Math 6100 - Final Exam, December 17, 2016 Name:____

Instructions: Work on any 5 problems. Circle the problems you want to be graded:

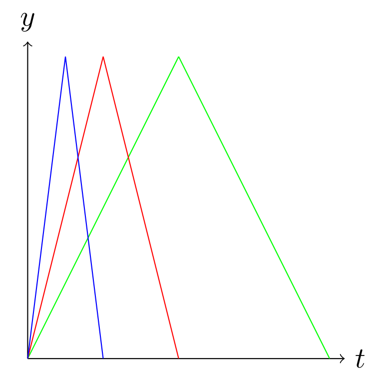
1 2 3 4 5 6 7

Problem 1 (20 points). Suppose K is a compact Hausdorff space and $\{f_n\}_n$ is a sequence of continuous complex-valued functions on K such that f_n converges pointwise to a continuous function f. Does it follow that f_n converges to f uniformly. Prove or give a counterexample.

Solution. It does not follow. Suppose $K = [0, 1]$, and $f_n : [0, 1] \to [0, 1]$ is given by

$$f_n(t) = \begin{cases} 2^{n+1}t & \text{if } 0 \le t \le 2^{-n-1}; \\ -2^{n+1}t + 2 & \text{if } 2^{-n-1} \le t \le 2^{-n}; \\ 0 & \text{if } 2^{-n} \le t \le 1. \end{cases}$$

Here are the graphs of f_1, f_2 and f_3 in green, red, and blue respectively:



We have $f_n(t) \to 0$ pointwise, however $\|f_n\|_\infty = 1$ for each n and so $f_n \not\to 0$ uniformly. □

Problem 2 (20 points). Let $(X, \mathcal{M}, \mu)$ be a finite measure space and suppose $f \in L^\infty(X, \mu)$. Set $a_n = \int |f|^n \, d\mu$. Show that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \|f\|_\infty$.

Solution. There is a very slick proof of this fact using the Cauchy-Schwarz inequality. Although we have not covered this yet and so we give a direct proof instead.

First note that if $\|f\|_\infty \neq 0$ then $\frac{a_{n+1}}{a_n} = \|f\|_\infty \left(\int (\frac{f}{\|f\|_\infty})^{n+1} \, d\mu\right) / \left(\int (\frac{f}{\|f\|_\infty})^n \, d\mu\right)$, and so by replacing f with $f/\|f\|_\infty$ it is enough to consider the case when $\|f\|_\infty = 1$. Also, by replacing μ with the measure $\nu(E) = \mu(E)/\mu(X)$, we may assume that $\mu(X) = 1$.

Suppose therefore that $\|f\|_\infty = 1$. Then $|f(x)|^{n+1} \leq |f(x)|^n$ and so we see that $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} \leq 1$. It then suffices to show that $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \geq 1$. For this we fix $\varepsilon > 0$ and set $E = \{x \in X \mid |f(x)| \geq 1-\varepsilon\}$. Take $\delta > 0$ so that if $F = \{x \in X \mid |f(x)| \geq 1-\varepsilon-\delta\}$ then $\mu(F \setminus E) < \varepsilon\mu(E)$.

Then for large enough n we have $\left(\frac{(1-\varepsilon-\delta)}{1-\varepsilon}\right)^n \leq \varepsilon\mu(E)$, so that $(1-\varepsilon-\delta)^n \leq \varepsilon(1-\varepsilon)^n\mu(E)$. Then we have

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &\geq \left(\int_E |f(x)|^{n+1} \, d\mu\right) / \left(\int_F |f(x)|^n \, d\mu + \int_{F^c} |f(x)|^n \, d\mu\right) \\
&\geq (1-\varepsilon)\left(\int_E |f(x)|^n \, d\mu\right) / \left(\int_F |f(x)|^n \, d\mu + \varepsilon\mu(E)(1-\varepsilon)^n\right) \\
&\geq (1-\varepsilon)\left(\int_E |f(x)|^n \, d\mu\right) / \left(\int_E |f(x)|^n \, d\mu + 2\varepsilon\mu(E)(1-\varepsilon)^n\right)
\end{aligned}$$

If $c > 0$, then the function $f(t) = \frac{t}{t+c}$ is increasing for $t > 0$. Also, if $b_n = \int_E |f(x)|^n \, d\mu$ then $b_n \geq (1-\varepsilon)^n\mu(E)$. Hence $b_n/(b_n + 2\varepsilon(1-\varepsilon)^n\mu(E)) \geq 1/(1+2\varepsilon)$. Therefore, we have shown that for large enough n we have $\frac{a_{n+1}}{a_n} \geq \frac{1-\varepsilon}{1+2\varepsilon}$. As $\varepsilon > 0$ was arbitrary it then follows that $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \geq 1$. □

Problem 3 (20 points). Let μ be a finite, positive, Borel measure on $\mathbb{R}^2$, and let $\mathcal{G}$ be the family of finite unions of squares of the form

$$S = \{(x, y) \mid j2^{-n} \leq x \leq (j+1)2^{-n}; k2^{-n} \leq y \leq (k+1)2^{-n}\},$$

where j, k, and n are integers. Prove that the set of linear combinations of characteristic functions of elements from $\mathcal{G}$ is dense in $L^1(\mathbb{R}^2, \mu)$.

Solution. First note that for any point $(a, b) \in \mathbb{R}^2$, and any open neighborhood U of (a, b), there exists $E \in \mathcal{G}$ which is a neighborhood on (a, b) and such that $E \subset U$. Now suppose that $K \subset O \subset \mathbb{R}^2$ is such that K is compact and O is open. Then for each point $(a, b) \in K$ we take such a set $E_{(a,b)}$. This then gives a cover of K by neighborhoods and by compactness we then have a finite subcover. Hence there exists an element $E \in \mathcal{G}$ so that $K \subset E \subset O$.

μ is a finite Borel measure and hence for any Borel set $A \subset \mathbb{R}^2$ and any $\varepsilon > 0$ there exists a compact set K and an open set O so that $K \subset A \subset O$ and $\mu(O \setminus K) < \varepsilon$. If we take $E \in \mathcal{G}$ as above so that $K \subset E \subset O$ then we have that $\|1_E - 1_A\|_1 = \mu(A \Delta E) < \mu(O \setminus K) < \varepsilon$. Thus, the closure of the linear combinations of characteristic functions from elements in $\mathcal{G}$ contains arbitrary characteristic functions of Borel sets, and hence must contain all of $L^1(\mathbb{R}^2, \mu)$ since the span of characteristic functions of all Borel sets is dense in $L^1(\mathbb{R}^2, \mu)$. □

Problem 4 (20 points). Let $f : [0,1] \to \mathbb{C}$ be Borel. Assume $fg \in L^1([0,1],\lambda)$ whenever $g \in L^1([0,1],\lambda)$, where λ is Lebesgue measure on $[0,1]$. Prove that $f \in L^\infty([0,1],\lambda)$.

Solution. Suppose $f \notin L^\infty([0,1],\lambda)$. Then for each $n \in \mathbb{N}$ there exists $E_n \subset [0,1]$ a Borel set of positive measure such that $|f(x)| \geq n$ for all $x \in E_n$. Therefore if $g_n = \frac{1}{\mu(E_n)} 1_{E_n}$ then we have $\|g_n\|_1 = 1$, $g_n \geq 0$, and $\int |f| g_n \, d\mu \geq n$.

We set $h(x) = f(x)/|f(x)|$ if $f(x) \neq 0$ and $h(x) = 0$ if $f(x) = 0$. We also set $g = \sum_{k=1}^\infty 2^{-k} g_{k2^k}$, so that $\|hg\|_1 \leq \|g\|_1 = 1$. For each $l > 0$ we have

$$\int fhg \, d\mu = \sum_{k=1}^\infty \int |f| 2^{-k} g_{k2^k} \, d\mu \geq \int |f| 2^{-k} g_{l2^l} \, d\mu \geq l.$$

Thus, $fhg \notin L^1([0,1],\lambda)$ and the result then follows by contraposition. □

Problem 5 (20 points). Is the Banach space $\ell^\infty(\mathbb{N})$ separable? Prove your answer is correct.

Solution. This space is not separable. The proof is similar to Cantor's proof that the reals are not countable. Specifically, suppose $f_n \in \ell^\infty(\mathbb{N})$ gives a countable subset. For each $n \in \mathbb{N}$ take $a_n \in [-1, 1]$ so that $|a_n - f_n(n)| \geq 1$. Then $a = (a_1, a_2, \ldots)$ defines a sequence in $\ell^\infty(\mathbb{N})$ and for each n we have that $\|a - f_n\|_\infty \geq |a_n - f_n(n)| \geq 1$. Therefore, $\{f_n\}_{n=1}^\infty$ is not dense in $\ell^\infty(\mathbb{N})$. $\square$

Problem 6 (20 points). Suppose $E \subset \mathbb{R}$ is a Borel set which has positive Lebesgue measure. Show that the set $E - E := \{x - y \mid x, y \in E\}$ contains an open interval $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Hint: Suppose $\lambda(E) < \infty$ and consider the function $f(y) = \int_{\mathbb{R}} 1_E(x) 1_E(y + x)\, d\lambda(x)$.

Solution. Restricting to a finite positive measure subset we may assume that $\lambda(E) < \infty$. If we consider the function f defined above then $f(0) = \mu(E) > 0$ and $f(y) = \lambda(E \cap (E + y))$ is continuous. (This was a homework problem assigned this semester, recall that the key step in the proof of this fact was to approximate E by a finite union of intervals). Thus, for some $\varepsilon > 0$ we have $\lambda(E \cap (E + y)) = f(y) > 0$ whenever $|y| < \varepsilon$. In particular, it then follows that for $|y| < \varepsilon$ we have $E \cap (E + y) \neq \emptyset$ and hence $y \in E - E$. $\square$

Problem 7 (20 points). Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and suppose $\mathcal{F} \subset L^\infty(X,\mu)$ is a family such that for each $g \in L^1(X,\mu)$ we have $\sup_{f\in\mathcal{F}} |\int fg\,d\mu| < \infty$, prove that $\sup_{f\in\mathcal{F}} \|f\|_\infty < \infty$. Hint: First show that for some n the set $X_n = \{g \in L^1(X,\mu) \mid \sup_{f\in\mathcal{F}} |\int fg\,d\mu| \le n\}$, contains an L^1-open ball $B(g_0,\varepsilon)$. Then, writing $g = \frac{1}{\varepsilon}(\varepsilon g + g_0 - g_0)$, use this to show that $\sup_{f\in\mathcal{F}} \|f\|_\infty \le 2n/\varepsilon$.

Solution. This is really the Uniform Boundedness Principal in disguise. We will cover this next semester.

First note that the sets X_n defined above are closed and by hypothesis we have that $\cup_n X_n = L^1(X,\mu)$. Since $L^1(X,\mu)$ has the Baire property there then exists some n so that X_n has non-empty interior, i.e., there exists $g_0 \in L^1(X,\mu)$ and $\varepsilon > 0$ so that X_n contains the open ball $B(g_0,\varepsilon)$.

If $g \in L^1(X,\mu)$ with $\|g\|_1 \le 1$ then $\varepsilon g + g_0, g_0 \in B(g_0,\varepsilon) \subset X_n$ and hence

$$|\int fg\,d\mu| \le \frac{1}{\varepsilon}\left(|\int f(\varepsilon g + g_0)\,d\mu| + |\int fg_0\,d\mu|\right) \le 2n/\varepsilon.$$

Therefore,

$$\|f\|_\infty = \sup_{g\in L^1(X,\mu), \|g\|_1\le 1} |\int fg\,d\mu| \le 2n/\varepsilon < \infty.$$

□