pointwise multiplication.

In fact, $L^\infty(X,\mu)' = L^\infty(X,\mu)$.

Ex: Γ a group $L\Gamma = \lambda(\Gamma)'' \subset \mathcal{B}(\ell^2\Gamma)$.

Ex: Γ a group, (X,μ) standard prob. space, $\Gamma \curvearrowright (X,\mu)$ quasi-invariant (preserves null sets).

Koopman rep $\sigma^0 : \Gamma \to \mathcal{U}(L^2(X,\mu))$

$$\sigma^0_t \xi = \xi \circ t^{-1} \left(\frac{d\mu}{dt\mu} \circ t^{-1} \right)^{1/2}$$

$$\underline{L^\infty(X,\mu) \rtimes \Gamma} := \{ L^\infty(X,\mu) \otimes \mathbb{C},\ \underbrace{\sigma^0_t \otimes \lambda_t}_{= U_t},\ t \in \Gamma \}''$$

$$\subset \mathcal{B}(L^2(X,\mu) \overline{\otimes} \ell^2\Gamma)$$

the group-measure space construction.

the crossed product of $L^\infty(X,\mu)$ with Γ.

Fell's absorptions: $L\Gamma \hookrightarrow L^\infty(X,\mu) \rtimes \Gamma$

Remk: If $f \in L^\infty(X,\mu)$, $t \in \Gamma$ $\quad \underline{U_t^* f U_t} = \underline{f \circ t^{-1}}$

Fell's absorption lemma: If Γ a gp. $\sigma^0 : \Gamma \to \mathcal{U}(\mathcal{H})$ a rep then

$$\sigma^0 \otimes \lambda \sim 1 \otimes \lambda$$

Proof:

define $F : \mathcal{H} \bar{\otimes} \ell^2\Gamma \to \mathcal{H} \bar{\otimes} \ell^2\Gamma$ by

$$F(\xi \otimes \delta_t) = \sigma^0_{t^{-1}} \xi \otimes \delta_t.$$

$$\boxed{(1 \otimes \lambda_s) F}(\xi \otimes \delta_t) = (1 \otimes \lambda_s)\, \sigma^0_{t^{-1}} \xi \otimes \delta_t$$
$$= \sigma^0_{t^{-1}} \xi \otimes \delta_{st}$$
$$= F\, \sigma^0_s \xi \otimes \delta_{st}$$
$$= \boxed{F(\sigma^0_s \otimes \lambda_s)}\, (\xi \otimes \delta_t). \quad \square$$

$\{ \mathcal{B}(\mathcal{H}) \simeq TC(\mathcal{H})^*$ $\quad \langle T, A \rangle = Tr(TA)$

$$\boxed{\ell^\infty \mathbb{N} \simeq (\ell^1 \mathbb{N})^*}$$

wk^*-top on $\mathcal{B}(\mathcal{H})$ is the σ-WOT, this is the same as the WOT on bounded sets.

$M \subset \mathcal{B}(\mathcal{H})$ a vN algo then M is wk^*-closed.

$\therefore M \simeq (M_*)^*$.

Sakai: M_* is unique. and Moreover, a unital C^*-alg. A is isomorphic to a vN alg iff A is the dual of a Banach space.

cor: If M and N are vN algs and $\Theta : M \to N$ is a $*$-isomorphism, then Θ preserves the weak*-topologies.

A state $\tau : M \to \mathbb{C}$ is normal if it is cont wrt the wk*-top. ($\underline{\text{WOT}}$, SOT).

τ is faithful if $\tau(x^*x) = 0 \Rightarrow x = 0$.

τ is tracial if $\tau(xy) = \tau(yx) \ \forall\, x, y \in M$.

Ex: $M = M_n(\mathbb{C}) = \mathcal{B}(\mathbb{C}^n)$, $\tau = \frac{1}{n}\mathrm{Tr}$.

- $L^\infty(X, \mu)^{\mu \text{ prob measure}}$ $\tau = \int \cdot \, d\mu$
- $L\Gamma$ $\tau(x) = \underline{\langle x\delta_e, \delta_e \rangle}$
- $\Gamma \curvearrowright (X, \mu)$ $\underline{\text{probability}}$ $\underline{\text{measure-preserving}}$,

$$\tau(x) = \langle x(\hat{1} \otimes \delta_e), (\hat{1} \otimes \delta_e) \rangle.$$

measure-preserving $\Rightarrow \tau$ tracial.

$\underline{\tau\text{-trace}}$ (normal, faithful) GNS-rep gives the $\underline{\text{Standard representation}}$

$$M \subset \mathcal{B}(L^2(M, \tau))$$

$$\underline{L^2(M, \tau) := \overline{M}^{\langle \cdot, \cdot \rangle_\tau}} \quad \langle x, y \rangle_2 = \tau(y^* x).$$

$$J : L^2(M, \tau) \to L^2(M, \tau)$$

$$J\hat{x} = \widehat{x^*} \quad \text{for } x \in M$$

$$\|J\hat{x}\|_2^2 = \|\widehat{x^*}\|_2^2 = \tau(xx^*) = \tau(x^*x) = \|\hat{x}\|_2^2$$

If $a \in M$, $x \in M$

$$[Ja^*J]\hat{x} = Ja^*\widehat{x^*} = J\widehat{a^*x^*} = \widehat{xa}$$

$\therefore JMJ$ is a vN algebra in $\underbrace{M' \cap \mathcal{B}(L^2(M, \tau))}$

Prop: we have $JMJ = M'$ and $JM'J = M$.

Prop: we have $JMJ = M'$ and $JM'J = M$

Proof:

$JMJ \subset M'$, we already showed.

Suppose $T \in M'$ then $\exists\, a_n \in M$

s.t. $T\hat{1} = \lim_{n\to\infty} \hat{a_n}$ where $a_n \in M$

$$\langle \hat{x}, T^*\hat{1} \rangle = \langle Tx\hat{1}, \hat{1} \rangle$$
$$= \langle T\hat{1}, \hat{x^*} \rangle$$
$$= \lim_{n\to\infty} \tau(x a_n)$$
$$= \lim_{n\to\infty} \langle \hat{x}, \hat{a_n^*} \rangle \quad \forall x \in M$$

remark: $\lim_{n\to\infty} |\langle \hat{x}, \hat{a_n^*} \rangle| = |\langle \hat{x}, T^*\hat{1} \rangle| \le \|T\|\, \|x\|_2$

$\therefore T^*\hat{1} = \lim_{n\to\infty} \hat{a_n^*}$

Similarly if $S \in (JMJ)'$

$$S\hat{1} = \lim_{n\to\infty} Jb_nJ\hat{1} \text{ then}$$
$$S^*\hat{1} = \lim_{n\to\infty} Jb_n^*J\hat{1}.$$

$$\langle TS\hat{1}, \hat{1} \rangle = \lim_{n\to\infty} \lim_{m\to\infty} \langle Jb_nJ\hat{1}, \hat{a_m^*} \rangle$$
$$= \lim_{n\to\infty} \lim_{m\to\infty} \tau(a_m b_n^*)$$
$$= \lim_{n\to\infty} \lim_{m\to\infty} \langle \hat{a_m}, Jb_n^*J\hat{1} \rangle$$
$$= \langle ST\hat{1}, \hat{1} \rangle$$

If $x, y \in M$ then

$$\langle TS\hat{x}, \hat{y} \rangle = \langle y^* T S J x^* J \hat{1}, \hat{1} \rangle$$
$$= \langle \underbrace{(TJx^*J)}_{\in M'} \underbrace{(y^*S)}_{(JMJ)'} \hat{1}, \hat{1} \rangle$$
$$= \langle y^* S T J x^* J \hat{1}, \hat{1} \rangle = \langle ST\hat{x}, \hat{y} \rangle$$

$\therefore TS = ST \quad \therefore M' \subset (JMJ)'' = JMJ \quad \square$

3 classical facts:

- If M is an abelian vNA then $M \simeq L^\infty(X,\mu)$ for some measure space.

- If M is separable (separable predual) then $\underbrace{M = \int_X^{\oplus} M_x \, d\mu(x)}$ where M_x is a factor $Z(M_x) = \mathbb{C}$.

- If $Z(M)$ is separable then M has a normal faithful trace iff $\underbrace{v^*v = 1} \Longrightarrow \underbrace{vv^* = 1}$ for $v \in M$

(finite von Neumann algebras).

check: $L^2(L^\infty(X,\mu), \int) \simeq L^2(X,\mu)$

$L^2(L\Gamma, \tau) \simeq \ell^2\Gamma$

$L^2(L^\infty(X,\mu) \rtimes \Gamma, \tau) \simeq L^2(X,\mu) \bar{\otimes} \ell^2\Gamma$.

$\longrightarrow$ Dixmier's Book: Von Neumann Algebras

Thm: (Kadison's inequality) If A, B are a unital C^*-alg and $\phi : A \to B$ is UCP then $\forall\, x \in A$ we have

$$\phi(x)^* \phi(x) \leq \phi(x^* x).$$

Proof: Assume $B \subset B(\mathcal{H})$. By Stinespring's theorem there exists $V : \mathcal{H} \to \mathcal{K}$ an isometry and $\pi : A \to B(\mathcal{K})$ ST

$$\phi(x) = V^* \pi(x) V$$

then

$$\phi(x^* x) - \phi(x)^* \phi(x)$$

$$= V^* \pi(x^* x) V - V^* \pi(x)^* V V^* \pi(x) V$$

$$= \underbrace{V^* \pi(x^*)}\,\underbrace{(1 - VV^*)}_{\text{projection}}\,\underbrace{\pi(x) V}_{} \geq 0.$$

hence ≥ 0

Suppose (M, τ) is a tracial vN alg. $\phi : M \to M$ UCP ST $\tau \circ \phi(x^* x) \leq \tau(x^* x)$ $\forall\, x \in M$ (i.e., ϕ is subtracial)

then

$$\|\phi(x)\|_2^2 = \tau(\phi(x^*)\phi(x))$$
$$\leq \tau(\phi(x^* x))$$
$$\leq \tau(x^* x) = \|x\|_2^2$$

$\therefore$ ϕ defines a contraction

$$T_\phi : L^2(M, \tau) \longrightarrow L^2(M, \tau)$$

given by $T_\phi \hat{x} = \widehat{\phi(x)}$.

Thm: (M,τ) tracial vN alg.

Suppose $\phi: M \to M$ ucp normal subtracial.

then there exists a normal Hilbert bimodule $\mathcal{H}$ (ie we have normal representation $\pi: M \to B(\mathcal{H})$ and $\rho: M^{op} \to B(\mathcal{H})$ st

$$\pi(x)\rho(y^{op}) = \rho(y^{op})\pi(x) \quad \forall x, y \in M.$$

we write $\pi(x)\rho(y^{op})\xi = x\xi y$)

and a vector $\xi_\phi \in \mathcal{H}$ st $\forall x, y \in M$ we have

$$\langle x\xi_\phi y, \xi_\phi \rangle = \tau(\phi(x)y).$$

Note: $\boxed{\|x\xi_\phi y\|^2} = \tau(y^*\overbrace{\phi(x^*x)}^{a^*a}y) \le \|y\|^2 \tau(\phi(x^*x)) \le \|y\|^2 \|x\|_2^2$

$\le \|\phi(x^*x)\| \, \|y\|_2^2$

Proof: (sketch)

on $M \otimes_{alg} M$ we define an inner-product by $\langle a\otimes b, x\otimes y\rangle_\phi = \tau(y^*\phi(x^*a)b)$

ϕ cp $\Rightarrow \langle \cdot,\cdot\rangle_\phi \ge 0$.

$$x\cdot(a\otimes b)\cdot y = xa\otimes by$$

$$\xi_\phi = 1\otimes 1$$

$$\langle x\cdot\xi_\phi\cdot y, \xi_\phi\rangle = \langle x\otimes y, 1\otimes 1\rangle_\phi = \tau(\phi(x)y)$$

$x_i \to 0$ SOT a bounded net

$$\|x_i\cdot(a\otimes b)\|_\phi^2 = \tau(b^*\underline{\phi(a^*x_i^*x_ia)}b) \to 0$$

$$\|(a\otimes b)\cdot x_i\|_\phi^2 = \tau(x_i^* b^*\phi(a^*a)bx_i) \le \|b^*\phi(a^*a)b\| \, \tau(x_i^*x_i) \to 0.$$

Def: • (M,τ) has property (T) if whenever $\phi_n : M \to M$ are normal ucp subtracial st
$$\|\phi_n(x) - x\|_2 \xrightarrow[n\to\infty]{} 0 \quad \forall\, x \in M,$$
then
$$\sup_{x\in (M)_1} \|\phi_n(x) - x\|_2 \xrightarrow[n\to\infty]{} 0.$$

• (M,τ) has the Haagerup property if there exists a net $\{\phi_i\}_i$ of normal ucp subtracial maps s.t.
$$\|\phi_i(x) - x\|_2 \xrightarrow[i\to\infty]{} 0 \quad \forall\, x \in M$$
and $T_{\phi_i} \in \mathcal{K}(L^2(M,\tau))$

• (M,τ) has the compact approximation property if
and $\phi_i((M)_1)$ is precompact in $L^2(M,\tau)$.

Connes–Jones '85

Thm: If Γ is a group then Γ has Property (T) (resp. the Haagerup property) iff $L\Gamma$ also does.

Choda '83?

Proof:

If $\varphi : \Gamma \to \mathbb{C}$ is of pos. type, then $m_\varphi : L\Gamma \to L\Gamma$ is ucp and trace preserving.

If $x \in \mathbb{C}\Gamma$, say $x = \sum_t \alpha_t \lambda_t$
$$\|m_\varphi(x) - x\|_2^2 = \Big\|\sum_t \alpha_t(\varphi(t)-1)\lambda_t\Big\|_2^2 = \sum_t |\alpha_t|^2 \|\varphi(t)-1\|^2$$

In general if $\|x-y\|_2^2 < \varepsilon$ then $\|m_\varphi(x-y)\|_2^2 \le \|x-y\|_2^2$

$\therefore$ if $\varphi_i \to 1$ pointwise then $m_{\varphi_i} \to \mathrm{id}$ pointwise in $\|\cdot\|_2$.

If Γ does not have (T)

$\exists\ \varphi_i : \Gamma \to \mathbb{C}$ st $\varphi_i \to 1$ ptwise but not unif.

$\therefore\ m_{\varphi_i} \to id$ pointwise in $\|\cdot\|_2$ but not uniformly in $\|\cdot\|_2$ on $(\underline{L\Gamma})_1$

$\therefore\ L\Gamma$ does not have (T).

If Γ has (T), let $\phi_i : M \to M$ be a net of ucp normal subtracial maps st $\phi_i \to id$ pointwise in $\|\cdot\|_2$.

$\exists$ normal Hilbert M-M bimodules $\mathcal{H}_i$ and $\xi_i \in \mathcal{H}_i$ st

$$\tau(\phi_i(x)y) = \langle x\,\xi_i\,y, \xi_i \rangle$$

Now $\|\xi_i\| = 1$

define $\pi_i : \Gamma \to \mathcal{U}(\mathcal{H}_i)$ by

$$\pi_i(t)\xi = \lambda_t \xi \lambda_t^*$$

then $\langle \underline{\lambda_t \xi_i \lambda_t^*}, \underline{\xi_i} \rangle - \|\xi_i\|_2^2$

$$= \tau(\phi_i(\lambda_t)\lambda_t^*) - 1$$

$$= \tau((\phi_i(\lambda_t) - \lambda_t)\lambda_t^*)$$

$$\le \|\phi_i(\lambda_t) - \lambda_t\|_2 \xrightarrow[i\to\infty]{} 0$$

$\therefore\ \|\underline{\lambda_t \xi_i \lambda_t^*} - \xi_i\| \to 0$

$\therefore\ \exists\ \eta_i \in \mathcal{H}_i$ st $\boxed{\lambda_t \eta_i \lambda_t^* = \eta_i}$

and $\|\xi_i - \eta_i\| \to 0$

$$\left\{\begin{array}{ll} \lambda_t \eta_i = \eta_i \lambda_t & \forall\ t \in \Gamma \\ \therefore\ x\eta_i = \eta_i x & \forall\ x \in \mathbb{C}\Gamma \subset L\Gamma \\ \therefore\ x\eta_i = \eta_i x & \forall\ x \in L\Gamma. \end{array}\right.$$

$$\|\phi(x) - x\|_2^2 = \|\phi(x)\|_2^2 + \|x\|_2^2 - 2\operatorname{Re}\tau(\phi(x)x^*)$$

$$\le 2(\|x\|_2^2 - \operatorname{Re}\tau(\phi(x)x^*))$$

$$= 2\langle \xi_i x x^* - x\xi_i x^*, \xi_i \rangle$$

$$\le \|x\| \cdot 2\|\xi_i x - x\xi_i\|$$

$$\|\phi_i(x)-x\|_2^2 \le 2\|x\|\,\|x\xi_i - \xi_i x\| \le 4\|x\|^2\|\xi_i-\eta_i\| \xrightarrow[i\to\infty]{} 0.$$

($+x\eta_i - \eta_i x$)

Hence $L\Gamma$ has (T).

obs: If $\varphi \in C_0(\Gamma)$ then

$$T_{m_\varphi}(\hat{\lambda}_t) = \widehat{m_\varphi(\lambda_t)} = \underline{\varphi(t)}\,\hat{\lambda}_t$$

$$\therefore\ T_{m_\varphi} = \sum_{t\in\Gamma} \varphi(t)\,\mathrm{Proj}_{\mathbb{C}\delta_t}$$

this is in $\mathcal{K}(\ell^2\Gamma)$ iff $\varphi \in C_0(\Gamma)$.

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if $\phi: L\Gamma \to L\Gamma$ UCP then

$$\underline{\varphi(t)} := \tau(\underline{\phi(\lambda_t)\lambda_t^*})$$ is pos type.

$\square$

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obs: If Γ has (T) and $\pi: \Gamma \to \mathcal{U}(M)$ rep then $\pi(\Gamma)''$ has (T)

Cor: $\underline{L(PSL_2\mathbb{Z})} \not\cong L(PSL_3\mathbb{Z})$.

In fact $\underline{L(PSL_3\mathbb{Z})}$ is not isomorphic to any von Neumann subalgebra of $\underline{L(PSL_2\mathbb{Z})}$.

Lemma: If (M,τ) a tracial vN algebra and $B\subset M$ a vN subalgebra then then exists a trace-preserving $\underbrace{\text{conditional expectation}}$ normal UCP idempotent and B-bimodular

$E: M \to B$.

$b_1E(x)b_2 = E(b_1xb_2)$
$b_1,b_2\in B$ $x\in M$

Proof. $M \subset \mathcal{B}(L^2(M,\tau))$.

Let $\underline{e_B}$ denote the orthogonal $\underline{\text{projection}}$ from $\underline{L^2(M,\tau)}$ to $\underline{L^2(B,\tau)}$

Define $E: M \to \mathcal{B}(L^2(B,\tau))$ by

$$E(x) = \underbrace{e_B x e_B}$$

claim: E is our conditional expectation.

$[b, e_B] = 0$

Fix $x \in M$ $\quad b \in B$, $c, d \in B$

$$\langle JbJ E(x)\hat{c}, \hat{d}\rangle = \langle e_B x e_B \hat{c}, \widehat{db}\rangle$$
$$= \langle x\hat{c}, \widehat{db}\rangle$$
$$= \tau(b^* d^* x c)$$

$$\langle E(x) JbJ \hat{c}, \hat{d}\rangle = \langle e_B x e_B \widehat{cb^*}, \hat{d}\rangle$$
$$= \tau(d^* x c b^*)$$

$\therefore E(x) \in JBJ' \cap B(L^2(B, \tau)) = B.$

If $x \in M$, $\tau(E(x)) = \langle E(x)\hat{1}, \hat{1}\rangle$
$= \langle x\hat{1}, \hat{1}\rangle = \tau(x)$ $\square$

Cor: If M has the Haagerup property and $B \subset M$ is a vN subalgebra, then B has the Haagerup property.

proof: $\phi_n : M \to M$ ucp normal subtracial

$\|\phi_n(x) - x\|_2 \xrightarrow[n\to\infty]{} 0 \quad x \in M$

$T_{\phi_n} \in \mathcal{K}(L^2(M, \tau))$

$E \circ \phi_n|_B : B \to B$ ucp normal subtracial

$\|E(\phi_n(b)) - b\|_2$
$\leq \|\phi_n(b) - b\|_2 \longrightarrow 0$

$T_{E\circ\phi_n|_B} = e_B T_{\phi_n} e_B \in \mathcal{K}(L^2(B, \tau))$

Thm (Connes '80) If M is a separable II_1 factor with property (T), then $\mathrm{Out}(M)$ is countable.

Proof: (Note: II_1 factors have ! traces)

$\mathrm{Aut}(M)$ is a Polish group with the topology of pointwise WOT convergence, ie $\alpha_i \to \alpha$ if $\alpha_i(x) - \alpha(x) \xrightarrow{WOT} 0$ $\forall x \in M$.

Note if $\alpha_i \to \alpha$ pt WOT then for $x \in M$

$$\|\alpha_i(x) - \alpha(x)\|_2^2$$
$$= \|\alpha_i(x)\|_2^2 + \|\alpha(x)\|_2^2 - 2\mathrm{Re}\langle \alpha_i(x), \alpha(x)\rangle$$
$$= \|x\|_2^2 + \|x\|_2^2 - 2\mathrm{Re}\langle \alpha_i(x), \alpha(x)\rangle_\tau \to 0$$

Claim: $\mathrm{Inn}(M)$ is an open subgroup of $\mathrm{Aut}(M)$. (then $\mathrm{Aut}(M)/\mathrm{Inn}(M)$ is a discrete Polish group)

Pf of claim:

Suppose $\alpha_i \in \mathrm{Aut}(M)$ st $\alpha_i \longrightarrow \alpha \in \mathrm{Inn}(M)$, $\alpha = \mathrm{Ad}(w)$

then $\mathrm{Ad}(w^*) \circ \alpha_i = \beta_i \longrightarrow id$

$\therefore \|\beta_i(x) - x\|_2 \xrightarrow[i\to\infty]{} 0 \quad \forall x \in M$

Property (T) $\Rightarrow \sup_{x \in (M)_1} \|\beta_i(x) - x\|_2 \xrightarrow[i\to\infty]{} 0$

Suppose $\boxed{\sup_{x\in(M)_1} \|\beta_i(x) - x\|_2 \leq \frac{1}{2}}$ (*)

consider $\mathcal{K}_i = \overline{co}^{\|\cdot\|_2}\{\beta_i(u)u^* \mid u \in \mathcal{U}(M)\} \subset L^2(M,\tau) \cap \hat{M}$

(*) $\|\beta_i(u)u^* - 1\|_2 = \|\beta_i(u) - u\|_2 \leq \frac{1}{2} \quad \forall u \in \mathcal{U}(M)$

Let $z \in \mathcal{K}_i$ be the ! element of minimal $\|\cdot\|_2$.

If $v \in \mathcal{U}(M)$ then $\beta_i(v)\mathcal{K}_i v^* = \mathcal{K}_i$ & $x \mapsto \beta_i(v)xv^*$ is isometric.

$\therefore \beta_i(v) z v^* = z$

$\therefore \beta_i(v) z = z v \quad \forall v \in \mathcal{U}(M)$

$\therefore \beta_i(x) z = z x \quad \forall x \in M.$

$\therefore z^* \beta_i(x) = x z^* \quad \forall x \in M.$

$\therefore z^*\beta_i(x) z = x z^* z$

$z^* z x = z^* \beta_i(x) z = x z^* z$

$\therefore z^* z \in M \cap M' = Z(M) = \mathbb{C}$

Note $\forall x \in \mathcal{K}_i$ we have $\|x - 1\|_2 \leq \frac{1}{2}$

$\therefore \|z - 1\|_2 \leq 1/2 \Rightarrow z \neq 0$

$\therefore \exists \lambda \in \mathbb{C}$ st $w = \lambda z \in \mathcal{U}(M)$

$\beta_i(x) w = w x \quad \forall x \in M$

$\therefore \beta_i = \mathrm{Ad}(w) \in \mathrm{Inn}(M)$ $\square$

Fact: $\mathcal{F}(M) \hookrightarrow \mathrm{Out}(M \bar{\otimes} M)$

Fact: If M has (T) then $M \bar{\otimes} M$ has (T)

conclusion: If M is a II_1 factor with (T) then $\mathcal{F}(M)$ is countable.

Connes's rigidity conjecture: • If Γ is $\boxed{\text{i.c.c.}}$ property (T) then $L\Gamma \simeq L\Lambda \Rightarrow \Gamma \simeq \Lambda$. $\boxed{\Gamma = SL_3\mathbb{Z}}$

↳ each non-trivial conjugacy class is infinite.

conjecture: If M is a II_1 factor with (T) then $\mathcal{F}(M) = \{0\}$.

Ex: (Popa) $\Gamma = \left\{ \begin{pmatrix} 1 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in SL_5\mathbb{Z} \right\}$

$$Z(\Gamma) \simeq \underline{\mathbb{Z}} = \left\{ \begin{pmatrix} 1 & & 0^* \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$$

$\underline{L\Gamma}$ has (T)

$$\underbrace{L\Gamma \simeq \int_X^{\oplus} M_x \, d\mu(x)}$$ vN's direct integral decomposition.

If $\underbrace{M_x \simeq M_y}_{\text{w/ } \mu(A) > 0}$ for $x, y \in A \subset X$

then $\underline{1_A} L\Gamma \simeq \underbrace{L^\infty(A, \mu|_A) \bar{\otimes} M_0}_{\hookrightarrow \text{ does not have (T)}}$

$\therefore \nexists$ a pos measure subset $A \subset X$ st $M_x \simeq M_y \ \forall \ x, y \in A$.

Question: If M has (T) what can be said about the ER on (X, μ) in the integral decomposition given by isomorphism?

Def: (Murray–vN) A tracial vN alg M has property (Gamma) if there exists a net $(u_i) \subset \mathcal{U}(M)$ st. $u_n \to 0$ weakly and $\|[x, u_i]\|_2 \xrightarrow[i\to\infty]{} 0$ $\forall\ x \in M$.

Ex: If $M \simeq N \bar{\otimes} R$ $\quad R = \overline{\bigcup M_n(\mathbb{C})}$

then M has property (Gamma).

$$R = \overline{\bigcup_n \underbrace{M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \cdots \otimes M_2(\mathbb{C})}_{n\text{-times}}} \quad \simeq M_{2^n}(\mathbb{C})$$

take u_n to be

$$= \underbrace{\mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{Id} \otimes \cdots \otimes 1}_{n\text{-times}} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes 1 \otimes 1 \cdots$$

then $[x, u_n] = 0$ for large n on a $\|\cdot\|_2$-dense subset of R.

LF_2 does not have (Gamma).

If M a finite vN alg τ_1 τ_2 normal faithful traces then there exists a positive invertible operator a "affiliated" with $Z(M)$ st

$\tau_2(x) = \tau_1(xa)$.

C. Anantharaman-Delaroche + S. Popa

Def: (Effros) Γ is inner-amenable if it is finite or then exists a conj invariant state φ on $\ell^\infty\Gamma$ st $\varphi|_{c_0(\Gamma)} \equiv 0$.

Exercise: Show that every amenable group is inner-amenable.

Thm (Effros) If $L\Gamma$ has property (Gamma) then Γ is inner-amenable.

Proof:
$\exists\ u_n \in \mathcal{U}(L\Gamma)$ st $u_n \to 0$ WOT and $\|[u_n, x]\|_2 \xrightarrow[n\to\infty]{} 0 \quad \forall x \in L\Gamma$

$$L^2(L\Gamma, \tau) \xrightarrow{\sim} \boxed{\ell^2\Gamma}$$
$$\hat{\lambda}_t \longmapsto \delta_t$$

$\therefore\ \hat{u}_n$ give unit vectors in $\ell^2\Gamma$

$$\underbrace{\|\lambda_t u_n \lambda_t^* - \hat{u}_n\|_2}\xrightarrow[n\to\infty]{} 0 \quad \forall t \in \Gamma$$

$$= \underbrace{\|\lambda_t \rho_t\, \hat{u}_n - \hat{u}_n\|_2} \xrightarrow[n\to\infty]{} 0 \quad \forall t \in \Gamma$$

define state φ_n on $\ell^\infty\Gamma \subset B(\ell^2\Gamma)$ by

$$\varphi_n(f) = \langle f \hat{u}_n, \hat{u}_n \rangle$$

$$\varphi_n(L_t R_t(f)) = \langle \lambda_t \rho_t f \lambda_t^* \rho_t^* \hat{u}_n, \hat{u}_n \rangle$$
$$= \langle f \lambda_t^* \rho_t^* \hat{u}_n, \lambda_t^* \rho_t^* \hat{u}_n \rangle$$

$$|\varphi_n(L_t R_t(f)) - \varphi_n(f)|$$
$$\leq 2\|f\|_\infty \|\hat{u}_n - \lambda_t \rho_t \hat{u}_n\|_2$$

Let φ be any wk*-accumulation pt. of φ_n then $\varphi(L_t R_t(f)) - \varphi(f) = 0$

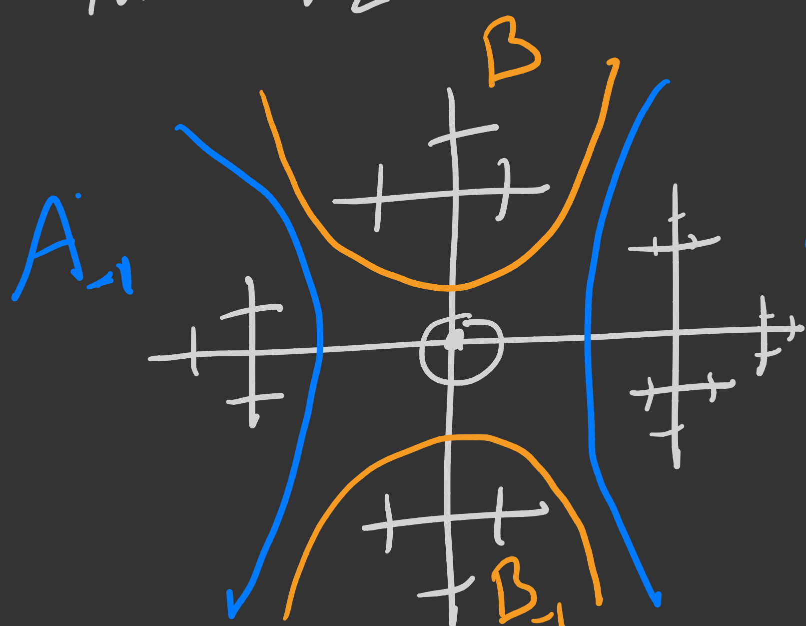
Also, if $f \in C_0(\Gamma)$ $\varphi(f) \approx \underbrace{\langle f \cdot \hat{u}_n, \hat{u}_n \rangle} \xrightarrow[n\to\infty]{} 0$

2012: Vaes: An example of an icc group Γ st Γ is inner-amenable but $L\Gamma$ does not have (Gamma).

Remark: • If $\Gamma \curvearrowright S$ a set then TFAE ① $\exists$ a Γ inv state φ on $\ell^\infty S$ st $\underbrace{\varphi|_{c_0 S} \equiv 0.}$

② $\exists$ $\xi_n \in \ell^2 S$ $\|\xi_n\| = 1$ $\xi_n \to 0$ weakly st $\|\gamma \cdot \xi_n - \xi_n\|_2 \to 0$ $\forall \gamma \in \Gamma$.

Thm: $\mathbb{F}_2$ is not inner-amenable.



$A = \{$words in reduced form $w = aw'\}$

If $\varphi \in \ell^\infty(\mathbb{F}_2)^*$ is a conj. inv state, then

$$B_{-1} \cup B \subset a A_{-1} a^{-1} \qquad A_{-1} \cup A \subset b B_{-1} b^{-1}$$

$$\therefore \varphi(1_{B_{-1} \cup B}) \leq \varphi(A_{-1}) \leq \varphi(1_{B_{-1}}) \leq \varphi(1_{B_{-1} \cup B}) = \varphi(1_{B_{-1}}) + \underbrace{\varphi(1_B)}$$

$$\therefore \varphi(1_B) = 0$$

Similarly $\varphi(1_{B_{-1}}) = \varphi(1_A) = \varphi(1_{A_{-1}}) = 0$

hence $\varphi(1_{\{e\}}) = 1$.

So $\varphi|_{c_0 \mathbb{F}_2} \not\equiv 0$.

$SL_2\mathbb{Z}$ is not inner-amenable.

Proof: $\Gamma = SL_2\mathbb{R} \curvearrowright \mathbb{R}P^1 \cong \mathbb{R} \cup \{\infty\}$ by FLT.

If $g \in SL_2\mathbb{R}$, write $g = K \cdot h$ where $K \in SO(2)$ and h is positive definite. h is diagonalizable and hence $\underline{g = K_1 a K_2}$

where $a = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$ with $\lambda \geq 1$.

$$\underbrace{SL_2\mathbb{R} = K A_+ K}, \quad K = SO(2)$$

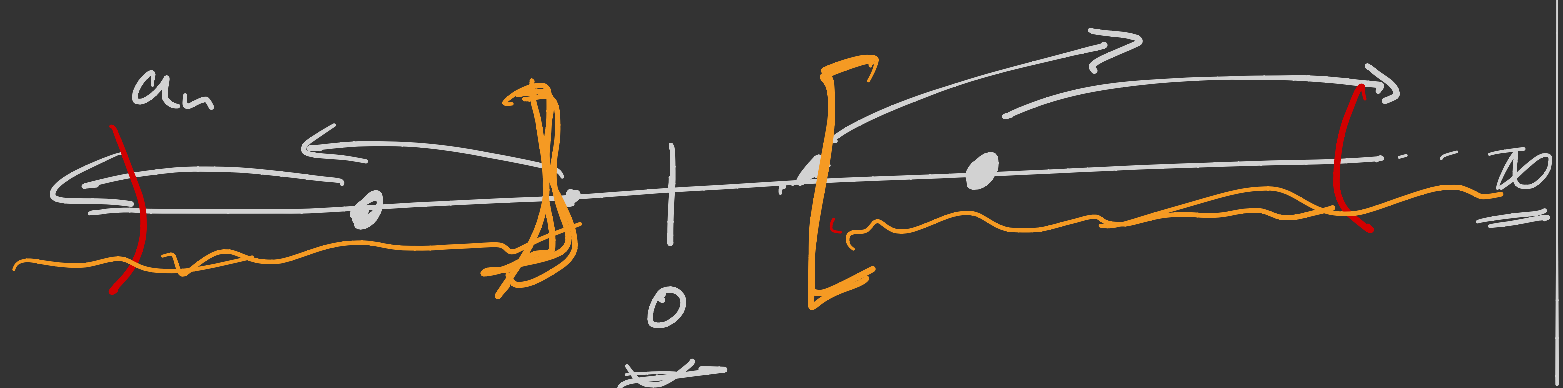
A_+ = diagonal matrices with positive diagonal entries in decreasing order.

If $t_n \in SL_2\mathbb{Z}$ st $t_n \to \infty$ and

$$t_n = k_n \underbrace{\begin{pmatrix} \lambda_n & 0 \\ 0 & \frac{1}{\lambda_n} \end{pmatrix}}_{= a_n} \tilde{k}_n \quad \therefore \quad \lambda_n \to \infty,$$

assume $k_n \to k_\infty \in SO(2)$ $\quad \tilde{k}_n \to \tilde{k}_\infty \in SO(2)$

$$\boxed{\begin{pmatrix} \lambda_n & 0 \\ 0 & \frac{1}{\lambda_n} \end{pmatrix}} \cdot z = \frac{\lambda_n z + 0}{0 + \frac{1}{\lambda_n}} = \lambda_n^2 z$$



North-South dynamics: If $t_n \in SL_2\mathbb{Z}$ st $t_n \to \infty$ then after passing to a subsequence there exists $a, b \in \mathbb{R} \cup \{\infty\}$ st $t_{n_k} \cdot x \to b$ $\forall\, x \neq a$.

In fact if O_a and O_b are nbhds of a and b respectively then there exists $K \geq 1$ st for $k \geq K$ we have

$$\underbrace{t_{n_k} \cdot O_a^c} \subset \underbrace{O_b}.$$

$\therefore$ If we let μ to be any prob. measure w/o atoms on $\mathbb{R} \cup \{\infty\} \simeq \mathbb{T}$

then $t_{n_k} \mu \xrightarrow[wk^*]{} \delta_{\{b\}}$

In particular if $g \in SL_2\mathbb{Z}$ then

$$t_{n_k} g \mu \xrightarrow[wk^*]{} \delta_{\{b\}}$$

$\therefore$ If $\mu \in \mathrm{Prob}(\mathbb{R} \cup \{\infty\})$ without atoms

and if $\{t_n\}_n \subset \Gamma = SL_2\mathbb{R}$ $\quad t_n \to \infty$

then there exists a subsequence st

$\forall\ g \in SL_2\mathbb{Z}$ we have

$$t_{n_k}\mu - t_{n_k} g\mu \xrightarrow[wk^*]{} 0$$

$$\boxed{\text{In fact, } \quad t_n\mu - t_n g\mu \xrightarrow[wk^*]{} 0 \quad \forall g \in \Gamma.}$$

define $\phi : C(\mathbb{R}\cup\{\infty\}) \longrightarrow \ell^\infty\Gamma$ by

$$\phi(f)(x) = \int f \, dx\mu$$

$$L_tR_t(\phi(f))(x) = \phi(f)(t^{-1}xt)$$

$$= \int f \, d\underbrace{t^{-1}xt\mu}$$

$$= \underbrace{\int f\circ t \; dxt\mu}$$

If $\varphi \in (\ell^\infty\Gamma)^*$ is a conj inv state

st $\underline{\varphi|_{c_0\Gamma} \equiv 0}$, then

$\varphi\circ\phi$ gives a state on $\boxed{C(\mathbb{R}\cup\{\infty\})}$

st $\varphi\circ\phi(f\circ t^{-1})$

$$= \underbrace{\varphi(L_tR_t\,\phi(f))}$$

$$= \varphi\circ\phi(f)$$

then we would have that Γ has

a invariant prob. measure on $\mathbb{R}\cup\{\infty\}$.

$\therefore$ Γ fixes either a point or a

pair of points in $\mathbb{R}\cup\{\infty\}$,

giving a contradiction.

Last time we showed that the action $\Gamma = SL_2\mathbb{Z} \curvearrowright \underset{K}{\mathbb{RP}^1}$ is a convergence action, i.e., whenever $\{t_n\}_n \subset \Gamma$ st $t_n \xrightarrow[n\to\infty]{} \infty$ then there exists a subsequence $\{t_{n_k}\}_{k=1}^{\infty}$ and two points $a, b \in K$ st $\forall$ nbhds A of a B of b

$t_{n_k}(K \setminus A) \subset B$ for k large. $\forall\ x \in K \setminus \{a\}$.

Equivalently for any nbhds A of a and B of b we have

$$t_{n_k}(K \setminus A) \subset B \quad \text{for } k \text{ large enough.}$$

$\therefore$ If $\mu \in Prob(K)$ then

$$t_{n_k}\mu \longrightarrow \delta_{\{b\}} \quad \text{if } \mu(\{a\}) = 0.$$

Bowditch: 1999: convergence groups and configuration spaces.

Lemma: If $\Gamma \curvearrowright K$ is a convergence action, and $\mu \in Prob(K)$ without atoms. then whenever $t_n \in \Gamma$ st $\boxed{t_n \to \infty}$ ($\forall F \subset \Gamma$ finite $\exists N$ st $t_n \notin F$ $n \geq N$.) and $\gamma \in \Gamma$, then $t_n\mu - t_n\gamma\mu \xrightarrow[wk^*]{} 0$.

Proof: Suppose $t_n\mu - t_n\gamma\mu \underset{wk^*}{\not\to} 0$

$\therefore \exists\ f \in C(K)$ and $c > 0$ st.

$$\overline{\lim_{n\to\infty}} \left| \int f\, d\underbrace{t_n\mu} - \int f\, d\underbrace{t_n\gamma\mu} \right| \geq c > 0$$

Taking a subsequence we can assume this is a limit.

Taking a further subsequence $\{t_{n_k}\}_{k=1}^{\infty}$ we have that $t_{n_k}\mu \to \delta_{\{b\}}$

also $t_{n_k}\gamma\mu \to \delta_{\{b\}}$

$$\therefore \left| \int f\, dt_n\mu - \int d\, t_n\gamma\mu \right| \to |f(b) - f(b)| = 0.$$

$\square$

Def: An infinite group Γ is a convergence group if it has a convergence action $\Gamma \curvearrowright K$ st this action does not fix a point or a pair of points, equivalently $\Gamma \curvearrowright K$ does not have an invariant prob. measure.

Ex: • $\mathbb{Z} \curvearrowright \mathbb{Z} \cup \{\infty\}$ a convergence action

• $D_\infty \curvearrowright \mathbb{Z} \cup \{-\infty\} \cup \{\infty\}$. a convergence action.

Def: An infinite group Γ is properly proximal if there does not exist a left Γ-invariant state on $\left(\ell^\infty\Gamma / c_0\Gamma\right)^{\Gamma_R}$

$= \{$right Γ-invariant functions in $\ell^\infty\Gamma / c_0\Gamma\}$.

Equivalently, if we consider

$$\mathcal{A} = \{f \in \ell^\infty\Gamma \mid f - R_t(f) \in c_0(\Gamma) \ \forall\, t \in \Gamma\}$$

this is a C^*-subalgebra of $\ell^\infty\Gamma$

Γ is properly proximal iff $\nexists$ a left Γ-inv state on $\mathcal{A}$.

$$\mathcal{A} \xrightarrow{\pi} \left(\ell^\infty\Gamma / c_0\Gamma\right)^{\Gamma_R}$$

$\ker(\pi) = c_0\Gamma$

if $\varphi \in \mathcal{A}^*$ a Γ-inv state then $\varphi|_{c_0\Gamma} \equiv 0$

$\therefore$ φ defines a state on $\left(\ell^\infty\Gamma / c_0\Gamma\right)^{\Gamma_R}$

Thm: If Γ is a convergence group then Γ is properly proximal.

Proof: suppose Γ is not properly proximal. suppose $\Gamma \curvearrowright K$ is a convergence action. Fix $\underline{\mu \in Prob(K)}$ w/o atoms. define $\phi: C(K) \to \ell^\infty\Gamma$ by

$$\phi(f)(t) = \int f \circ t \, d\mu = \int f \, dt\mu$$

Note $\phi(\underbrace{f \circ s^{-1}})(t) = \int f \circ s^{-1} t \, d\mu$

$$= \phi(f)(s^{-1}t)$$

$$= \underbrace{L_s(\phi(f))}(t)$$

Also, $R_s(\phi(f))(t)$

$$= \phi(f)(ts)$$

$$= \int f \, dts\mu$$

$\int f \circ t^{-1} d\mu = \int f \, dt\mu$

$t\mu(E) = \mu(t^{-1}E)$

$\therefore (R_s(\phi(f)) - \phi(f))(t)$

$$= \int f \, d(\underline{ts\mu - t\mu}) \xrightarrow[t \to \infty]{} 0$$

$\therefore \phi(f) \in \underline{A}$

If $\psi \in A^*$ is a left-invariant state then $\psi \circ \phi$ is a Γ-invariant state on $C(K)$

Riesz rep. thm gives a Γ-inv measure on K. $\square$

Thm: If Γ is inner-amenable then Γ is not properly proximal.

Proof: suppose $\ell^\infty\Gamma$ has a conj. inv state φ st $\varphi|_{c_0\Gamma} \equiv 0$.

then $\varphi|_A$ gives a state on A and if $t \in \Gamma$ and $f \in A$ then

$$\varphi(L_t(f)) - \varphi(f) = \varphi(\underbrace{R_{t^{-1}}(f) - f}_{\in c_0(\Gamma)}) = 0. \quad \square$$

Thm: Γ is properly proximal iff $\exists$ $\Gamma \curvearrowright K$ cpt Hausdorff st this action has no invariant measure, and st $\exists$ $\mu \in Prob(K)$ st

$$\underset{t\to\infty}{\text{wk}^*\text{-lim}}\ \underline{t\mu} - ts\mu = 0 \quad \forall s \in \Gamma.$$

Proof: $(\Longleftarrow)$ see the proof before.

$(\Longrightarrow)$ $\mathcal{A} = \{ f \in \ell^\infty\Gamma \mid f - R_t(f) \in C_0(\Gamma)\ \forall t \in \Gamma \}$

Set $\Delta\Gamma = \sigma(\mathcal{A})$ so that $\mathcal{A} = C(\Delta\Gamma)$

$\Gamma \curvearrowright \Delta\Gamma$ we have $C_0(\Gamma) \subset \mathcal{A}$

and hence $\Gamma \hookrightarrow \Delta\Gamma$ has dense range

Lemma: If $t_n \in \Gamma$ st $t_n \to w \in \underline{\Delta\Gamma \setminus \Gamma}$

then $\forall$ $s \in \Gamma$ we have $t_n s \to w$

ie this compactification is "small at infinity".

Pf: of lemma:

Suppose not & take a subnet st

$$t_n s \longrightarrow \tilde{w} \neq w$$

take $f \in C(\Delta\Gamma)$ st $f(w) \neq f(\tilde{w})$

$$\cong \mathcal{A} \subset \ell^\infty\Gamma$$

$$f(w) = \lim_{n\to\infty} f(t_n)$$

$$f(\tilde{w}) = \lim_{n\to\infty} f(t_n s) = R_s(f(t_n))$$

$$\therefore \quad f(w) - f(\tilde{w}) = \lim_{n\to\infty} \big(\underbrace{f - R_s(f)}_{\in C_0(\Gamma)}\big)(t_n) = 0. \quad \square$$

$\therefore$ If $\mu = \delta_{\{e\}} \in Prob(\Delta\Gamma)$

then $(\underline{t_n\mu} - \underline{t_n s\mu}) \xrightarrow[n\to\infty]{} 0$ wk*

There exists a Γ-inv prob measure on $\Delta\Gamma$

iff $\exists$ a Γ-inv State on $\mathcal{A}$. $\square$

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we will prove:

$\mathbb{F}_2 \times \mathbb{F}_2$ is properly proximal.

Q: what is the natural action?

2018 Boutonnet, Ioana, P.

Def: If Γ a group. Then a boundary piece is a closed subset $X \subset \underline{\beta\Gamma} \setminus \Gamma$ that is invariant under left and right multiplication.

Ex: • $X = \beta\Gamma \setminus \Gamma$.

• If $\Sigma < \Gamma$ consider

$$X^c = \bigcup_{\substack{F_1, F_2 \in \Gamma \\ \text{finite}}} \underbrace{\overline{F_1 \Sigma F_1}} \subset \beta\Gamma.$$

• $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ a representation

$\pi : \beta\Gamma \longrightarrow (\mathcal{B}(\mathcal{H}))_1$ continuous.

$X = \underline{\pi^{-1}(\{0\})}$

Note: π is mixing iff $X = \beta\Gamma \setminus \Gamma$.

$X = \phi$ if π is not weak-mixing.

• $\Gamma < SL_n \mathbb{R} = G$ $\quad G = K A_+ K$

$K = SO(n)$ $\quad A_+ = \left\{ \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix} \,\middle|\, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0 \right\}.$

For each $1 \leq k < n$ and $t \geq 0$

consider $\Sigma_{k,t} = \{ \gamma \in \Gamma \mid \lambda_k / \lambda_{k+1} \leq t \}$

$$X_k^c = \bigcup_{t > 0} \overline{\Sigma_{k,t}},$$

ie if we have a net $\gamma_i \in \Gamma$

st. $\gamma_i \longrightarrow \omega \in \beta\Gamma \setminus \Gamma,$

then $\gamma_i \in X_k$ iff

$$\lim_{i \to \infty} \lambda_k^i / \lambda_{k+1}^i = \infty.$$

Note: $\underline{\lambda_1^i / \lambda_2^i} \longrightarrow \infty$ iff $\exists\ s_i > 0$

$\underline{s_i \gamma_i}$ converges to a rank 1 operator in $\underline{M_n(\mathbb{R})}$.

$s_i = \underline{\lambda_1^{i}}^{-st}$

X_K is indeed a boundary piece for $1 \leq K < n$.

Remark: $X_K^{-1} = X_{n-K}$.

For $SL_2\mathbb{Z} = \Gamma$ $\gamma = K\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{pmatrix}\tilde{K}$

$\underline{X_1} = \underline{\beta\Gamma \setminus \Gamma}$

$SL_n\mathbb{R} \curvearrowright \underline{\mathbb{R}P^{n-1}}$ $\gamma = K\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix}\tilde{K}$

if λ_1/λ_n tends to infinity, renormalizing the γ's tend to a rank $\underline{1 \text{ operator}}$.

$\therefore$ If we have $\Gamma < SL_n\mathbb{R}$ discrete and $\gamma_i \in \Gamma$ st $\gamma_i \to w \in X_1$

then passing to a sub-net there exists a $n-1$ dimensional subspace $\underline{W_0} \subset \mathbb{R}^n$ and a unit vector $v_0 \in \mathbb{R}^n$ st $\forall$ open nbhds A of $[W_0] \subset \mathbb{R}P^{n-1}$ and B of $[v_0] \in \mathbb{R}P^{n-1}$ we have $\underline{\gamma_i A^c} \subset B$ for i large.

As a consequence. If $\mu \in \mathrm{Prob}(\underline{\mathbb{R}P^{n-1}})$ st $\mu([W_0]) = 0$ for all $W_0 \subset \mathbb{R}^n$ with $\dim(W_0) < n$, then

$$\gamma_i \mu \longrightarrow \delta_{\{v_0\}} \quad wk^*.$$

As a consequence: If $\gamma_i \in \Gamma$ st $\underbrace{\gamma_i \to w \in X}_{}$, then $\forall\ t \in \Gamma$

$$\underbrace{\gamma_i \mu - \gamma_i t\mu \xrightarrow[i\to\infty]{} 0 \text{ wk}^*}_{}.$$

Def: If X is a boundary piece for a group Γ, then we say that Γ is properly proximal relative to X, if there does not exist a left Γ-invariant state φ on

$$\left(\ell^\infty\Gamma / \underline{I_0(X)}\right)^{\Gamma_R}.$$

where $I_0(X) \subset \underline{\ell^\infty\Gamma} \cong C(\beta\Gamma)$ is the ideal of functions that vanish on X.

Note: $X = \beta\Gamma \setminus \Gamma$ this give the definition from before.

Also note: $A_X := \{f \in \ell^\infty\Gamma \mid f - R_t(f) \in I_0(X)$ for all $t \in \Gamma\}$.

then Γ is properly proximal rel. to X iff $\nexists$ a Γ-inv state on A_X.

$$\underline{A_X} \longrightarrow \underline{\left(\ell^\infty\Gamma / I_0(X)\right)^{\Gamma_R}}$$

If A_X has a Γ-inv state φ then either $\varphi|_{I_0(X)} \equiv 0$ giving a state on $A_X / I_0(X) = \underline{\left(\ell^\infty\Gamma / I_0(X)\right)^{\Gamma_R}}$

or else we get a left-invariant state on $\ell^\infty\Gamma$ by setting

$$\varphi(f) = \lim_{n\to\infty} \varphi(f a_n) \quad \text{where}$$

$a_n \in I_0(X)$ is an approximate identity. st $\|L_t(a_n) - a_n\| \xrightarrow[n\to\infty]{} 0\ \forall t \in \Gamma$.

$\implies \Gamma$ amenable $\implies$ not prop prox rel X.

Ex: $\Gamma < SL_n\mathbb{R}$ lattice then Γ is properly proximal rel to X_1

Pf: As before, given $\mu \in Prob(\mathbb{R}P^{n-1})$ we get a ucp map

$$\phi : C(\mathbb{R}P^{n-1}) \to \ell^\infty\Gamma \text{ by}$$

$$\phi(f)(t) = \int f \, dt\mu$$

then since $t\mu - ts\mu = 0$ weak*-lim $t \to X_1$ $\forall s \in \Gamma$, this shows that

$$\phi : C(\mathbb{R}P^{n-1}) \to A_X$$

give a Γ-inv prob measure on $\mathbb{R}P^{n-1}$, giving a contradiction.

In fact: By looking at the action $\Gamma \curvearrowright Gr(k, \mathbb{R}^n)$, we see that Γ is properly proximal relative to X_k for all $1 \leq k < n-1$.

Remark: $X_1 \cup X_2 \cup \cdots \cup X_{n-1} = \beta\Gamma \setminus \Gamma.$

Ex: If Γ_1 is properly proximal then $\Gamma_1 \times \Gamma_2$ is properly proximal relative to X_{Γ_2}.

In particular, if both Γ_1 and Γ_2 are properly proximal then $\Gamma = \Gamma_1 \times \Gamma_2$ is properly proximal rel to X_{Γ_1} and X_{Γ_2}. Note:

$$X_{\Gamma_1} \cup X_{\Gamma_2} = \beta\Gamma \setminus \Gamma.$$

Thm: If Γ is properly proximal rel to X_1 and X_2, then Γ is properly proximal rel. to $X_1 \cup X_2$.

Cor: lattices in $SL_n\mathbb{R}$ are properly proximal.

Cor: Proper proximality is closed under finitely many direct products.

Key lemma: Γ a countable group, $X \subset \beta\Gamma \setminus \Gamma$, a boundary piece, then TFAE:

① Γ is properly proximal rel. to X, i.e.,

$\nexists$ a left-Γ-invariant state on

$$\left(\ell^\infty\Gamma / I_0(X)\right)^{\Gamma_R}$$

② $\nexists$ a left-Γ-invariant state on

$$\underline{\left(\ell^\infty\Gamma / I_0(X)\right)}^{\text{an}\,\Gamma_R}$$

Pf of Thm from the lemma:

If X_1 & X_2 boundary pieces then we have an

$$\Bigg\{\underbrace{\left(\ell^\infty\Gamma / I_0(X_1 \cup X_2)\right)}^{\Gamma_R} \simeq C(X_1 \cup X_2)^{\Gamma_R}\Bigg\}$$

we therefore get an embedding

$$\underline{\mathrm{Borel}^\infty(X_1 \cup X_2)^{\Gamma_R}} \longrightarrow \underline{\left(\ell^\infty\Gamma / I_0(X_1 \cup X_2)\right)^{\text{an}\,\Gamma_R}}.$$

so if Γ is not properly prox rel $X_1 \cup X_2$ then $\exists$ a Γ-inv state φ on $\mathrm{Borel}^\infty(X_1 \cup X_2)^{\Gamma_R}\}$

$$\underline{\mathrm{Borel}^\infty(X_i)^{\Gamma_R}} \ni f \longmapsto \tilde{f} \qquad \tilde{f}(\omega) = 0 \text{ for } \omega \notin X_i.$$

since $1_{X_1} + 1_{X_2} \geq 1_{X_1 \cup X_2}$ we have that $\varphi(1_{X_1}) \neq 0$ or $\varphi(1_{X_2}) \neq 0$. $\square$

Lemma: Suppose X is a bdry piece. $I_0(X) \subset \ell^\infty\Gamma$ the corresponding ideal, then there exists an approximate identity for $I_0(X)$ $\{\alpha_i\}$ s.t. α_i is increasing $0 \le \alpha_i \le 1$, and

$$\|R_t(\alpha_i) - \alpha_i\| \xrightarrow[i\to\infty]{} 0 \quad \forall\, t \in \Gamma$$

$$\|L_t(\alpha_i) - \alpha_i\| \xrightarrow[i\to\infty]{} 0 \quad \forall\, t \in \Gamma.$$

Proof: (Arveson)

Take any increasing approximate identity $\{\beta_i\}_{i\in I}$ for $I_0(X)$

consider $\big(\underbrace{I_0(X)}_{A}\big)^{**} \subset \big(\underbrace{\ell^\infty\Gamma}_{B}\big)^{**}$

A^{**} is a weak*-closed ideal in a vN alg. B^{**}

$\therefore$ $\exists$ a central projection $p \in Z(B^{**})$

$$A^{**} = p(B^{**}).$$

$$A^{**} \subset pB^{**} \subset B^{**} \rtimes (\Gamma \times \Gamma).$$

$$L_t(p) = R_t(p) = p \quad \forall\, t \in \Gamma$$

$$p = \text{WOT-}\lim \beta_i$$

$$\therefore\ L_t(\beta_i) - \beta_i \longrightarrow 0 \ \text{WOT}$$

$$R_t(\beta_i) - \beta_i \longrightarrow 0 \ \text{WOT}$$

By taking ~~convex~~ convex combination of β_i we can insure that

$\alpha_i \in \mathrm{con}\{\beta_j\}$ then

$$\|L_t(\alpha_i) - \alpha_i\| \longrightarrow 0$$

$$\|R_t(\alpha_i) - \alpha_i\| \longrightarrow 0.$$

Pf of the lemma that $\left(\ell^\infty\Gamma/I_0(X)\right)^{\Gamma_R}$ does not have an inv state iff $\left(\left(\ell^\infty\Gamma/I_0(X)\right)^{\ast\ast}\right)^{\Gamma_R}$ does not.

$$\left(\ell^\infty\Gamma/I_0(X)\right)^{\Gamma_R} \xhookrightarrow[\text{embedding}]{\text{unital}} \left(\left(\ell^\infty\Gamma/I_0(X)\right)^{\ast\ast}\right)^{\Gamma_R}.$$

Suppose $\nexists$ a Γ-inv state on $\left(\ell^\infty\Gamma/I_0(X)\right)^{\ast\ast\Gamma_R}$

then $\nexists$ any non-zero left Γ-inv linear functional.

By Hahn-Banach this means that

$$\operatorname{sp}\left\{ f - L_t(f) \mid t\in\Gamma \ f\in\left(\ell^\infty\Gamma/I_0(X)\right)^{\ast\ast\Gamma_R}\right\}$$

is norm dense.

$\therefore \exists$ $f_1, f_2, \ldots, f_n \in \left(\ell^\infty\Gamma/I_0(X)\right)^{\ast\ast\Gamma_R}$

$\exists$ $t_1, \ldots, t_n \in\Gamma$ st

$$\left\| 1 - \left(\sum f_k - L_{t_k}(f_k)\right)\right\| < \frac{1}{2}$$

choose nets $g_k^i \in \ell^\infty\Gamma$ st

$g_k^i \xrightarrow[i\to\infty]{} f_k$ wk* and $\|g_k^i\| \le \|f_k\|$.

Note $g_k^i - R_t(g_k^i) \longrightarrow f_k - R_t(f_k) = 0$ wk*

By taking convex combinations of g_k^i

we may assume

$$\|g_k^i - R_t(g_k^i)\| \xrightarrow[i\to\infty]{} 0 \ \forall t\in\Gamma$$

Take lifts $h_k^i \in \ell^\infty\Gamma$ of g_k^i

take a lift $b^i \in \ell^\infty\Gamma$ of

$$1 - \left(\sum_k g_k^i - R_{t_k}(g_k^i)\right)$$

Enumerate $\Gamma = \bigcup_n B_n$. Choose an approximate identity $\{\alpha_n\}_{n=1}^{\infty}$ for $I_0(X)$ s.t.

$\forall n$ we can choose $i(n)$ st

$$\| h_\kappa^{i(n)} - R_t(h_\kappa^{i(n)}) \| < 2^{-n} \quad \forall t \in B_n,$$

$\alpha_n \to 1$

① $\| (1-\alpha_n)(h_\kappa^{i(n)} - R_t(h_\kappa^{i(n)})) \| < 2^{-n}$

② $\| \alpha_n - R_t(\alpha_n) \| < 2^{-n}$ $\quad t \in B_n$ $\in I_0(X)$

③ $\| (1-\alpha_n)(b^{i(n)} - (1 - \sum_\kappa h_\kappa^{i(n)} - L_{t_\kappa}(h_\kappa^{i(n)}))) \| < 2^{-n}$

④ $\| \alpha_n - L_t(\alpha_n) \| < 2^{-n}$ $\quad \forall t \in B_n$

Define h_κ, b by

$$h_\kappa = \sum_n (\alpha_{n+1} - \alpha_n) h_\kappa^{i(n)} \in \ell^\infty \Gamma$$

$$b = \alpha_1 + \sum_{n \geq 1} (\alpha_{n+1} - \alpha_n) b^{i(n)} \in \ell^\infty \Gamma$$

Note $\| b - 1 \| < \frac{1}{2}$

If $t \in B_n$

$$\Big\| \sum_{m \geq n} (\alpha_{m+1} - \alpha_m) h_\kappa^{i(m)} - R_t(\alpha_{m+1} - \alpha_m) h_\kappa^{i(m)} \Big\|$$

$$\underset{②}{\lesssim} \sum_{m \geq n} \| (\alpha_{m+1} - \alpha_m)(h_\kappa^{i(m)} - R_t(h_\kappa^{i(m)})) \|$$

$$\underset{①}{\lesssim} 2^{-n-1}$$

$$\therefore h_\kappa + I_0(X) \in (\ell^\infty \Gamma / I_0(X))^{\Gamma_R}$$

Similarly

$$\Big\| \sum_{m \geq n} \Big((\alpha_{m+1} - \alpha_m) b^{i(m)} - \sum_\kappa (\alpha_{m+1} - \alpha_m) \cdot h_\kappa^{i(m)} - L_{t_\kappa}(\alpha_{m+1} - \alpha_m) h_\kappa^{i(m)} \Big) \Big\|$$

$\underset{④}{\lesssim}$ $\underset{③}{\lesssim}$ $\leq 2^{-n-1}$

then $\left\| b - \left(1 - \left(\sum_R h_R - L_{t_R}(h_R)\right)\right)\right\| = 0$

in $\ell^\infty\Gamma / I_0(X)$

$h_R \in \left(\ell^\infty\Gamma / I_0(X)\right)^{\Gamma_R}$.

$\| b - 1 \| < 1/2$

$\therefore \nexists$ a left-inv state on

$\left(\ell^\infty\Gamma / I_0(X)\right)^{\Gamma_R}$. $\square$

Hahn-Banach separation:

If C is a convex subset of a Banach space B and $x \in B \setminus \overline{C}^{\|\cdot\|}$ then $\exists$ a $g \in B^*$ and $a \in \mathbb{R}$ st

$Re(g)(x) < a$ and

$Re(g)|_C \geq a$.

or vice-versa.

Hence if $C \subset B$ is convex st $0 \in \overline{C}^{wk}$

then $0 \in \overline{C}^{\|\cdot\|}$.

Open problem: $L(PSL_3\mathbb{Z}) \not\cong L(PSL_n\mathbb{Z})$? $n \geq 4$?

$\Gamma = PSL_2\mathbb{Z} \curvearrowright \underline{\mathbb{R}\cup\{\infty\}}$ convergence action.

This is an amenable action.

'04

Ozawa: $L\Gamma$ is solid, ie If $B \subset L\Gamma$ diffuse then $B' \cap L\Gamma$ is amenable.

Question: Suppose $B_1, B_2 \subset L(PSL_3\mathbb{Z})$ commuting and nonamenable then is

$$W^*(B_1, B_2)' \cap L(PSL_3\mathbb{Z}) \text{ amenable?}$$

if so $L(PSL_3\mathbb{Z}) \simeq L(PSL_6\mathbb{Z})$

$PSL_3\mathbb{Z} \curvearrowright \mathbb{RP}^2$
$PSL_3\mathbb{Z} \curvearrowright Gr(2, \mathbb{R}^3)$
weak convergence Furstenberg late 60's.

The corresponding boundary pieces cover $\beta(PSL_3\mathbb{Z}) \setminus (PSL_3\mathbb{Z})$.

$PSL_3\mathbb{Z} \curvearrowright \left(\text{Full Flags in } \mathbb{R}^3\right)$ amenable.

Ozawa–Popa: weakly cpt Cartan subalgebra

Thm (BIP) $L(\overbrace{PSL_3\mathbb{Z}}^{\Gamma})$ has no weakly cpt Cartan.

Also. If $\Gamma \curvearrowright (X, \mu)$ cpt then $L^\infty(X, \mu) \rtimes \Gamma$ has a ! weakly cpt Cartan subalg. up to unitary.

Cor: If $n \neq m$, $\Gamma = SL_n\mathbb{Z}$, $\Sigma = SL_m\mathbb{Z}$

$\Gamma \curvearrowright (X,\mu)$ cpt, $\Sigma \curvearrowright (Y,\nu)$ arbitrary

Then $L^\infty(X,\mu) \rtimes \Gamma \not\cong L^\infty(Y,\nu) \rtimes \Sigma$

This all holds for any gp that has actions $\Gamma \curvearrowright K_n$ (cpt Hausdorff), $\eta_n \in \mathrm{Prob}(K_n)$ no inv measure

st the corresponding boundary pieces cover $\beta\Gamma \setminus \Gamma$. finitely many K_n

Ozawa: This is equivalent to proper proximality.

Boutonnet, Ioana, Peterson: Properly proximal groups and their von Neumann algebras, 2018

$\{\alpha_i\}_{i \in I}$ increasing approximate identity

If we consider $\sum_{k=1}^n t_k \alpha_{i_k}$, $\sum_{k=1}^n t_k = 1$, $t_k \geq 0$.

Then exists $i_0 \in I$ st $i_k \leq i_0$ $\forall k$

$\beta_i = \sum_{k=1}^n t_k \alpha_{i_k} \leq \alpha_i$ $\forall$ $i \geq i_0$.

$F \subset \Gamma$ finite, $G \subset I_0(X)$ finite, and $i_0 \in I$

$j = (F, G, I, \varepsilon)$, β_j of $\{\alpha_i\}_{i \geq i_0}$

choose some convex comb

st
$$\|L_t(\beta_j) - \beta_j\| < \varepsilon \quad t \in F$$
$$\|R_t(\beta_j) - \beta_j\| < \varepsilon$$
$$\|\beta_j a - a\| < \varepsilon \quad a \in G$$

For each j $\exists$ $i_j \in I$ st $\beta_j \leq \alpha_i$ $\forall$ $i \geq i_j$

Hyperbolic groups

Let Γ be a connected graph.

we'll view this as a metric space with the path distance.

↳ a sequence of vertices

$\{\alpha(k)\}_{k=1}^{n}$ st $(\alpha(k), \alpha(k+1))$ is an edge in Γ $\forall\ 1 \leq k \leq n$.

a path α is geodesic if

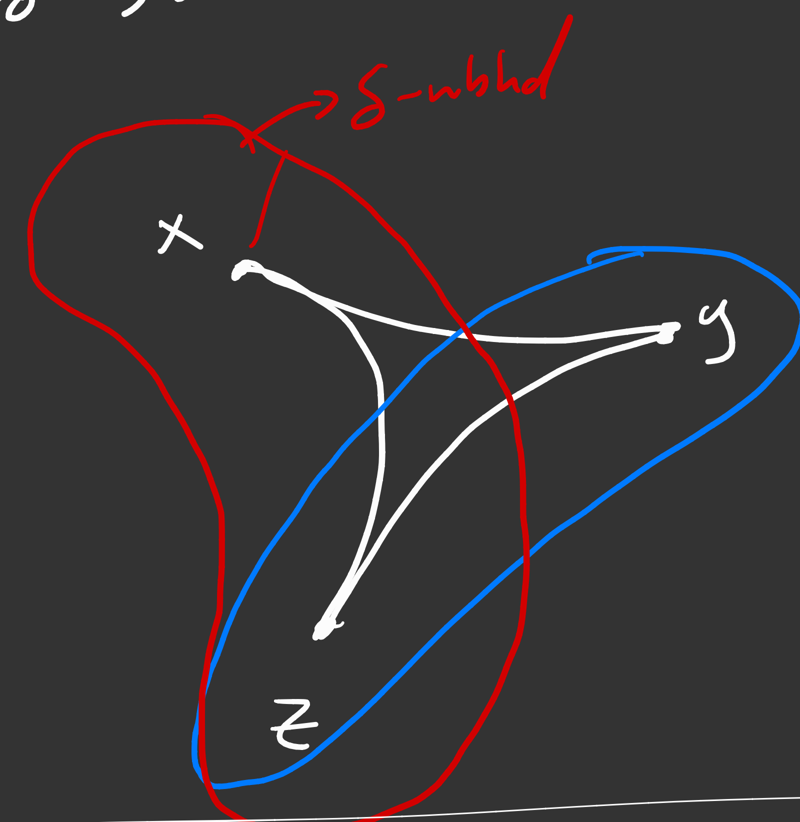
$$d(\alpha(n), \alpha(m)) = |n-m|.$$

There is some path connecting any two vertices.

we may write $[x,y]$ for a geodesic connecting a vertex x to a vertex y.

A geodesic triangle consists of three vertices x, y, z and three geodesics $[x,y]$, $[y,z]$, $[z,x]$

Def: A geodesic triangle $\Delta = [x,y] \cup [y,z] \cup [z,x]$ is δ-slim if the δ-nbhd of any two sides contains the third side.

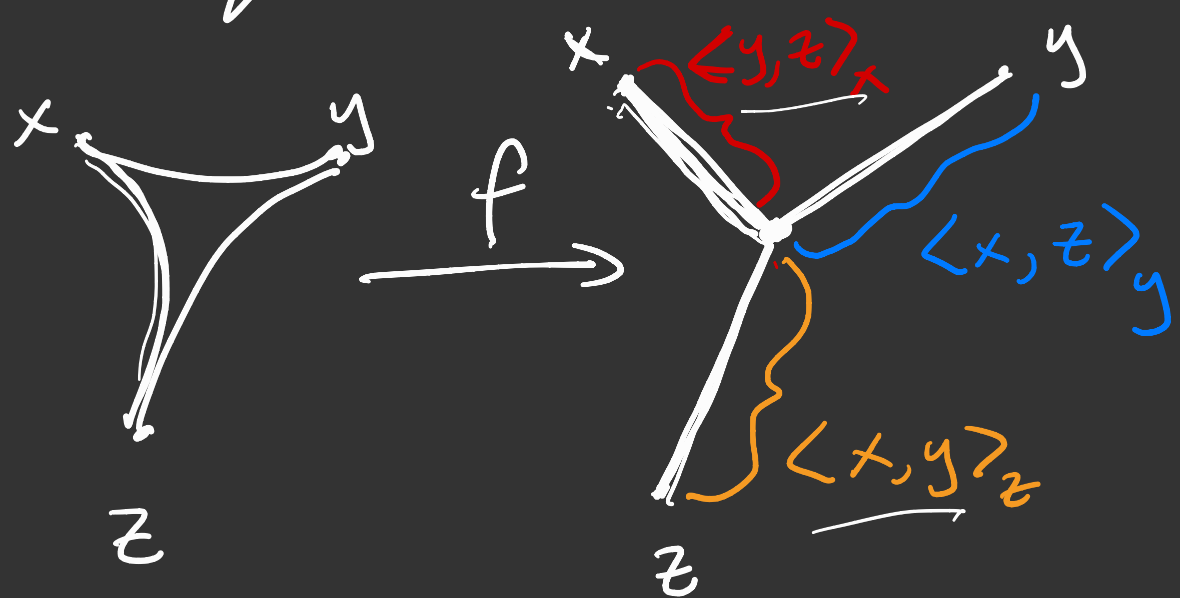


$$N_\delta(A) = \{x \in \Gamma \mid \exists\ a \in A \text{ with } d(x,a) \leq \delta\}.$$

If $x, y, z \in \Gamma$, then the Gromov product is

$$\langle y, z \rangle_x := \frac{1}{2}\left(d(y,x) + d(z,x) - d(y,z)\right)$$

For any geodesic triangle Δ there is a unique tripod T and a unique comparison map $f : \Delta \to T$



st $f|_{[x,y]}$ and $f|_{[y,z]}$ and $f|_{[x,z]}$ are isometries.

where the length of each of the legs of the tripod are given by the Gromov products.

Def: The geodesic triangle Δ is δ-thin if whenever u, v are points on the triangle st $f(u) = f(v)$ then we have

$$d(u,v) \leq \delta.$$

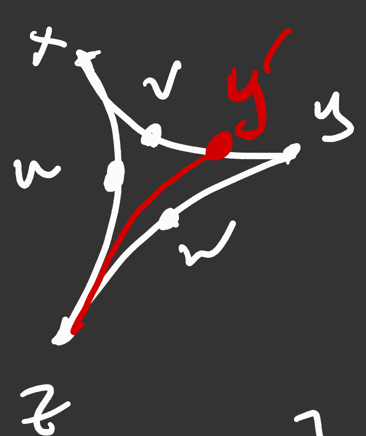
Note: If Δ is δ-thin then it is δ-slim.

Def: Γ is hyperbolic if there exists some $\delta \geq 0$ st every geodesic triangle is δ-slim.

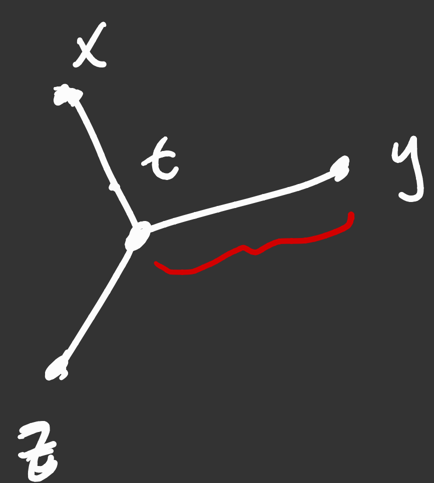
Thm: If every geodesic triangle in Γ is δ-slim then every geodesic triangle in Γ is 4δ-thin.

Proof:

Fix $x, y, z \in \Gamma$ and geodesics $[x,y]$, $[y,z]$, $[z,x]$



$\xrightarrow{\quad f \quad}$



Assume $\begin{matrix} u \in [x,z] \\ v \in [x,y] \end{matrix}$ st $f(u) = f(v)$.

<u>case 1:</u> $f(u) = f(v) = t$.

Since the triangle is δ-slim

$$u \in N_\delta([x,y] \cup [y,z])$$

If $u \in N_\delta([x,y])$ then since $f(u) = f(v)$ we have $d(u,v) < 2\delta$

Similarly if $v \in N_\delta([x,z])$ then $d(u,v) < 2\delta$.

If $u \in N_\delta([y,z])$ and $v \in N_\delta([y,z])$

take $w \in [z,y]$ st $f(w) = t$.

$\therefore d(u,w) < 2\delta$ and $d(v,w) < 2\delta$

$\therefore d(u,v) < 4\delta$.

general case:

Since $f(u) = f(v)$ is on the interval from x to t.

Take $y' \in [x,y]$ st $\underbrace{\langle y', z \rangle_x}_{= d(x,v) = d(x,u)}$

take any geodesic $[z, y']$ take the subpath of $[x,y]$ giving a geodesic from $[x, y']$.

In this new geodesic triangle we have that the new comparison map sends u and v to the triple point.

$\therefore d(u,v) < 4\delta$ by case 1. $\square$

If we have two connected graphs Γ an Γ' then a map

$f : \Gamma \longrightarrow \Gamma'$ is a quasi-isometric embedding if there exists $C > 0$ $r \geq 0$ ST

$$C^{-1} d(x,y) - r \leq d'(f(x), f(x)) \leq C d(x,y) + r.$$

A sequence $\{\alpha(n)\}_n$ is a (C,r)-quasi-geodesic if there exists ST

$$C^{-1} d(\alpha(m), \alpha(n)) - r \leq |m-n| \leq C d(\alpha(m), \alpha(n)) + r.$$

Rmk: If $f : \Gamma \to \Gamma'$ is a quasi-isometric embedding then f takes geodesics in Γ to quasi-geodesics in Γ'.

Prop: If Γ is hyperbolic, $C \geq 1$ $r \geq 0$ then there exists $D > 0$ ST any (C,r)-quasi geodesic α and any geodesic β having the same origin and terminal points, then we have $d_H(\alpha, \beta) < D$

i.e., $\forall$ points on α $\exists$ a point on β with distance at most D away and vice versa.

Cor: If $f:\Gamma\to\Gamma'$ is a quasi-isometric embedding and if Γ' is hyperbolic, then Γ is hyperbolic.

Def: If Γ is a f.g. group with gen set $\underline{S}=S^{-1}$, Γ is hyperbolic if the Cayley graph $Cay_S(\Gamma)$ is hyperbolic.

If $\underline{\tilde{S}}=\tilde{S}^{-1}$ is another generating set then $id:(\Gamma,d_S)\to(\Gamma,d_{\tilde{S}})$ is a quasi-isometric embedding.

Hence being hyperbolic does not depend on the generating set

Ex: $\Gamma=\mathbb{F}_n$ $S=\{a_1,\dots,a_n\}$ free generators.

$Cay_S(\mathbb{F}_n)$ is a tree. hence 0-hyperbolic.

Prop: If Γ is hyperbolic, $C \geq 1$ $r \geq 0$

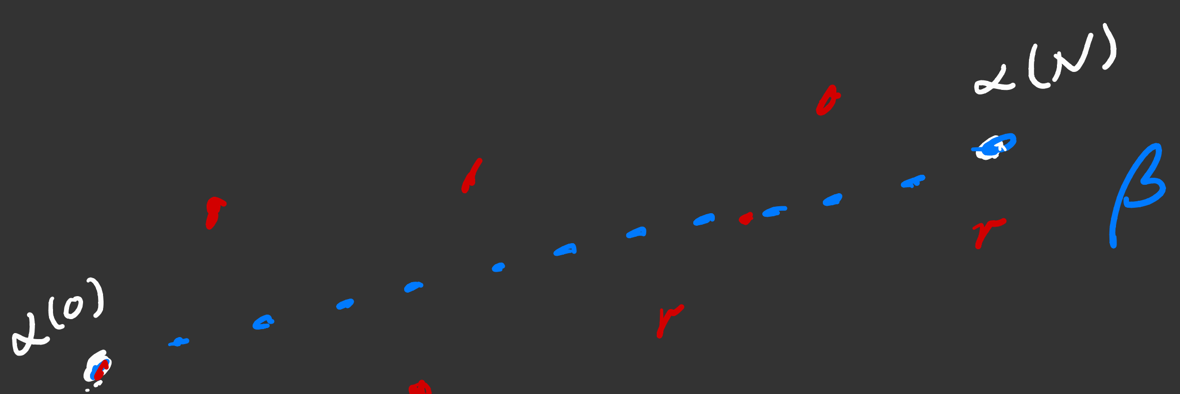
Then there exists $D > 0$ st. any (C,r)-quasi geodesic α and any geodesic β having the same origin and terminal points, then

we have $d_H(\alpha, \beta) < D$

i.e., $\forall$ points on α $\exists$ a point on β with distance at most D away and vice versa.

proof:

Take such an α and β.



let $D_0 = \max\{ d(p, \alpha) : p \text{ is a point on } \beta \}$

Fix ~~If~~ q_0 is some point on α.

For each u on β there is some u' on α st $d(u, u') \leq D_0$. $\boxed{u' = u \text{ if } u \text{ is an endpoint}}$

Either $d(q_0, \beta) \leq D_0$, or else there exist consecutive points u_0 u_1 on β st u_0' is before q_0 and u_1' is after q_0.

$$d(u_0', u_1') \leq 2D_0 + 1$$

the length of the subsequence from u_0' to u_1' in α is at most

$$C(2D_0+1) + r$$

$$\therefore\ d(u_0, q) \leq d(u_0, u_0') + d(u_0', q_0) \leq D_0 + C(C(2D_0+1)+2r).$$

To finish the prop we will find an upper bound for

$$D_0 = \{ d(p,\alpha) \mid p \text{ is a point on } \beta \}$$

Choose p_0 on β st $d(p_0,\alpha) = D_0$

Choose b_0 and b_1 on β st

$$d(p_0, b_0) = d(p_0, b_1) = 2D_0$$

(If we cannot choose such a b_0, b_1 take them to be the end points)

Choose a_0 and a_1 on α st

$$d(a_i, b_i) = d(\alpha, b_i).$$

choose geodesics γ_i connecting b_κ to a_κ, $\kappa = 0, 1$.

let α' be the subsequence of α from a_0 to a_1

Note: $d(p_0, \gamma_i) \geq D_0$

- $d(a_0, a_1) \leq d(a_0, b_0) + d(b_0, b_1) + d(b_1, a_1) \leq D_0 + 4D_0 + D_0 = 6D_0$.

$\therefore$ the length of α' is at most

$$6CD_0 + r$$

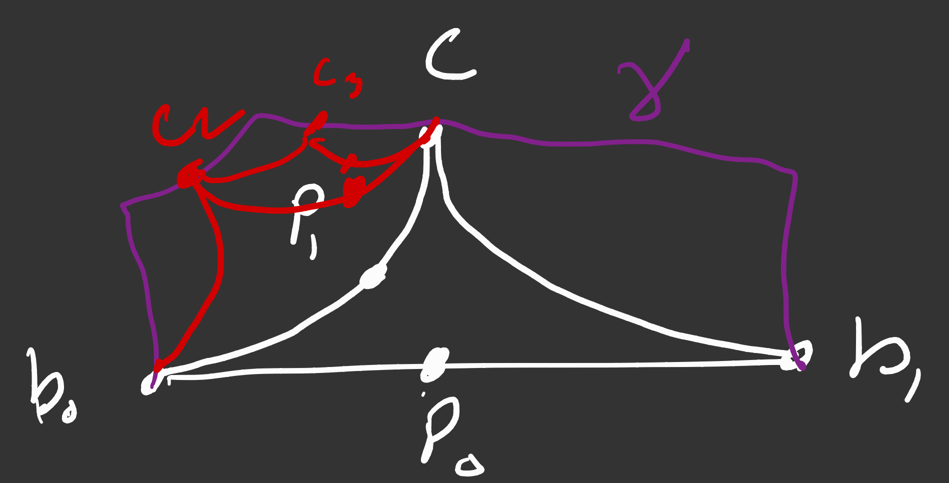
Define γ to be the sequence connecting b_0 to b_1 by taking γ_0 then α' then γ_1

- $d(p_0, \gamma) \geq D_0$
- $\text{length}(\gamma) \leq (6C+2)D_0 + r$
- $d(\gamma(\kappa), \gamma(\kappa+1)) \leq C(1+r)$

- $d(P_0, \gamma) \geq D_0$
- $\text{length}(\gamma) \leq (6C+2)D_0 + r$
- $d(\gamma(k), \gamma(k+1)) \leq \max\{C+r, 1\}$.

Let c be the midpoint (or next to it) of γ.

Consider a geodesic triangle Δ given by $[b_0, c]$, $[c, b_1]$, β



Since Γ is δ-hyperbolic there exists P_1 on either $[b_0, c]$ or $[c, b_1]$ st $d(P_0, P_1) < \delta$

If P_1 is on $[b_0, c]$

set $b_0' = b_0$, $b_1' = c$

otherwise, set $b_0' = c$, and $b_1' = b_1$

Take γ' the subsequence of γ connecting b_0' to b_1'

$$\text{length}(\gamma') \leq \frac{2}{3}\text{length}(\gamma)$$

Repeating this we will terminate after after ℓ steps with

$$\ell \leq \log(\text{length}(\gamma)) / \log(3/2).$$

and P_ℓ is on $[b_0^\ell, b_1^\ell]$ where b_0^ℓ and b_1^ℓ are consecutive terms in γ_ℓ

$\therefore$

$$D_0 \leq d(P_0, \gamma) \leq \ell \cdot \delta + d(b_0^\ell, b_1^\ell)$$

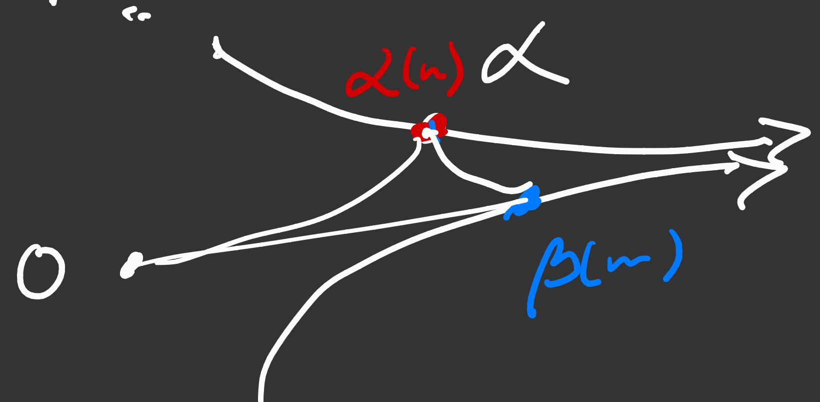
$$\leq \delta \log(3/2)^{-1} \log((6C+2)D_0 + r) + C(1+r)$$

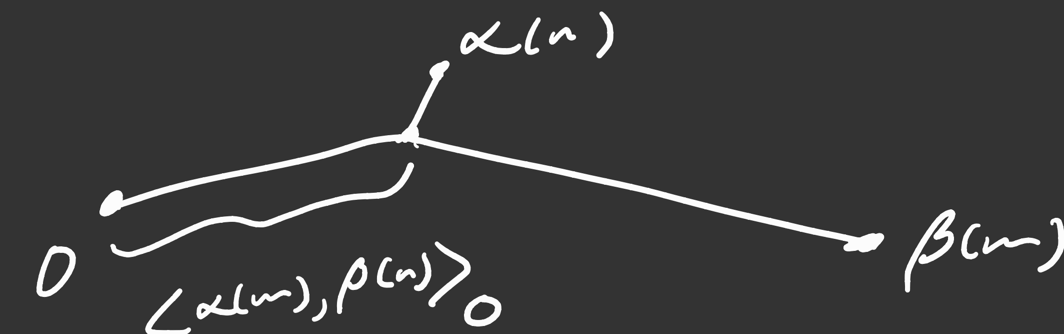
Since linear growth is faster than logarithmic growth, this gives a bound on D_0, in terms of C, r and δ. $\square$

Def: Γ a hyperbolic graph, two infinite geodesics α and β are equivalent if

$$\lim_{n,m\to\infty} \langle \alpha(m), \beta(n) \rangle_0 = \infty$$

where 0 is some fixed point in Γ.





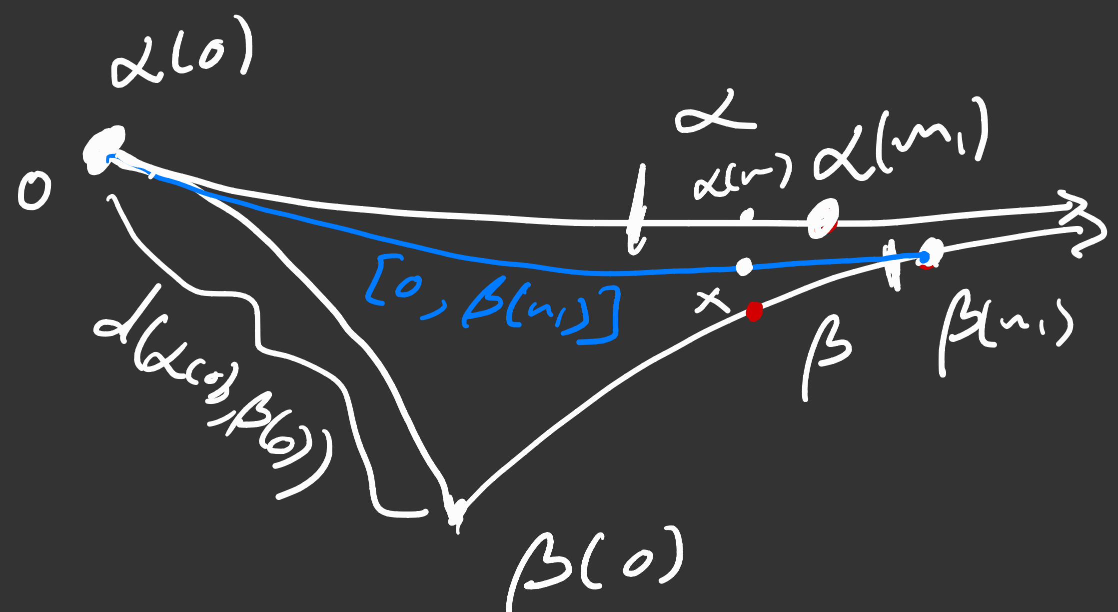
Lemma: There exists $C > 0$ s.t. if α and β are equivalent infinite geodesics and $m \geq d(\alpha(0), \beta(0))$ then there exists n with $|m - n| \leq d(\alpha(0), \beta(0))$ such that $d(\alpha(m), \beta(n)) < C$.

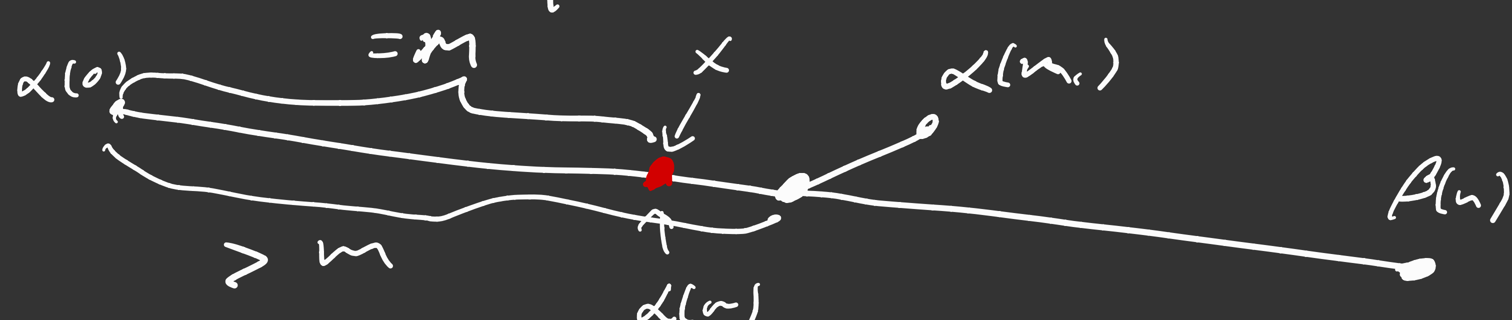
Therefore $\sup_m d(\alpha(m), \beta(m)) < \infty$.

The Gromov boundary of Γ is $\partial\Gamma$ the space of equivalence classes of infinite geodesics.

Lemma: There exists $C > 0$ s.t. if α and β are equivalent infinite geodesics and $m \geq d(\alpha(0), \beta(0)) + 2\delta$ then there exists n with $|m - n| \leq d(\alpha(0), \beta(0))$ such that $d(\alpha(m), \beta(n)) < C$.

Therefore $\underline{\sup_m d(\alpha(m), \beta(m)) < \infty}$.





Proof: Fix $m \geq d(\alpha(0), \beta(0))$

Set $o = \alpha(0)$

Find $m_1, n_1 \in \mathbb{N}$ st

$$\langle \alpha(m_1), \beta(n_1) \rangle_o > m.$$

Choose a geodesic $[o, \beta(n_1)]$

Take x on $[o, \beta(n_1)]$ such that

$d(o, x) = d(o, \alpha(m)) = m$

Because triangles are δ-thin

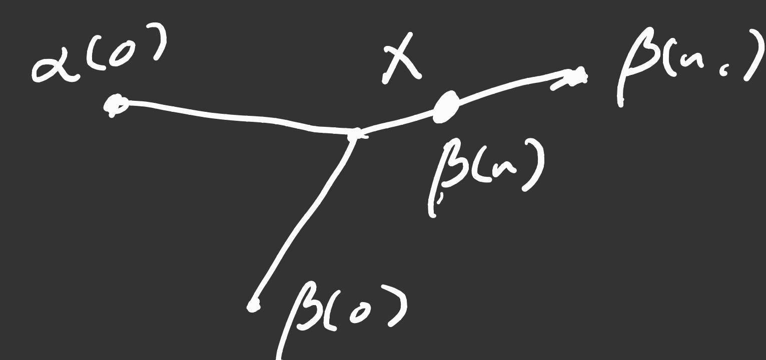
$$d(x, \alpha(m)) < \delta.$$

Take $n < n_1$ such that

$d(x, \beta(n_1)) = d(\beta(n), \beta(n_1))$

Again we have $d(x, \beta(n)) < \delta$

$\therefore d(\alpha(m), \beta(n)) < 2\delta$



$|n - m| \leq d(\alpha(0), \beta(0)) + 2\delta$

Conversely If $\sup_m d(\alpha(m), \beta(m)) < \infty$.

$$\Rightarrow d_H(\alpha, \beta) < \infty.$$

$n < m$

$$2\langle \alpha(m), \beta(n) \rangle_0 = d(\alpha(m), 0) + d(\beta(n), 0) - d(\alpha(m), \beta(n))$$

$$\geq m - d(\alpha(0), 0) + n - d(\beta(0), 0) - ((m-n) + d(\alpha(m), \beta(m)))$$

$$\geq 2n - \left(d(\alpha(0), 0) + d(\beta(0), 0) + d_H(\alpha, \beta)\right)$$

$$\xrightarrow[n,m]{} \infty. \qquad \square$$

Gromov boundary $\partial\Gamma$ is the set of equivalence classes of infinite geodesics.

$\overline{\Gamma} = \Gamma \cup \partial\Gamma$,

if a geodesic α represents $z \in \partial\Gamma$ write $\alpha_+ = z$.

Fix a base point $0 \in \Gamma$. For $z \in \partial\Gamma$ and $R > 0$ set

$$U(z, R) = \{ x \in \overline{\Gamma} \mid \exists \text{ geodesics } \alpha, \beta \text{ with } \alpha_+ = x,\ \beta_+ = z \text{ and } \lim_{m,n\to\infty} \langle \alpha(m), \beta(n) \rangle_0 > R \}.$$

$$U'(z, R) = \{ x \in \overline{\Gamma} \mid \forall \text{ geodesics } \alpha, \beta \text{ with } \alpha_+ = x,\ \beta_+ = z\ \lim_{m,n\to\infty} \langle \alpha(m), \beta(n) \rangle_0 > R \}.$$

$$U'(z, R) \subset U(z, R).$$

$$U(z, R) \subset U'(z, R - C).$$

Lemma: There exists $C \geq 0$ st

if α, α' β, β' are geodesics

with $\alpha_+ = \alpha'_+$ and $\beta_+ = \beta'_+$

then

$$\varliminf_{n,m \to \infty} \langle \alpha'(m), \beta'(n) \rangle_0$$
$$\geq \varliminf_{n,m \to \infty} \langle \alpha(m), \beta(n) \rangle_0 - \underline{C}.$$

Proof:

$$\langle \alpha'(m'), \beta'(n') \rangle_0$$
$$\geq \langle \alpha(m), \beta(n) \rangle_0$$
$$- \Big(d(\alpha'(m'), \alpha(m)) + d(\beta'(n'), \beta(n)) \Big)$$
$$\geq \langle \alpha(m), \beta(n) \rangle_0 - 2C$$

where C is from the previous lemma $\square$

Lemma: $\forall R \geq 0$ $\exists$ $S > 0$ st.

if $y, z \in \partial\Gamma$ with $y \in U(z, S)$

then $\underline{U(y, S)} \subset U(z, R)$.

Proof:

It suffices to show that if $N \geq 0$

and $\underline{y \in U'(z, N)}$ and $\underline{x \in U'(y, N)}$

then $\underline{x \in U(z, \underline{N - \delta})}$.

The geodesic paths $\underline{\alpha}, \beta, \gamma$

connecting 0 to $\underline{x}, y, \underline{z}$ respectively

$$\varliminf_{n,m \to \infty} \langle \gamma(\underline{n}), \beta(\underline{m}) \rangle_0 > N \quad = 2N + 2\delta$$

then $d(\gamma(N), \beta(N)) < \delta$

similarly $d(\alpha(N), \beta(N)) < \delta$

$\therefore$ $d(\gamma(N), \alpha(N)) < 2\delta$

$\therefore$ $2\langle \alpha(m), \gamma(n) \rangle_0 = m + n - d(\alpha(m), \gamma(n))$

$\geq m + n - (m - \underline{N} + d(\underline{\alpha(N), \gamma(N)}) + n - \underline{N})$

Def: We equip $\overline{\Gamma}$ with a topology by declaring a subset $O \subset \overline{\Gamma}$ to open iff $\forall z \in \partial\Gamma \cap O$ $\exists R > 0$ st $U(z,R) \subset O$.

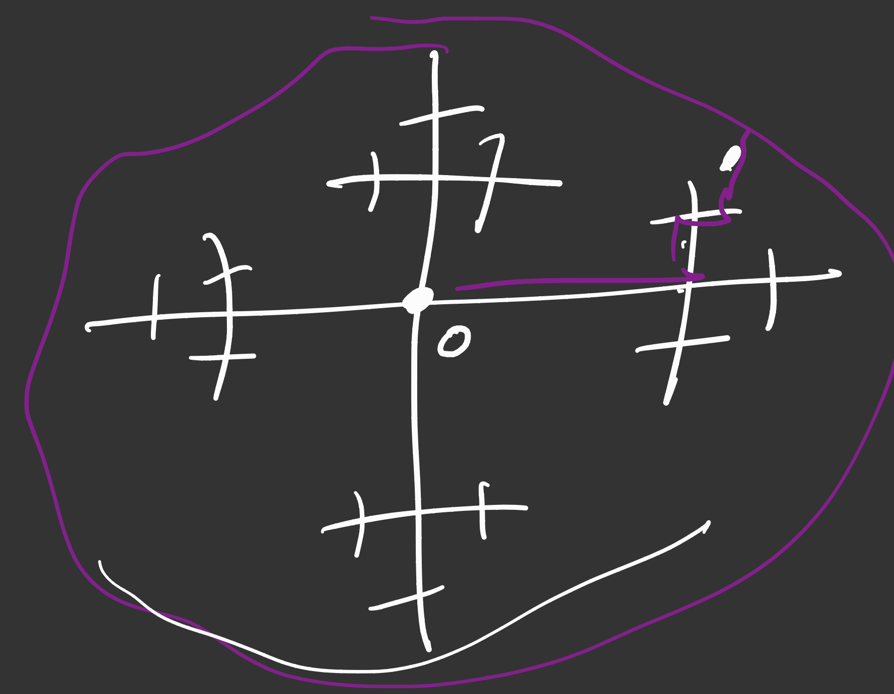
i.e. $U(z,R)$ are all neighborhoods of z.

Note: Γ is open and dense in $\overline{\Gamma}$.

This topology is Hausdorff.

Ex: consider T a tree,

e.g. $\mathrm{Cay}(\mathbb{F}_2, \{a,b\})$



Thm: If the graph Γ satisfies the property that balls are finite, then $\partial\Gamma$ is compact.

Pf: Fix $o \in \Gamma$. The topology on $\overline{\Gamma}$ is separable, so to see that $\partial\Gamma$ is compact it suffices to show that every sequence has a convergent subsequence.

Take $\{x_n\}_{n=1}^{\infty}$ a sequence in $\partial\Gamma$ take α_n geodesics from o st $\alpha_{n+} = x_n$. Since $B(1,o)$ is finite

$\exists\ a_1\ d(a_1, o) = 1$ st

$a_1 = \alpha_n(1)$ for a subsequence

$\{\alpha_{n_k}\}_{k=1}^{\infty}$ There exists a_2 $d(a_2, a_1) = 1$

st $a_2 = \alpha_{n_k}(2)$ for ∞ many k.

In this way we construct an

∞ geodesic $\{a_n\}_n$,

and this is clearly an accumulation

point of $\{\alpha_n\}_{n=1}^{\infty}$. $\square$

Ex: Tree T st. each vertex has

countably infinitely many edges

attached.

Then ∂T is not compact.

If Γ a connected graph γ an ∞ geodesic

and $o \in \Gamma$ then $\exists$ an

∞ geodesic α with $\alpha(0) = o$

such that $\alpha(n)$ is on γ for

n large.

Consider $d(o, \gamma(n)) - n$

Fix n where it achieves its min

consider $\alpha = [o, \gamma(n)] \cup [\gamma(n), \gamma_+]$.

Ex: $\operatorname{cay}(\mathbb{F}_\infty, \{a_1, a_2, \dots\})$.

Γ hyperbolic graph.

$\partial\Gamma$ = equivalence classes of infinite geodesics.

$$\overline{\Gamma} = \Gamma \cup \partial\Gamma$$

Thm (Tukia '94) If Γ is an infinite hyperbolic group then $\Gamma \curvearrowright \overline{\Gamma}$ is a convergence action.

Proof:

Take $t_n \in \Gamma$ st $t_n \to \infty$.

Fix $o \in \Gamma$. After passing to a subsequence, assume $\underline{t_n \cdot o} \to \underline{a} \in \partial\Gamma$ and $\underline{t_n^{-1} \cdot o} \to \underline{b} \in \partial\Gamma$

Fix $R > 0$ we consider $U(b,R)$

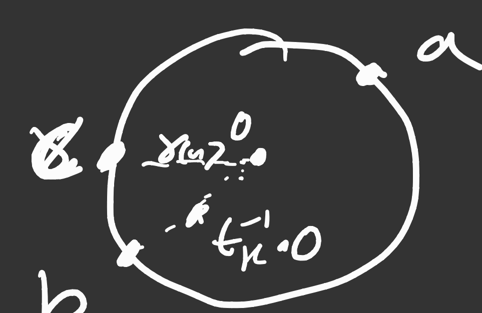
$$= \{ x \in \overline{\Gamma} \mid \exists \text{ geodesics } \alpha, \beta \text{ with } \alpha_+ = x \ \ \beta_+ = b \text{ and } \lim_{n,m\to\infty} \langle \alpha(n), \beta(m) \rangle_o \geq R \}.$$

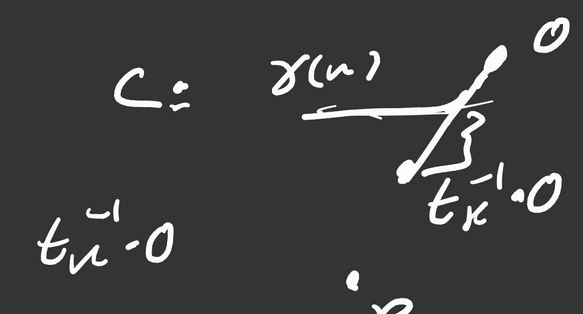
If $c \notin U(b,\underline{R})$, and α is a geodesic st $\alpha_+ = a$, $\gamma_+ = c$ then

$$\lim_{n,m\to\infty} \langle \alpha(m), t_k \cdot \gamma(n) \rangle$$

$$\geq \min\{ \underline{\lim_{m\to\infty} \langle \alpha(m), t_k \cdot o \rangle_o}, \lim_{n\to\infty} \langle \underline{t_k \cdot o, t_k \cdot \gamma(n)} \rangle_o \} - \delta.$$

$$\lim_{n\to\infty} \langle t_k \cdot o, t_k \cdot \gamma(n) \rangle_o = \lim_{n\to\infty} \underline{\langle o, \gamma(n) \rangle_{t_k^{-1} \cdot o}} \xrightarrow{k\to\infty} \infty$$





$\square$

Def: If Γ is a hyperbolic graph and $t \in Isom(\Gamma)$ then t is loxodromic if the mapping $\mathbb{Z} \ni t^k \longmapsto t^k \cdot o$ is a quasi-isometric embedding.

Thm: If $t, s \in Isom(\Gamma)$ are loxodromic such that their 4 boundary limits are distinct, then for n, m large enough we have that $\underline{\langle t^n, s^m \rangle \text{ is free} \cong \mathbb{F}_2}$ and

$$\langle t^n, s^m \rangle \ni w \longmapsto w \cdot o \in \Gamma$$

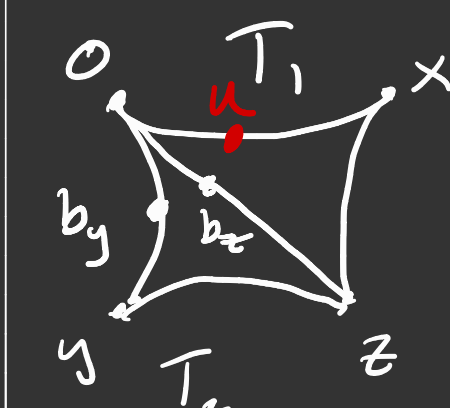
is a quasi-isometric embedding.

in particular we have $\partial \mathbb{F}_2 \subseteq \partial \Gamma$ in this case.

Lemma: If Γ is δ-hyperbolic (triangles are δ-thin) then $\forall\ o, x, y, z \in \Gamma$ we have

$$d(x,y) + d(o,z) \leq \max\big(d(x,o) + d(y,z),\ d(x,z) + d(y,o)\big) + 2\delta.$$

Proof: Assume $\underline{\langle y, z \rangle_o \leq \langle x, z \rangle_o}$



T_1: geodesic triangle with vertices o, x, z

T_2: geodesic triangle " o, y, z

" geodesic from o to z coinciding

Take $b_y \in [o, y]$ $b_z \in [o, z]$ st b_y and b_z map to the triple point of the comparison tripod for T_2, i.e. $d(o, b_y) = \langle y, z \rangle_o = d(o, b_z)$.

Take $u \in [o, x]$ st $d(o,u) = \langle y, z\rangle_o$

then $d(b_y, u) < d(b_y, b_z) + d(b_z, u) < 2\delta$.

$\therefore$

$$
\begin{aligned}
d(x,y) &\le d(x,u) + 2\delta + d(b_y, y) \\
&= d(o,x) - \underbrace{d(o,u)} + 2\delta + d(o,y) - \underbrace{d(o,b_y)} \\
&= d(o,x) + d(o,y) + 2\delta - \underbrace{2\langle y,z\rangle_o}_{d(y,o)+d(z,o)-d(y,z)}
\end{aligned}
$$

$\therefore d(x,y) + d(o,z) \le \{d(o,x) + d(y,z)\} + 2\delta$ $\square$

Lemma: If $\{x_j\}_{j=1}$ is a sequence in Γ such that

$(*)$ $d(x_{n+2}, x_n) \ge \max(d(x_{n+2}, x_{n+1}), d(x_{n+1}, x_n)) + 2\delta + 1$,

then $(**)$ $d(x_n, x_p) \ge |n-p|$ for all n, p.

Proof: (by induction on $K = |n-p|$).

$$
d(x_{n+2}, x_n) \le \max(d(x_{n+2}, x_{n+1}), d(x_{n+1}, x_n)) + \min(d(x_{n+2}, x_{n+1}), d(x_{n+1}, x_n))
$$

$\therefore \min(d(x_{n+2}, x_{n+1}), d(x_{n+1}, x_n)) \ge 2\delta + 1$ by $(*)$.

Suppose $K \ge 2$ and $(**)$ holds whenever $|n-p| < K$

we use δ-hyperbolicity from the previous lemma applied to

$(x_n, x_{n+2}, x_{n+1}, x_{n+1+K})$

Lemma: If $\{x_j\}_{j=1}$ is a sequence in Γ such that

$$(*) \quad d(x_{n+2},x_n) \ge \max\big(d(x_{n+2},x_{n+1}),\ d(x_{n+1},x_n)\big) + 2\delta + A$$

Then $(**)$ $d(x_n,x_p) \ge |n-p|\cdot A$ for all n,p.

Proof: (by induction on $k=|n-p|$).

$$d(x_{n+2},x_n) \le \max\big(d(x_{n+2},x_{n+1}),\ d(x_{n+1},x_n)\big) + \min\big(d(x_{n+2},x_{n+1}),\ d(x_{n+1},x_n)\big)$$

$\therefore \min\big(d(x_{n+2},x_{n+1}),\ d(x_{n+1},x_n)\big) \ge 2\delta + A$ by $(*)$.

Suppose $k\ge 1$ and $(**)$ holds whenever $|n-p|\le k$

we use δ-hyperbolicity from the previous lemma applied to

$$(x_n,\ x_{n+2},\ x_{n+1},\ x_{n+1+k})$$

Then

$$d(x_n,x_{n+2}) + d(x_{n+1},x_{n+1+k}) \le \max\big\{d(x_n,x_{n+1}) + d(x_{n+2},x_{n+1+k}),\ d(x_n,x_{n+1+k}) + d(x_{n+2},x_{n+1})\big\} + 2\delta$$

Note: $d(x_n,x_{n+2}) + d(x_{n+1},x_{n+k+1}) \ge d(x_n,x_{n+1}) + 2\delta + 1 + d(x_{n+2},x_{n+k+1}) + A$

$$\therefore \boxed{d(x_n,x_{n+2}) + d(x_{n+1},x_{n+1+k})} \le d(x_n,x_{n+1+k}) + d(x_{n+2},x_{n+1}) + 2\delta$$

$$\ge d(x_{n+2},x_{n+1}) + 2\delta + A + d(x_{n+1},x_{n+1+k})$$

$$\therefore d(x_n,x_{n+1+k}) \ge d(x_{n+1},x_{n+1+k}) + 2\delta + A$$

$(***)$: If

$d(x_k,x_p) \ge d(x_{k-1},x_p) + A$

$d(x_k,x_p) \ge d(x_k,x_{p-1}) + A$

holds for $k-p$

Thm: If $t, s \in \mathrm{Isom}(\Gamma)$ are loxodromic such that their 4 boundary limits are distinct, then for n, m large enough we have that

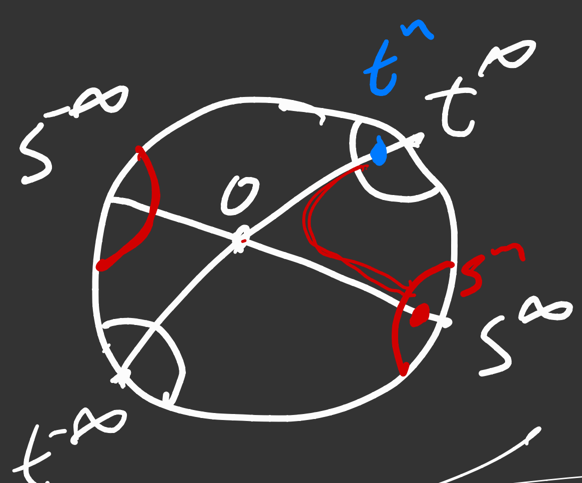
$\underline{\langle t^n, s^m \rangle \text{ is free} \simeq \mathbb{F}_2}$

and

$\langle t^n, s^m \rangle \ni w \longmapsto w \cdot 0 \in \Gamma$

is a quasi-isometric embedding.

Proof:



For n, m large we have

$$\underline{d(t^{2n} \cdot 0, 0) - d(t^n \cdot 0, 0)} \geq \frac{n}{c} - r \xrightarrow{n \to \infty} \infty$$

$$\begin{cases} d(t^{\pm 2n} \cdot 0, 0) \geq d(t^{\pm n} \cdot 0, 0) + 2\delta + 1 \\ d(s^{\pm 2m} \cdot 0, 0) \geq d(s^{\pm m} \cdot 0, 0) + 2\delta + 1 \end{cases}$$

$$d(t^{\pm n} \cdot 0, s^{\pm m} \cdot 0)$$
$$\geq \max\left(d(t^{\pm n} \cdot 0, 0), d(s^{\pm m} \cdot 0, 0)\right) + 2\delta + 1.$$

$\therefore$ If w is any reduced word in t^n, s^m then we may apply the previous lemma to this word to conclude

$$d(w \cdot 0, 0) \geq |w|_{\mathbb{F}_2} \neq 0$$

$$\leq |w|_{\mathbb{F}_2} \max\left(d(t^n \cdot 0, 0), d(s^m \cdot 0, 0)\right)$$

$\therefore$ the embedding

$\mathbb{F}_2 \simeq \langle t^n, s^m \rangle \ni w \longmapsto w \cdot 0$ is a quasi-isometry.

Bounded Cohomology

Γ a group, B a Banach space

$\Gamma \curvearrowright B$ Isometric action.

$$B \xrightarrow{\delta^1} \underline{\ell^\infty}(\Gamma; B) \xrightarrow{\delta^2} \underline{\ell^\infty}(\Gamma \times \Gamma; B) \to \cdots$$

$\delta^k(f)(t_1, \ldots, t_k) =$

$t_1 f(t_2, t_3, \ldots, t_k)$

$- f(t_1 t_2, t_3, \ldots, t_k)$

$+ f(t_1, t_2 t_3, t_4, \ldots, t_k)$

$\cdots + (-1)^{k-1} f(t_1, t_2, \ldots, t_{k-1} t_k)$

$+ (-1)^k f(t_1, t_2, \ldots, t_{k-1})$

Obs! $\delta^k \delta^{k-1} \equiv 0$.

$$H_b^k(\Gamma; B) := \ker(\delta^{k+1}) / \operatorname{Im}(\delta^k) = \frac{Z_b^k(\Gamma; B)}{B_b^k(\Gamma; B)}.$$

Ex: $c \in \ker(\delta^2)$ iff

$0 = \delta^2(c)(s,t)$

$= s \cdot c(t) - c(st) + c(s)$

i.e., $c(st) = s \cdot c(t) + c(s)$ $\forall\, s,t \in \Gamma$.

If $b \in B$

$(\delta^1 b)(t) = t \cdot b - b$ inner.

$H_b^1(\Gamma; B)$ bounded 1-cocycles / inner-cocycles.

Note: If B is a Hilbert space.

$$H_b^1(\Gamma; B) = \{0\}.$$

Def: If $\mathbb{B}$ is a Banach Γ-module, then we say that this is a dual Banach Γ-module if $\mathbb{B} \simeq (\mathbb{B}_*)^*$ and if the action of Γ on $\mathbb{B}$ is dual to the action of Γ on $\mathbb{B}_*$.

Thm: Γ is amenable iff for every dual Banach Γ-module $\mathbb{B}$ we have $H_b^1(\Gamma; \mathbb{B}) = \{0\}$.

Proof: ($\Leftarrow$)

consider $\underline{\mathbb{B} = \{\varphi \in (\ell^\infty\Gamma)^* \mid \varphi(1) = 0\}}$. A dual Banach Γ-module.

define a cocycle $c : \Gamma \to \mathbb{B}$ by

$$c(t) = \delta_{\{t\}} - \delta_{\{e\}} \in \mathbb{B}$$
$$= t \cdot \delta_{\{e\}} - \delta_{\{e\}}$$

If this is inner, then there exists some $\varphi \in \mathbb{B}$ st

$$c(t) = t \cdot \varphi - \varphi$$

$\therefore\ \varphi - \delta_{\{e\}} \in \ell^\infty\Gamma^*$ is fixed by Γ
$0 \neq$

Taking the real (or imaginary) part we get a symmetric Γ-inv lin functional that is non-zero.

Taking Hahn-decomposition gives a non-zero Γ-inv pos linear function, showing Γ is amenable.
non-zero

($\Rightarrow$)

Suppose now that Γ is amenable and $c:\Gamma\to B$ is a bounded cocycle into a dual Banach Γ-module.

Set $C = \overline{\mathrm{co}}^{wk^*}\{c(t) : t\in\Gamma\}$, bdd. Hence wk^*-cpt.

We define an wk^*-continuous affine action of Γ on C by

$$s\cdot\xi = s\xi + c(s)$$

$$s\cdot c(t) = sc(t) + c(s) = c(st)\in C$$

$\therefore$ this action maps C into C.

Since Γ is amenable there is a fixed point η, then

$$\eta = s\cdot\eta = s\eta + c(s),\ \text{ie } c(s) = \eta - s\eta$$

$\square$.

Remark: It is unknown if $H_b^n(\mathbb{F}_2,\mathbb{R})\neq\{0\}$ for any $n\geq 4$.

Def: B a Banach Γ-module. A map $q:\Gamma\to B$ is a quasi-cocycle if there exists $C>0$ st $\forall\, s,t\in\Gamma$ we have

$$\|\underbrace{q(st)-(sq(t)+q(s))}_{\delta^1(q)(s,t)}\| < C.$$

$\delta^2\delta^1(q) = 0$.

$\therefore\ \delta^1(q)\in Z_b^k(\Gamma, B)$.

If $\delta^1(q)$ is a coboundary then $\delta^1(q) = \delta^1(\tilde q)$ where $\tilde q$ is bounded.

ie $(q-\tilde q)$ is a (perhaps unbounded) 1-cocycle

$\therefore\ \widetilde{QC} = \{\text{quasi-cocycles}\} \big/ \ell^\infty(\Gamma; B) + \text{Cocycles}$

$\hookrightarrow H^2_b(\Gamma, B).$

Homogeneous chain complex:

B a Banach Γ-module

$\Gamma \curvearrowright X$ X is a set (or a measure-space, locally cpt)

$$B^\Gamma \xrightarrow{\partial^0} \ell^\infty(X; B)^\Gamma \xrightarrow{\partial^1} \ell^\infty(X \times X; B)^\Gamma \xrightarrow{\partial^2} \cdots$$

where

$\underline{\ell^\infty(X^k; B)^\Gamma}$

$= \{ f \in \ell^\infty(X^k; B)$ st $f(tx_1, tx_2, \ldots, tx_k) = t f(x_1, x_2, \ldots, x_k) \}$

$$(\partial^k f)(x_0, x_1, \ldots, x_k) = f(x_1, x_2, \ldots, x_k) - f(x_0, x_2, \ldots, x_k) + \cdots + (-1)^k f(x_0, \ldots, x_{k-1})$$

$$= \sum_{i=0}^{k} (-1)^i f(x_0, \ldots, \hat{x}_i, \ldots, x_k).$$

$\partial^k \partial^{k-1} \equiv 0$

$$H^n_b(\underline{\Gamma \curvearrowright X}; B) = \ker(\partial^{n+1}) \big/ \mathrm{Im}(\partial^n)$$

Ex: $X = \Gamma$ with left multiplication,

then

$$H^n_b(\Gamma \curvearrowright \Gamma; B) \cong H^n_b(\Gamma; B).$$

Homogeneous

$\ell^\infty(\Gamma^{n+1}; B)^\Gamma$

$\check{f} \longmapsto \hat{f}(x_1, x_2, \ldots, x_n)$

$= f(1, x_1, x_1x_2, x_1x_2x_3, \ldots, x_1x_2\cdots x_n)$

$\tilde{g} \longleftarrow g$

$\hat{g}(x_0, x_1, \ldots, x_n)$

$= x_0 \cdot g(x_0^{-1}x_1, x_1^{-1}x_2, x_2^{-1}x_3, \ldots, x_{n-1}^{-1}x_n)$.

Inhomogeneous

$\ell^\infty(\Gamma^n; B)$

$\delta\hat{f} = \widehat{\partial f}$

and $\partial\tilde{g} = \widetilde{\delta g}$.

---

will show: $H_b^2(\mathbb{F}_2; \ell^\infty \mathbb{F}_2) \neq \{0\}$.

Ex: (Brooks '78) Fix a reduced word w in $\mathbb{F}_2$, $|w| \geq 3$.

define $q(t) := \#\, w$ occurs as a substring of t $- \#\, w$ occurs as a substring of t^{-1}.

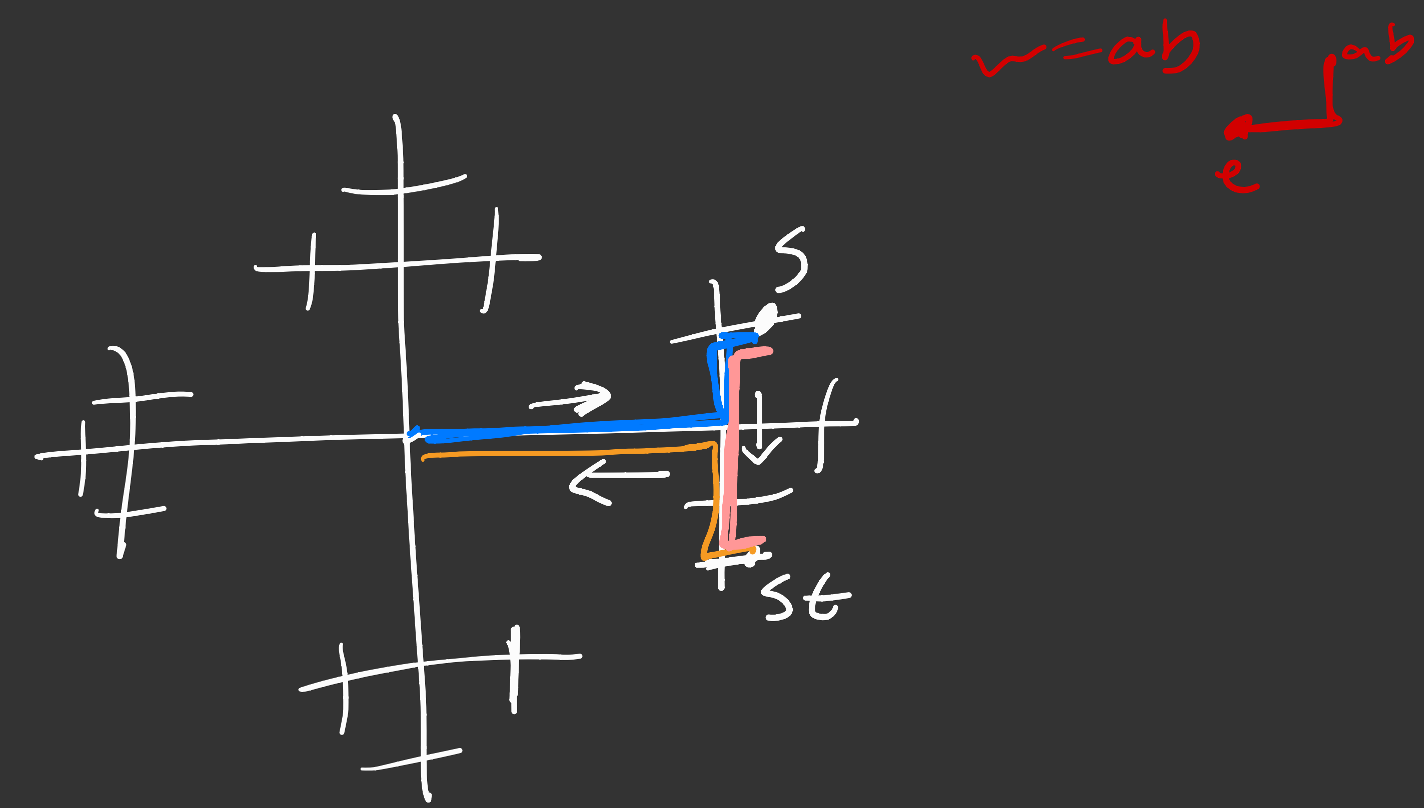
$$q_w((w w_0)^n) \xrightarrow[n\to\infty]{} \infty$$

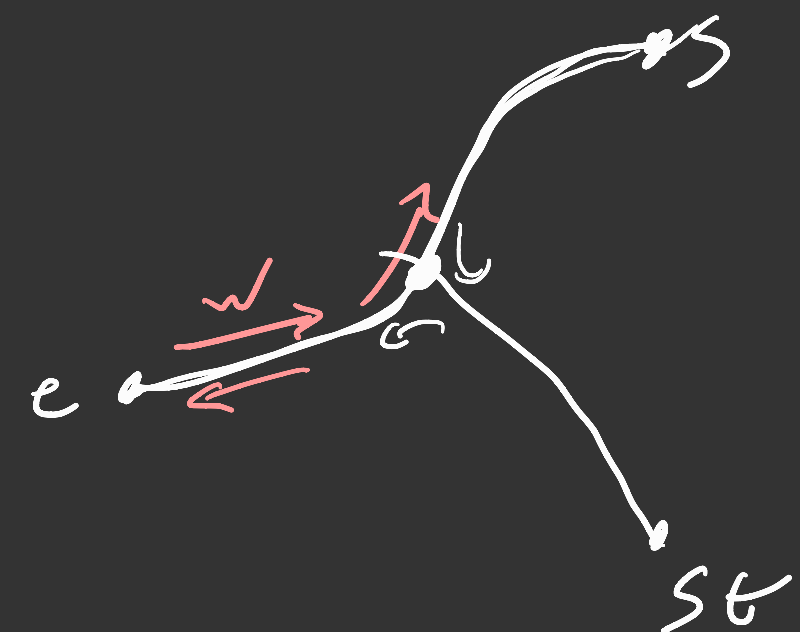
Ex: $w = abab$
$$q(w^n) = 2n-1 \xrightarrow[n\to\infty]{} \infty$$

This is a quasi-~~cocycle~~ morphism

$$\underbrace{|q(s) - q(st) + q(t)|}$$
$$\leq 6(|w|-1)$$

$\therefore q = q_w$ is a quasi-cocycle





Ex: $w = abab$. claim: q_w is not a bounded distance away from a homomorphism. If not $\Theta : \mathbb{F}_2 \to \mathbb{R}$ a homomorphism st $|q_w(t) - \Theta(t)| \leq K$

for $w = abab$ then $q_w(a^n) = q_w(b^n) = 0$

$\therefore |\Theta(a^n)|, |\Theta(b^n)| \le K$

$\therefore \Theta(a) = \Theta(b) = 0 \Rightarrow \Theta \equiv 0$

$\Rightarrow q_w$ is bounded.

$\therefore q_w$ represents a nontrivial class in $\widetilde{QC}(\Gamma;\mathbb{R}) \hookrightarrow H^2_b(\Gamma;\mathbb{R})$

$\therefore H^2_b(\mathbb{F}_2;\mathbb{R}) \neq \{0\}$.

From the homogeneous perspective.

If w is a reduced word with $|w| \geq 1$ we consider the map

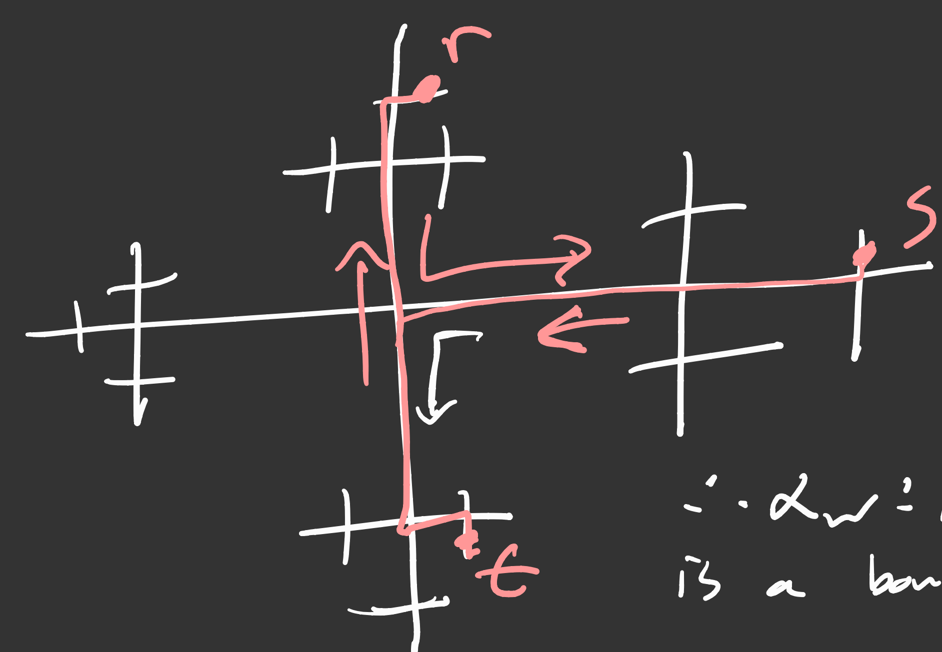
$$\xi_w : (\mathbb{F}_2)^2 \to \ell^\infty \mathcal{E}_{|w|}$$

where $\mathcal{E}_{|w|} = \{\text{geodesics of length } |w|\}$.

by
$$\xi_w(s,t)(\gamma) = \begin{cases} 1 & \text{if } \gamma \text{ is a translate of } w \text{ and } \gamma \subset [s,t] \\ -1 & \text{if } \gamma \text{ is a translate of } w \text{ and } \gamma \subset [t,s] \\ 0 & \text{otherwise} \end{cases}$$

this is $\Gamma = \mathbb{F}_2$ equivariant.

$$\alpha_w = (\partial \xi_w)(r,s,t) = \xi_w(s,t) - \xi_w(r,t) + \xi_w(r,s).$$



$\|\alpha_w(r,s,t)\|_1 \le 6(|w|-1)$.

$\therefore \alpha_w : \Gamma^3 \to \ell^1 \mathcal{E}_{|w|}$ is a bounded cocycle.

The Brooks cocycle is just the composition

$$\alpha_w : \Gamma^3 \longrightarrow \underline{\ell^1 \Sigma_{|w|}} \xrightarrow[\text{sum the coefficients}]{} \mathbb{R}.$$

Note: α_w extend continuously to a separately continuous bounded cocycle

$$\alpha_w : \overline{\Gamma}^3 \longrightarrow \ell^1 \Sigma_{|w|}.$$

<u>Bestvina–Bromberg–Fujiwara</u> (2014).

Thm: (BBF '14) If $|w| \geq 3$ then $\underline{\alpha_w : (\mathbb{F}_2)^3 \longrightarrow \ell^2 \Sigma_{|w|}}$ is a non-trivial bounded cocycle.

Proof: Consider the corresponding quasi-cocycle

$$q : \mathbb{F}_2 \to \ell^1 \Sigma_{|w|} \quad \text{by}$$

$$q(s)(\gamma) = \begin{cases} 1 & \text{if } \gamma \subset [e,s] \\ -1 & \text{if } \gamma \subset [s,e] \\ 0 & \text{otherwise, or if } \gamma \text{ is not a translate of } w. \end{cases}$$

$$\| q(s) - q(ts) + t \cdot q(s) \|_1 \leq 6(|w|-1).$$

Note that q is unbounded, again by finding explicit elements.

Suppose $c : \mathbb{F}_2 \to \ell^2 \Sigma_{|w|}$ is a 1-cocycle such that

$$\| \underline{q(s) - c(s)} \|_2 \leq K$$

Step 1: Find two subgroups $\Gamma_1, \Gamma_2 < \mathbb{F}_2$ such that $q|_{\Gamma_i} \equiv 0$ and $|\Gamma_1 \cap \Gamma_2| = \infty$.

$\mathbb{F}_2 = \langle \Gamma_1, \Gamma_2 \rangle$

Ex: If $w \neq a^n b^m$ or $b^n a^m$ $n,m \in \mathbb{Z}$.

Then take $\Gamma_1 = \langle a^p, b \rangle$

$\Gamma_2 = \langle a^q, b \rangle$ where p and q are distinct primes such that p & q are larger than any occurrence of powers of a in w.

$\therefore q_w|_{\Gamma_1} \equiv 0 \equiv q_w|_{\Gamma_2}$.

Step 2: $c|_{\Gamma_i}$ is bounded

Hence there exist $\xi_1, \xi_2 \in \ell^2 \mathcal{E}_{|w|}$

such that

$c(s) = \xi_i - s\xi_i$ for $s \in \Gamma_i$

Remark: $\Gamma \curvearrowright \ell^2 \mathcal{E}_{|w|} \cong \bigoplus \ell^2 \Gamma$

$\boxed{w = abab}$ For a fixed word δ_s

$\alpha : \Gamma^3 \to \ell^\infty \{ s[e,w] \} \simeq \ell \mathbb{F}_2$

If $\xi, \eta \in \ell^2 \Gamma^{\oplus}$ then

$\langle \pi(s)\xi, \eta \rangle \xrightarrow[s \to \infty]{} 0$

Note if $s \in \Gamma_1 \cap \Gamma_2$ then

$\xi_1 - s\xi_1 = c(s) = \xi_2 - s\xi_2$

$\therefore (\xi_1 - \xi_2) = s(\xi_1 - \xi_2)$

$\therefore \xi_1 = \xi_2$

$\therefore c(s) = \xi_1 - s\xi_1 \quad \forall s \in \langle \Gamma_1, \Gamma_2 \rangle = \mathbb{F}_2$.

$\Longrightarrow q_w$ is bounded giving a contradiction. $\square$

In particular this shows that $\therefore \underbrace{\widetilde{QC}(\mathbb{F}_2, \ell^2\mathbb{F}_2)}_{\neq 0}$

$$\boxed{H_b^2(\mathbb{F}_2, \ell^2 \mathbb{F}_2) \neq \{0\}.}$$

Remark: Monod–Shalom: If $\mathcal{H}$ sep Hilbert space.

$\alpha : \mathbb{F}_2^3 \to \mathcal{H}$ extends continuously to a non-zero map

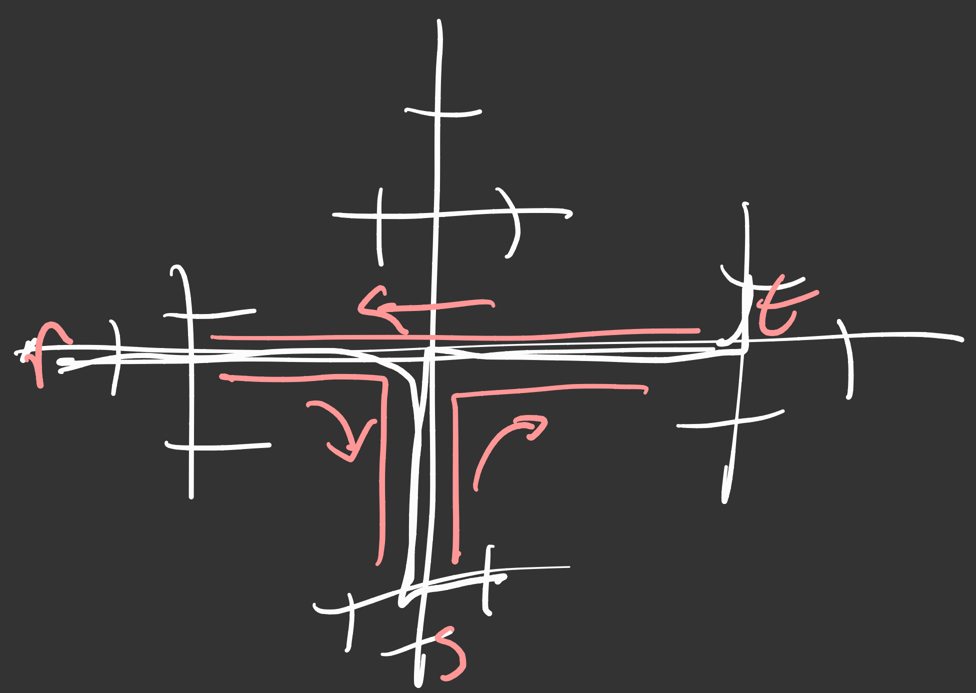
$\alpha : (\partial \mathbb{F}_2)^3 \to \mathcal{H}$

Then $H_b^2(\mathbb{F}_2, \mathcal{H}) \neq \{0\}$.

Ex: $\Gamma = \mathbb{F}_2$ $\quad \xi : \mathbb{F}_2^2 \longrightarrow \ell^2 E_2$

$$\xi(s,t)(\gamma) = \begin{cases} 1 & \text{if } \gamma \subset [s,t] \\ -1 & \text{if } \gamma \subset [t,s] \\ 0 & \text{otherwise} \end{cases}$$

$\alpha = \partial \xi$ $\quad$ is this a non-trivial cocycle?



T a tree, $\mathcal{W}$ a collection of geodesics $\mathcal{W} \subset T^2$ such that $1 \leq \sup_{\gamma \in \mathcal{W}} |\gamma| \leq K$

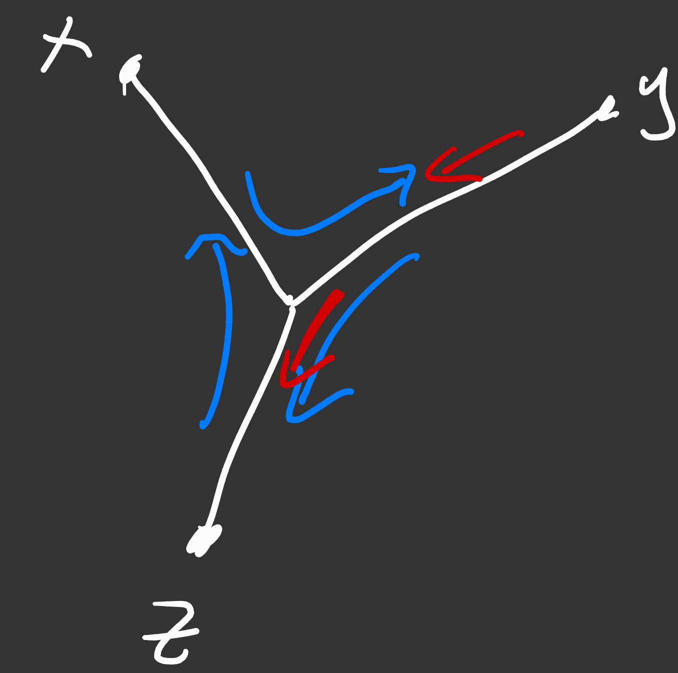
define $\xi_{\mathcal{W}} : T^2 \to \ell^\infty(T^2)$

by

$$\xi_{\mathcal{W}}(x,y)(\gamma) = \begin{cases} 1 & \text{if } \gamma \subset [x,y] \\ -1 & \text{if } \gamma \subset [y,x] \\ 0 & \text{otherwise.} \end{cases} \quad \gamma \in \mathcal{W}$$

Define $\alpha_{\mathcal{W}} : T^3 \to \ell^\infty(T^2)$

by $\alpha_{\mathcal{W}} = \partial \xi_{\mathcal{W}}$



$$\alpha_{\mathcal{W}}(x,y,z) = \xi_{\mathcal{W}}(y,z) - \xi_{\mathcal{W}}(x,z) + \xi_{\mathcal{W}}(x,y).$$

$$\alpha_{\mathcal{W}}(x,y,z)(\gamma) = \begin{cases} 1 & \text{if } \gamma \subset [y,z] \text{ and } t \in \gamma \\ & \text{or } \gamma \subset [z,x] \text{ and } t \in \gamma \\ & \text{or } \gamma \subset [x,y] \text{ and } t \in \gamma \\ -1 & \text{-- --} \qquad \text{and } t \in \gamma \\ 0 & \text{otherwise.} \end{cases}$$

$$\|\alpha_{\mathcal{W}}(x,y,z)\|_1 \leq 6\left(\sum_{j=2}^{K} (j-1)\right)$$

$\therefore$ $\alpha_{\mathcal{W}}$ is a bounded cocycle.

Note If $\Gamma \curvearrowright T$ isometrically and if $\Gamma \cdot \mathcal{W} = \mathcal{W}$ then $\alpha_{\mathcal{W}}$ is Γ-equivariant.

Ex: $T = \mathrm{Cay}(\mathbb{F}_2, \{a, b\})$

$\underline{w}$ any reduced word in $\mathbb{F}_2$

we consider $\partial W = \mathbb{F}_2 \cdot [e, w]$.

This gives the Brooks cocycle

we discussed last time.

Thm: If $\underline{\Gamma} = \mathbb{F}_2$ and if

$\alpha : \Gamma^3 \longrightarrow \ell^2(\Gamma^2)$ is a

bounded 2-cocycle such that

α extends to a bounded map

$\overline{\alpha} : \overline{\Gamma}^3 \longrightarrow \ell^2(\Gamma^2)$ that

is continuous on the subspace

of distinct triples and if

$\underline{\overline{\alpha}|_{(\partial\Gamma)^3}} \not\equiv 0$ then α is not a coboundary.

Let T be a regular tree, with degree

$\underline{3 \leq n} < \infty$.

Let $G = \mathrm{Aut}(T)$, with the topology

given by $g_i \longrightarrow g$ iff

$g_i \cdot o \longrightarrow g \cdot o$ for all $o \in T$.

This is a separable group.

If we fix $o \in T$

consider $K = G_o = \{g \in G \mid \underline{go = o}\}$.

$\underline{K}$ is an open subgroup.

And K is compact.

Therefore G is a second countable

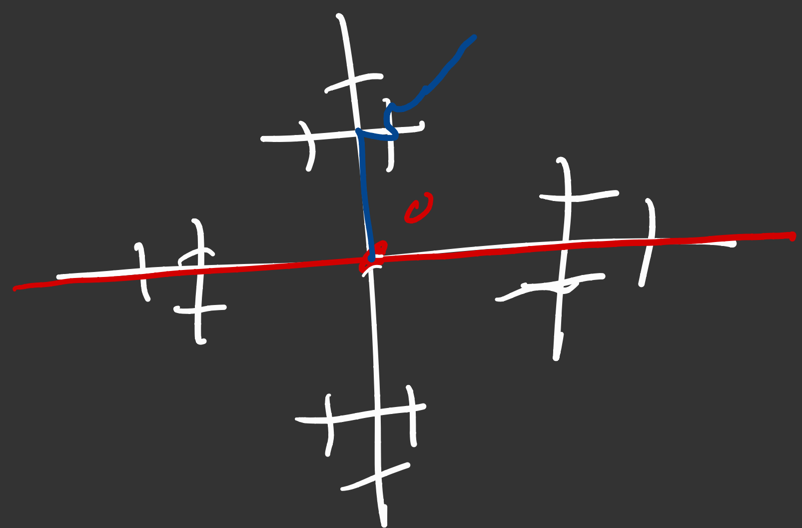
lctd group.

$\underline{\mathrm{Aut}(T)}$ is analogous to $\underline{SL_2(\mathbb{R})}$.

Fix a bi-infinite geodesic γ containing o.

Fix $t \in Aut(T)$ st

$$t \cdot \gamma(n) = \gamma(n+1)$$



Ex: $T = Cay(\mathbb{F}_2)$

take $\gamma(n) = a^n$

$t = a$

$$\mathbb{Z} \cong \langle t \rangle < Aut(T), \quad \langle t \rangle \cong A$$

$$\boxed{G = K A_+ K}$$ where $A_+ = \{t^n \mid n \geq 0\}$

Since if $g \in G$

there exists an ∞-geodesic λ

containing $[o, g \cdot o]$

Then exists some $k_1 \in K$ st

$k_1 \lambda = \gamma|_{\mathbb{Z}_+}$, ie $k_1 g \cdot o = \gamma(+n)$

then $t^{-n} k_1 g \cdot o = o$

$\therefore\ t^{-n} k_1 g = k_2 \in K$

$\therefore\ g = k_1^{-1} t^n k_2 \in K A_+ K$.

Define

$B_+ = \{g \in Aut(T) \mid g$ pointwise fixes some neighborhood of $\gamma_+\} < G$

$B_- = \{g \in Aut(T) \mid g$ pointwise fixes some nbhd of $\gamma_-\}$

Exercise: $G = \langle B_-, B_+ \rangle$

since the degree of G is ≥ 3.

In fact $\underbrace{B_- B_+ B_-} = G$.

Thm: (Burger–Mozes) $Aut(T)$ has the Howe–Moore property,

i.e., If $\pi : Aut(T) \to \mathcal{U}(\mathcal{H})$ is any cont. unitary representation without invariant vectors, then π is mixing, i.e. $\forall\ \xi, \eta \in \mathcal{H}$

$$Aut(T) \ni g \mapsto \langle \pi(g)\xi, \eta \rangle$$

is in $C_0(Aut(T))$.

equivalently, if $g_n \in Aut(T)$ is any sequence s.t. $g_n \to \infty$ then $\pi(g_n) \to 0$ WOT.

Proof:

Fix $\pi : Aut(T) \to \mathcal{U}(\mathcal{H})$ w/o inv. vectors.

Note if $g_n \in Aut(T)$

Then as $Aut(T) = K A_+ K$ we write $g_n = k_n a_n \tilde{k}_n$, and taking a subsequence we may assume that $k_n \to k_\infty$, $\tilde{k}_n \to \tilde{k}_\infty$

$$\therefore \langle \pi(g_n)\xi, \eta \rangle = \langle \pi(a_n)\pi(\tilde{k}_n)\xi, \pi(k_n^{-1})\eta \rangle$$
$$\approx \langle \pi(a_n)\pi(\tilde{k}_\infty)\xi, \pi(k_\infty^{-1})\eta \rangle$$

$\therefore$ π is a mixing rep iff $\pi|_A$ is a mixing rep.

Take $a_n \in A$, $a_n \to \infty$.

let S be any WOT-accumulation pt of $\pi(a_n)$, take a subsequence and assume $\pi(a_n) \to S$ WOT

we assume $a_n = t^{k_n}$ with $k_n \to \infty$

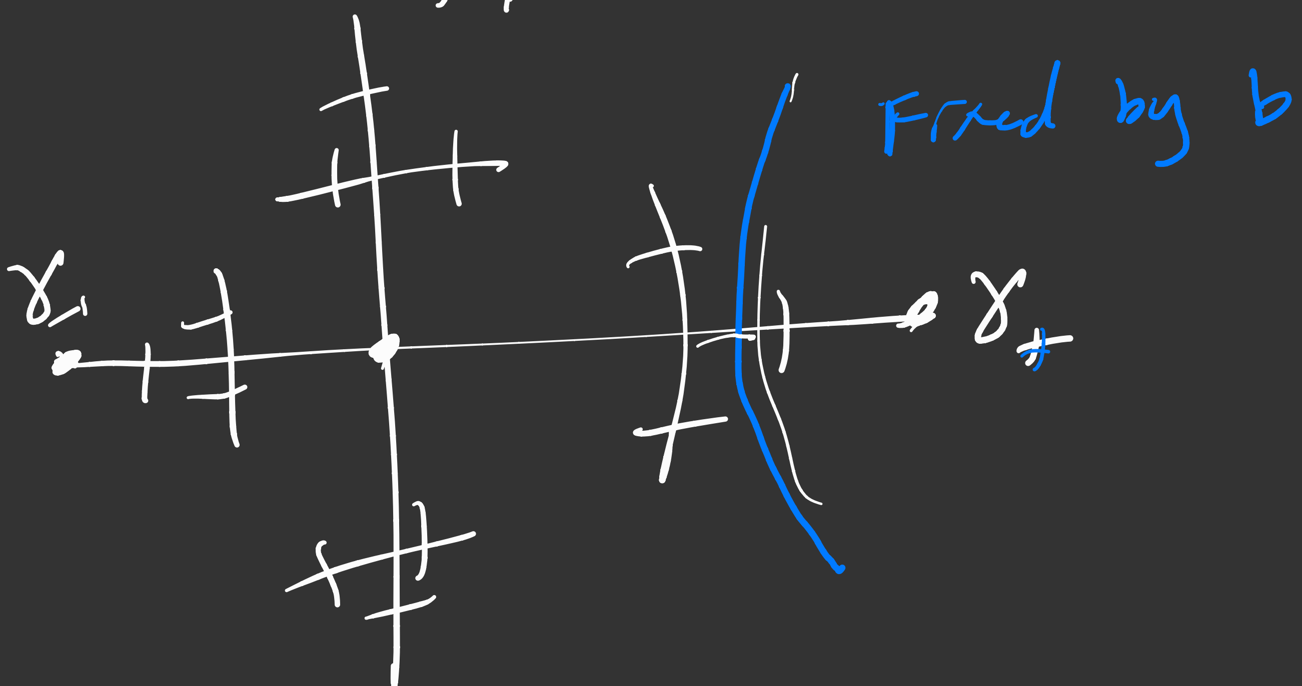
π has no G-inv vectors.

$\underline{\underline{\pi(t^{n_k})}} \xrightarrow[WOT]{} \underline{S}$ $\quad n_k \to \infty$

we want to show that $S = 0$.

$$B_+ = \{ g \in G \mid g \text{ ptwise fixes a nbhd of } \gamma_+ \}$$

take $b \in B_+$



Note If b fixes any vertex beyond $\gamma(l)$

then $t^{-n_k} b t^{n_k}$ fixes any vertex beyond $\gamma(l-n_k)$

$\therefore$ $t^{-n_k} b t^{n_k} \longrightarrow e$ in $Aut(T)$.

$\therefore$

$$\begin{aligned}\pi(b)S &= \operatorname*{WOT-lim}_{k\to\infty} \pi(b t^{n_k}) \\ &= \operatorname*{WOT-lim}_{k\to\infty} \underbrace{\pi(t^{n_k})}_{\xrightarrow[WOT]{} S} \pi\big(\underbrace{t^{-n_k} b t^{n_k}}_{\xrightarrow[SOT]{} 1}\big) \\ &= S\end{aligned}$$

$\therefore$ $\pi(B_+)$ fixes the range of S

Note $\underline{\pi(t^{-n_k})} \xrightarrow[WOT]{} S^*$

Similarly $\pi(c) S^* = S^*$ for all $c \in B_-$

Note $S, S^* \in \pi(A)''$ abelian.

$\therefore$ $\pi(b) S^* S = \pi(b) S S^* = S S^* = S^* S$

$\pi(c) S^* S = S^* S$

Since $\mathrm{Aut}(T) = \langle \underline{B_-}, \underline{B_+} \rangle$

the range of S^*S consists

of G-inv vectors

$\therefore$ $S^*S = 0 \Longrightarrow S = 0$. $\square$