TREES AND ULTRAMETRIC SPACES: A CATEGORICAL EQUIVALENCE

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Abstract. There is a well-known correspondence between infinite trees and ultrametric spaces that comes from considering the end space of the tree. The correspondence is interpreted here as an equivalence between two categories, one of which encodes the geometry of trees at infinity and the other encodes the micro-geometry of complete ultrametric spaces.

Contents

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1. Introduction

A close relationship between trees and ultrametric spaces has been observed for a long time. Trees model branching processes; the branching occurs as one moves away from the root of the tree towards its ends. Ultrametric spaces are those metric spaces with the unusual property that one of any two intersecting balls will contain the other. Thus, if one starts with the family of all balls of a given diameter and starts shrinking the diameter, the resulting parameterized family of balls forms a hierarchical system completely analogous to the branching in a tree. In trees the branching occurs towards infinity, whereas in ultrametric spaces the branching occurs near points.

This paper establishes a categorical equivalence, thus making the correspondence between trees and ultrametric spaces precise. Categories are introduced that capture the geometry of trees at infinity and the micro-geometry of ultrametric spaces. Thus, an important aspect of this work is to reveal the appropriate morphisms for these geometries. For trees we use isometries that need be defined only away from the root. For ultrametric spaces we use maps that change scale locally.

Since metric balls may be of any diameter, the splitting of balls in an ultrametric space may occur at any diameter. Therefore, we work in the context of so-called real trees, or $\mathbb{R}$-trees, that allow branching at all points, not just at a discrete set of points as with classical trees.

Main Theorem. *There is an equivalence from the category of geodesically complete, rooted* $\mathbb{R}$*-trees and equivalence classes of isometries at infinity, to the category of complete ultrametric spaces of finite diameter and local similarity equivalences.*

Complete definitions and proofs are in the body of the paper (the proof is completed in §6). An isometry at infinity between two rooted trees is just an isometry that only need be defined away from the roots; two such are equivalent if they agree sufficiently far from the root. A local similarity equivalence between metric spaces is a homeomorphism that is a similarity, or scale change, near each point (the modulus of similarity is allowed to vary from point to point).

The functor from trees to ultrametric spaces comes from end theory. The end space of a classical, locally finite simplicial tree is simply its end point, or Freudenthal, compactification with a natural metric. In general, the end space of a geodesically complete, rooted $\mathbb{R}$-tree is the set of geodesic rays emanating from the root, and the set is given a natural metric.

A specialization of the morphisms in the two categories leads to an important corollary (see §7). For isometries at infinity between rooted trees, "uniform" indicates that the isometry must be defined on the complement of a metric ball centered at the root. For local similarities, "uniform" means that the modulus of similarity does not vary from point to point.

Corollary 1. *There is an equivalence from the category of geodesically complete, rooted* $\mathbb{R}$*-trees and equivalence classes of uniform isometries at infinity, to the category of complete ultrametric spaces of finite diameter and uniform local similarity equivalences.*

It might seem more appropriate to study the large scale structure of trees in the context of Gromov's asymptotic geometry in which case the morphisms would be the quasi-isometries (see [Gr2], [Gr3]). Quasi-isometries of trees are indeed quite interesting, but they do not capture the geometry discussed here. Consider the following example (see §9).

Example (Cantor v. Fibonacci). The Cantor tree C and the Fibonacci tree F are quasi-isometric (specifically, bi-Lipschitz homeomorphic), but not isometric at infinity. Their end spaces $end(C)$ and $end(F)$ are homeomorphic (via a Hölder continuous homeomorphism), but not isometric. In fact, there is no homeomorphism between $end(C)$ and $end(F)$ that is a local similarity (i.e., there is no local similarity equivalence). Thus, from the point of view of asymptotic geometry, C and F are the same, but from the point of view of this paper, they are quite different.

Figure 1

Isometries of rooted trees induce isometries on their end spaces, and the techniques used in proving the Main Theorem and its Corollary are used in §8 to establish the following result.

Corollary 2. *There is an equivalence from the category of geodesically complete, rooted $\mathbb{R}$-trees and rooted isometries, to the category of complete ultrametric spaces of diameter less than or equal to one and isometries.*

Also in §8, a category of complete ultrametric spaces of diameter ≤ 1 and local isometry equivalences is shown to be equivalent to a category of $\mathbb{R}$-trees. The relationship among the four pairs of categories studied in this paper is summarized in §8.

There are many related results in the literature. It is well-known that the end space of a classical tree is usually homeomorphic to a Cantor set (if it contains no isolated points), and a good reference for this is Baues and Quintero [BaQ], where the passage from trees to end spaces is also made into a categorical equivalence. They restrict attention to classical trees (as opposed to $\mathbb{R}$-trees) and the morphisms that they use, namely (proper homotopy classes of) proper homotopy equivalences, are much weaker than isometries at infinity (or quasi-isometries). As a consequence, they capture the topology, rather than the geometry, of trees at infinity, and the natural metric on the end spaces plays no role in their work.

Ghys and de la Harpe [GdH] did emphasize the natural metric on end spaces and showed that a quasi-isometry of trees induces a Hölder continuous and quasi-conformal homeomorphism on end spaces. However, they worked in the context of classical trees and did not establish a categorical equivalence.

Choucroun [Cho] used end spaces to illuminate the connection between trees and ultrametric spaces, but did not work in the full generality of $\mathbb{R}$-trees, nor establish a categorical equivalence. Grigorchuk, Nekrashevich and Sushchanskii [GNS] discussed some of the folklore in this area. Berestovskii [Ber] has established some connections between ultrametric spaces and $\mathbb{R}$-trees from a different perspective. Lemin [Lem] has recently studied categorical aspects of ultrametric spaces.

An alternative approach to investigating the phenomena studied here is provided by Terhalle [Ter] (see also Dress and Terhalle [DT1]). Terhalle established a one-to-one correspondence between geodesically complete $\mathbb{R}$-trees and complete ultrametric spaces (a result close to Theorem 8.5 in this paper), but did not take a categorical approach.

Trees occur in the study of evolutionary branching processes, and ultrametrics occur in the theory of phylogenetic tree reconstruction. To date the emphasis has been on finite processes and finite data sets. In the finite case, the correspondence implied by the Main Theorem is well-known. For information about this active area, see Dress, Huber and Moulton [DHM], Dress and Terhalle [DT2], Durbin, Eddy, Krogh and Mitchison [DEKM], Klein and Takahata [KlT] and Rammal, Toulouse and Virasoro [RTV]. Böcker and Dress [BöD] is especially relevant to the ideas here.

The p-adic numbers with the p-adic norm provide natural examples of ultrametric spaces. Holly [Hol] constructs trees associated with the p-adics in order to visualize them. In this special case, this construction essentially illustrates the categorical equivalence of the Main Theorem.

As illustrated above, this paper is closely related to many others in the literature. However, this paper is unique because of the combination of the following three elements:

(1) Not only are the objects of two categories in one-to-one correspondence, but the categories themselves are shown to be equivalent.
(2) When passing from trees to an ideal space at infinity using end theory, the natural metric on the end space is emphasized; therefore, we are studying the geometry of the tree rather than its topology.
(3) The results are set in $\mathbb{R}$-trees rather than more classical types of trees.

How noncommutative geometry can be used to study the micro-geometry of ultrametric spaces and the geometry of trees at infinity constitutes the theme of this paper and others. With the categorical equivalence in the Main Theorem and its Corollary now established, the projected papers will turn to the study of isometries of trees at infinity and to local similarities of ultrametric spaces. The new ingredient will be noncommutative geometry as developed by Connes [Con] and Renault [Ren]. An ultimate goal is to make progress on the following problem.

Problem. *Classify complete ultrametric spaces up to local similarity equivalence, and uniform local similarity equivalence.*

Note that, up to a scaling factor, uniform local similarity equivalence is the same as local isometry equivalence. The groupoid of local isometries on a compact ultrametric space is the subject of [Hug]. Further comments can be found in §9.

My own interest in end theory comes from high dimensional geometric topology (see [HuR]). It is expected that an analysis of the one-dimensional case (trees) will lead to new insights in higher dimensions.

This paper is organized as follows. The basic definitions related to $\mathbb{R}$-trees are recalled in §2 along with the notion of cut set. Cut sets are used in §3 to define isometries at infinity for $\mathbb{R}$-trees and their equivalence classes. The category $\mathbf{T}$ of trees appearing in the Main Theorem is defined in §3 as is the group $Isom_\infty(T, v)$ of automorphisms of the object (T, v). Facts about ultrametric spaces are recalled in §4 along with the definition of local similarity equivalence. Also, the second category $\mathbf{U}$ in the Main Theorem of ultrametric spaces is introduced along with the group $LSE(X)$ of automorphisms of the object X. The functorial passage from trees to ultrametric spaces is described in §5 along with a proof that the functor $\mathcal{E}$ of the Main Theorem is full and faithful. The rest of the proof of that $\mathcal{E}$ is an equivalence is given in §6 by showing how to naturally construct a tree from an ultrametric space. Corollary 1 is established in §7 where the categories $\mathbf{T_u}$, $\mathbf{U_u}$ and the groups $Isom^u_\infty(T, v)$ and $LSE^u(X)$ are introduced. Corollary 2 is established in §8 and the Cantor and Fibonacci trees are examined in §9.

2. Trees

In this section we recall the definition of an $\mathbb{R}$-tree and establish some terminology and facts. We introduce the notion of a cut set for a tree, a concept that will be used in the next section for defining isometries at infinity between trees.

See Bestvina [Bes] and Chiswell [Chi] for more information about $\mathbb{R}$-trees.

Definition 2.1. A *real tree*, or $\mathbb{R}$-*tree*, is a metric space (T, d) that is uniquely arcwise connected, and for any two points $x, y \in T$ the unique arc from x to y, denoted $[x, y]$, is isometric to the subinterval $[0, d(x, y)]$ of $\mathbb{R}$.

This is not the original definition of an $\mathbb{R}$-tree, but is a characterization provided by Morgan and Shalen [MoS].

Classical trees are one-dimensional, simply connected simplicial complexes. Such a tree when endowed with its natural metric (a length metric with every 1-simplex isometric to the unit interval $[0,1]$) is an example of an $\mathbb{R}$-tree.

Lemma 2.2. *If T is an $\mathbb{R}$-tree and $v, w, t \in T$, then there exists $x \in T$ such that $[v,w] \cap [w,t] = [w,x]$.*

Proof. Let $t_0 = d(w,t)$ and let $f : [0,t_0] \to T$ be the unique isometric embedding such that $f(0) = w$ and $f(t_0) = t$. Let $x_0 = \sup\{s \in [0,t_0] \mid f(s) \in [v,w]\}$ and let $x = f(x_0)$. Then $[w,x] = f([0,x_0])$. One may easily verify that $[w,x] = [v,w] \cap [w,t]$. □

Definition 2.3. A *rooted* $\mathbb{R}$*-tree* (T,v) consists of an $\mathbb{R}$-tree (T,d) and a point $v \in T$, called the *root*.

This paper is concerned with rooted trees that are purely infinite. That is, not only are the trees non-finite, but they also have no finite ends. The following definition reflects this.

Definition 2.4. A rooted $\mathbb{R}$-tree (T,v) is *geodesically complete* if every isometric embedding $f : [0,t] \to T$, $t > 0$, with $f(0) = v$, extends to an isometric embedding $\tilde{f} : [0,\infty) \to T$. In this case, we say $[v, f(t)]$ can be *extended to a geodesic ray*.

Lemma 2.5. *If (T,v) is a geodesically complete rooted $\mathbb{R}$-tree and $g : T \to T$ is an isometry, then $(T, g(v))$ is also a geodesically complete rooted $\mathbb{R}$-tree.*

Proof. If $f : [0,t] \to T$, $t > 0$, is an isometric embedding with $f(0) = g(v)$, then $g^{-1}f$ extends to an isometric embedding $h : [0,\infty) \to T$ and $\tilde{f} = gh$ is the desired extension of f. □

Because roots in geodesically complete rooted trees may be "dead ends," it need not be the case that (T,v) is geodesically complete implies that (T,w) is geodesically complete (e.g., compare $([0,\infty),0)$ and $([0,\infty),1)$).

Notation 2.6. If (T,v) is a rooted $\mathbb{R}$-tree and $x \in T$, let $||x|| = d(v,x)$. Sometimes $||x||_v$ is used if the root is not clear from the context. If $r > 0$, let

$$\begin{aligned} B(v,r) &= \{x \in T \mid ||x|| < r\}, \\ \bar{B}(v,r) &= \{x \in T \mid ||x|| \le r\}, \\ \partial B(v,r) &= \{x \in T \mid ||x|| = r\}. \end{aligned}$$

Cut sets are the next topic of discussion. They will be needed in the definition of isometries at infinity. The idea is that cut sets allow us to talk about "at infinity" in rooted trees.

Definition 2.7. A *cut set* C for a geodesically complete, rooted $\mathbb{R}$-tree (T,v) is a subset C of T such that $v \notin C$ and for every isometric embedding $\alpha : [0,\infty) \to T$ with $\alpha(0) = v$ there exists a unique $t_0 > 0$ such that $\alpha(t_0) \in C$.

In other words, to go to infinity from v you must pass through a unique point of C (the point is unique once the path to infinity is chosen).[1]

[1]It is possible to define cut sets in not necessarily geodesically complete trees by first defining *endpoints* of trees and then deciding how endpoints should behave with respect to cut sets.

Example 2.8. If (T, v) is a geodesically complete, rooted $\mathbb{R}$-tree and $r > 0$, then $\partial B(v, r)$ is a cut set for (T, v).

Definition 2.9. If C and C' are cut sets for (T, v), then C' is *larger than* C if for every $c \in C$, $[v, c] \cap C' \subseteq \{c\}$. C' is *strictly larger than* C if for every $c \in C$, $[v, c] \cap C' = \emptyset$.

Definition 2.10. If C_1 and C_2 are cut sets for the geodesically complete, rooted $\mathbb{R}$-tree (T, v), then define

$$\max\{C_1, C_2\} = \{c \in C_1 \mid [v, c] \cap C_2 \neq \emptyset\} \cup \{c \in C_2 \mid [v, c] \cap C_1 \neq \emptyset\}.$$

Lemma 2.11. $\max\{C_1, C_2\}$ *is a cut set for* (T, v) *larger than both* C_1 *and* C_2.

Proof. Let $f : [0, \infty) \to T$ be an isometric embedding such that $f(0) = v$. Then there exist unique $t_1, t_2 > 0$ such that $f(t_1) \in C_1$ and $f(t_2) \in C_2$. Assume without loss of generality that $t_1 \leq t_2$. Then $f(t_1) \in [v, f(t_2)]$ and so $f(t_2) \in \max\{C_1, C_2\}$. If $f(t_1) \in \max\{C_1, C_2\}$ and $t_1 \neq t_2$, then $f(t_1) \in C_1 \setminus C_2$ and so $[v, f(t_1)] \cap C_2 \neq \emptyset$, contradicting the uniqueness of t_2. Since $\max\{C_1, C_2\} \subseteq C_1 \cup C_2$, we have shown that there exists a unique $t_0 > 0$ such that $f(t_0) \in \max\{C_1, C_2\}$ (t_0 is t_1 or t_2). Hence, $\max\{C_1, C_2\}$ is a cut set. To see that it is larger than C_1 and C_2, it suffices to show that it is larger than C_1 (by symmetry). So suppose $c \in C_1$ and show $[v, c] \cap \max\{C_1, C_2\} \subseteq \{c\}$. If not, then there exists $p \neq c$, $p \in [v, c]$ and $p \in \max\{C_1, C_2\}$. It follows that $p \notin C_1$ (because C_1 is a cut set) so $p \in C_2$. Thus, $c \in \max\{C_1, C_2\}$. Since also $p \in \max\{C_1, C_2\}$, this is a contradiction to $\max\{C_1, C_2\}$ being a cut set. $\square$

Definition 2.12. If c is any point of the rooted $\mathbb{R}$-tree (T, v), the *subtree of* (T, v) *determined by* c is

$$T_c = \{x \in T \mid c \in [v, x]\}.$$

Note that T_c is indeed a subtree of T (that is to say, as a metric subspace of T, T_c is a tree).

Definition 2.13. If c is a point of a rooted $\mathbb{R}$-tree (T, v), then T_c is an *isolated ray* if T_c is isometric to $[0, \infty)$.

Lemma 2.14. *If* T *is an* $\mathbb{R}$*-tree,* $f_1 : [0, t_0] \to T$ *and* $f_2 : [t_0, \infty) \to T$ *are isometric embeddings such that* $f_1(s) = f_2(t)$ *if and only if* $s = t = t_0$*, then* $f : [0, \infty) \to T$ *defined by*

$$f(t) = \begin{cases} f_1(t) & \text{if } 0 \leq t \leq t_0 \\ f_2(t) & \text{if } t_0 \leq t \end{cases}$$

is an isometric embedding.

Proof. It suffices to show that if $0 \leq a \leq t_0$ and $t_0 \leq b$, then $d(f(a), f(b)) = b - a$. To this end, let $d_0 = d(f(a), f(b))$ and let $g : [0, d_0] \to [f(a), f(b)]$ be the unique isometry such that $g(0) = f(a)$ and $g(d_0) = f(b)$. Since T is uniquely arcwise connected, there exists $t_1 \in [0, d_0]$ such that $g(t_1) = f_1(t_0)$. Then $t_1 = d(g(0), g(t_1)) = d(f_1(a), f_1(t_0)) = t_0 - a$. Likewise, $d_0 - t_1 = d(g(t_1), g(d_0)) = d(f_2(t_0), f_2(b)) = b - t_0$. Adding these two together gives $d_0 = b - a$. $\square$

Lemma 2.15. *If T is an $\mathbb{R}$-tree, $\alpha, \beta : [0, \infty) \to T$ are two isometric embeddings such that $\alpha(0) = \beta(0)$ and there exist $t_0, t_1 > 0$ such that $\alpha(t_0) = \beta(t_1)$, then $t_0 = t_1$ and $\alpha(t) = \beta(t)$ whenever $0 \leq t \leq t_0$.*

Proof. First $t_0 = d(v, \alpha(t_0)) = d(v, \beta(t_1)) = t_1$. Next, since T is uniquely arcwise connected, if $0 \leq t \leq t_0$, then there exists t', $0 \leq t' \leq t_0$, such that $\alpha(t) = \beta(t')$. The first part of the proof implies $t = t'$. $\square$

Example 2.16. A geodesically complete, rooted $\mathbb{R}$-tree need not be complete. For example, let

$$T = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x < 1 \text{ and } y = 0, \text{ or } x = \frac{i}{i+1} \text{ and } y \geq 0 \text{ for } i = 1, 2, 3, \dots\}$$

with the length metric induced from the restriction of the standard metric on $\mathbb{R}^2$.

Figure 2

3. Isometries at infinity

In this section we introduce isometries at infinity between trees, define an equivalence relation on them, and prove that the resulting equivalence classes (which are essentially germs at infinity) form the morphisms in a category **T** whose objects are geodesically complete, rooted $\mathbb{R}$-trees. This is one of the two categories in the Main Theorem.

If we were considering only locally compact trees (e.g., locally finite, 1-dimensional simply connected simplicial complexes), then it would be much easier to talk about isometries at infinity (and their germs at infinity); they would be isometries defined on the complement of a compact subset of the tree (and two isometries would be equivalent if they agreed on the complement of a larger compact subset). The complement of a compact subset of a locally compact subset of a locally compact space is usually thought of as a neighborhood of infinity. In the absence of local compactness, we can use cut sets to talk about neighborhoods of infinity.

For basic information about, and terminology from, category theory, see Mac Lane [MaL].

Definition 3.1. Let (T, v) and (S, w) be geodesically complete, rooted $\mathbb{R}$-trees. An *isometry at infinity* from (T, v) to (S, w) is a triple (f, C_T, C_S) where C_T and C_S are cut sets of T and S, respectively, and $f : \cup\{T_c \mid c \in C_T\} \to \cup\{S_c \mid c \in C_S\}$ is a homeomorphism such that

(1) $f(C_T) = C_S$, and
(2) for every $c \in C_T$, $f| : T_c \to S_{f(c)}$ is an isometry.

We use the notation $(f, C_T, C_S) : (T, v) \to (S, w)$ to denote an isometry at infinity. Of course, C_S is completely determined by C_T so there is a bit of redundancy in the notation.

Example 3.2. Let T be an $\mathbb{R}$-tree and $v, w \in T$ such that (T, v) and (T, w) are geodesically complete. Then there exists a subset C of T that is a cut set of both (T, v) and (T, w) and (id_T, C, C) is an isometry at infinity. For example, we can take $C = \partial B(v, 1 + d(v, w))$.

Example 3.3. Let (T, v) be a geodesically complete, rooted $\mathbb{R}$-tree and let C be a cut set of (T, v). If $f : T \to T$ is any isometry, then $f(C)$ is a cut set for $(T, f(v))$ and $(f, C, f(C))$ is an isometry at infinity from (T, v) to $(T, f(v))$.

We need several facts about isometries at infinity and cut sets. The first is obvious and needs no further proof.

Lemma 3.4. *If $(f, C_T, C_S) : (T, v) \to (S, w)$ is an isometry at infinity, then $(f^{-1}, C_S, C_T) : (S, w) \to (T, v)$ is an isometry at infinity.* □

Lemma 3.5. *If $(f, C_T, C_S) : (T, v) \to (S, w)$ is an isometry at infinity and C is a cut set for (T, v) larger than C_T, then $f(C)$ is a cut set for (S, w) larger than C_S.*

Proof. We first show that $f(C)$ is a cut set for (S, w). Let $\alpha : [0, \infty) \to S$ be an isometric embedding such that $\alpha(0) = w$ and show that the image of α meets $f(C)$ in a unique point. Since $C_S = f(C_T)$ meets the image of α in a unique point, there exists a unique $c \in C_T$ such that $f(c) \in \alpha([0, \infty))$. Say $\alpha(t_0) = f(c)$. Since $f| : T_c \to S_{f(c)}$ is an isometry and $\alpha([t_0, \infty)) \subseteq S_{f(c)}$, $\beta = (f|)^{-1} \circ \alpha| : [t_0, \infty) \to T_c$ is an isometric embedding. Let $d_0 = d(v, c)$ and let $\gamma : [0, \infty) \to T$ be the isometric embedding such that $\gamma(0) = v$, $\gamma([0, d_0]) = [v, c]$ and $\gamma(t) = \beta(t - d_0 + t_0)$ if $t \geq d_0$ (γ is an isometric embedding by Lemma 2.14). Since C is a cut set for (T, v), there exists a unique $t_1 > 0$ such that $\gamma(t_1) \in C$. C is larger than C_T implies that $[v, c] \cap C \subseteq \{c\}$. Thus, $\gamma(t_1) \in T_c$ and $f\gamma(t_1) \in S_{f(c)}$. In fact, $f\gamma(t_1) \in \alpha([t_0, \infty))$ and hence, the image of α meets $f(C)$ in $f\gamma(t_1)$. Since $\alpha([0, \infty)) \cap f(C) \subseteq f(\gamma([d_0, \infty)) \cap C)$, the point is unique.

To show that $f(C)$ is larger than C_S, let $p \in C_S$ and show $[w, p] \cap f(C) \subseteq \{p\}$. Say $p = f(a)$ with $a \in C_T$ (a exists because $C_S = f(C_T)$). Suppose $q \in [w, p] \cap f(C)$. Say $q = f(b)$ with $b \in C$. Let $x \in C_T$ such that $b \in T_x$. Then $q = f(b) \in f(T_x) = S_{f(x)}$ and so $f(x) \in [w, q]$. Since $q \in [w, p]$, $[w, q] \subseteq [w, p]$. Thus, $f(x) \in [w, p]$ and $p \in S_{f(x)} = f(T_x)$. But $p \in S_p = S_{f(a)} = f(T_a)$. Hence, $T_x = T_a$, $x = a$ and $p = q$. □

The following result follows immediately from the previous two lemmas.

Corollary 3.6. *If $(f, C_T, C_S) : (T, v) \to (S, w)$ is an isometry at infinity and C is a cut set for (S, w) larger than C_S, then $f^{-1}(C)$ is a cut set for (T, v) larger than C_T.* □

Definition 3.7. Two isometries at infinity (f, C_T, C_S) and (f', C'_T, C'_S) from (T, v) to (S, w) are said to be *equivalent* if there exists a cut set C''_T for (T, v) larger than C_T and C'_T such that for every $c \in C''_T$:

(1) if T_c is not an isolated ray, then $f|T_c = f'|T_c$,
(2) if T_c is an isolated ray, then $f(T_c) \cap f'(T_c) \neq \emptyset$.

The second condition in Definition 3.7 is rather technical, but necessary. Consider the tree $T = [0, \infty)$ with root 0. Without condition (2) there would be infinitely many inequivalent isometries at infinity of T to itself; however, with condition (2) there is just one equivalence class.

The equivalence class of an isometry at infinity (f, C_T, C_S) is denoted by any of

$$[f, C_T, C_S] = [f, C_T] = [f],$$

the middle notation justified by the fact that C_S is determined by C_T and f; the notation $[f]$ being used only when the cut set C_T is clear from (or irrelevant to) the context.

The next result follows from the definitions.

Lemma 3.8. *If $(f, C_T, C_S) : (T, v) \to (S, w)$ is an isometry at infinity and C is a cut set for (T, v) larger than C_T, then $[f, C_T] = [f|, C]$ where $f| : \cup\{T_c \mid c \in C\} \to \cup\{S_{f(c)} \mid c \in C\}$.* $\square$

We now discuss composition of equivalence classes of isometries at infinity. Let (R, v), (S, w) and (T, x) be geodesically complete, rooted $\mathbb{R}$-trees and let $[f, C_R] : (R, v) \to (S, w)$ and $[g, C_S] : (S, w) \to (T, x)$ be equivalence classes of isometries at infinity. Let $C'_S = \max\{f(C_R), C_S\}$, $C'_R = f^{-1}(C'_S)$ and $C_T = g(C'_S)$. Finally, consider the restrictions $f| : \cup\{R_c \mid c \in C'_R\} \to \cup\{S_c \mid c \in C'_S\}$ and $g| : \cup\{S_c \mid c \in C'_S\} \to \{T_c \mid c \in C_T\}$.

Lemma 3.9. *With the notation just established, we have:*

(1) *$(f|, C'_R, C'_S) : (R, v) \to (S, w)$, $(g|, C'_S, C_T) : (S, w) \to (T, x)$, and $(g| \circ f|, C'_R, C_T) : (R, v) \to (T, x)$ are isometries at infinity,*
(2) *$[f, C_R] = [f|, C'_R]$ and $[g, C_S] = [g|, C'_S]$,*
(3) *$[g| \circ f|, C'_R]$ is well-defined in the sense that it depends only on $[f, C_R]$ and $[g, C_S]$.*

Proof. For item (1) note that Lemma 2.11 implies that C'_S is a cut set for (S, w), and Lemmas 3.4 and 3.5 imply that C'_R is a cut set for (R, v) (and it is larger than C_R). The rest of the conditions are easy to check. Items (2) and (3) follow from Lemma 3.8. $\square$

It follows from Lemma 3.9 that we may define the composition of $[f, C_R]$ and $[g, C_S]$ by

$$[g, C_S] \circ [f, C_R] = [g| \circ f|, C'_R].$$

Definition 3.10. If (T, v) is a geodesically complete, rooted $\mathbb{R}$-tree, let $Isom_\infty(T, v)$ denote the *group of equivalence classes of isometries at infinity from (T, v) to itself*.

Proposition 3.11. *$Isom_\infty(T, v)$ is a group.*

Proof. The identity is given by $[\mathrm{id}_T, C]$ where C is any cut set for (T, v). Multiplication is given by composition as defined above. If $[f, C] \in Isom_\infty(T, v)$, then Lemma 3.4 implies that we may define $[f, C]^{-1} = [f^{-1}, f(C)]$. $\square$

Definition 3.12. Let **T** be the *category of geodesically complete, rooted* $\mathbb{R}$*-trees and equivalence classes of isometries at infinity.* The objects of **T** are geodesically complete, rooted $\mathbb{R}$-trees and the morphisms are equivalence classes of isometries at infinity.

Theorem 3.13. **T** *is a category in which every morphism is an isomorphism.*

Proof. Identities, compositions and inverses are like those given in the proof of Proposition 3.11. In fact, for each object (T, v) of **T**, $Isom_\infty(T, v)$ is a subcategory of **T**. □

4. Ultrametric spaces and local similarity equivalences

In this section we recall the definition of an ultrametric and some of its elementary properties. Then we introduce local similarity equivalences between ultrametric spaces and prove that these form the morphisms in a category **U** whose objects are complete ultrametric spaces of finite diameter. This is the second category in the Main Theorem.

Definition 4.1. If (X, d) is a metric space and $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$, then d is an *ultrametric* and (X, d) is an *ultrametric space*.

If (X, d) is a metric space, $x \in X$ and $\epsilon > 0$, then we use the notation $B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$ for the *open ball about x of radius ϵ*, and $\bar{B}(x, \epsilon) = \{y \in X \mid d(x, y) \leq \epsilon\}$ for the *closed ball about x of radius ϵ*.

The following proposition lists some well-known properties of ultrametric spaces. They are readily verified.

Proposition 4.2 (Elementary properties of ultrametric spaces). *The following properties hold in any ultrametric space (X, d).*

1. *If two open balls in X intersect, then one contains the other.*
2. *If two closed balls in X intersect, then one contains the other.*
3. **(Egocentricity)** *Every point in an open ball is a center of the ball.*
4. **(Closed egocentricity)** *Every point in a closed ball is a center of the ball.*
5. *Every open ball is closed, and every closed ball is open.*
6. **(ISB)** *Every triangle in X is isosceles with a short base (i.e., if $x_1, x_2, x_3 \in X$, then there exists an i such that $d(x_j, x_k) \leq d(x_i, x_j) = d(x_i, x_k)$ whenever $j \neq i \neq k$).* □

Definition 4.3. A function $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is a *similarity* if there exists $\lambda > 0$ such that $d_Y(f(x), f(y)) = \lambda d_X(x, y)$ for all $x, y \in X$. In this case, λ is the *similarity constant* of f and f is a λ*-similarity*. A *similarity equivalence* is a similarity that is also a homeomorphism.

Definition 4.4. A homeomorphism $h : X \to Y$ between metric spaces is a *local similarity equivalence* if for every $x \in X$ there exist $\epsilon > 0$ and $\lambda > 0$ such that the restriction $h| : B(x, \epsilon) \to B(h(x), \lambda\epsilon)$ is a surjective λ-similarity.

Note that the similarity constants of the restrictions may vary from ball to ball.

Lemma 4.5. *The inverse of a local similarity equivalence is a local similarity equivalence. The composition of two local similarity equivalences is a local similarity equivalence.*

Proof. Let $h : X \to Y$ be a local similarity equivalence. If $y \in Y$, then there exist $\epsilon > 0$ and $\lambda > 0$ such that $h| : B(h^{-1}(y), \epsilon) \to B(y, \lambda\epsilon)$ is a surjective λ-similarity. It follows that $h^{-1}| : B(y, \lambda\epsilon) \to B(h^{-1}(y), \epsilon)$ is a surjective $(1/\lambda)$-similarity, showing that inverses of local similarity equivalences are local similarity equivalences.

If, in addition, $g : Y \to Z$ is a local similarity equivalence and $x \in X$, let $\epsilon_1, \epsilon_2 > 0$ and $\lambda_1, \lambda_2 > 0$ be such that $h| : B(x, \epsilon_1) \to B(h(x), \lambda_1\epsilon_1)$ and $g| : B(h(x), \epsilon_2) \to B(gh(x), \lambda_2\epsilon_2)$ are surjective λ_1- and λ_2-similarities, respectively. Let $\epsilon = \min\{\epsilon_1, \epsilon_2/\lambda_1\}$. Then $gh| : B(x, \epsilon) \to B(gh(x), \lambda_2\lambda_1\epsilon)$ is a surjective $(\lambda_2\lambda_1)$-similarity, showing that compositions of local similarity equivalences are local similarity equivalences. □

Definition 4.6. If (X, d) is a metric space, let $LSE(X)$ denote the *group of local similarity equivalences from X to itself*.

The following result follows immediately from Lemma 4.5.

Proposition 4.7. *$LSE(X)$ is a group.* □

Definition 4.8. Let **U** be the *category of complete ultrametric spaces of finite diameter and local similarity equivalences.* The objects of **U** are complete ultrametric spaces of finite diameter and the morphisms are local similarity equivalences.

Just as Proposition 4.7 follows from Lemma 4.5, so does the following result. In fact, for each object X of **U**, $LSE(X)$ is a subcategory of **U**.

Theorem 4.9. **U** *is a category in which every morphism is an isomorphism.* □

Lemma 4.10. *Let $h : X \to Y$ be a local similarity equivalence between two ultrametric spaces of finite diameter. Then there exist a subset $E \subset X$ and positive numbers $\lambda_x > 0$, $r_x > 0$ for each $x \in E$ such that*

1. *$h| : \bar{B}(x, r_x) \to \bar{B}(hx, \lambda_x r_x)$ is a surjective λ_x-similarity for each $x \in E$,*
2. *if $x, y \in E$, $x \neq y$, then $\bar{B}(x, r_x) \cap \bar{B}(y, r_y) = \emptyset$,*
3. *$X = \cup_{x \in E} \bar{B}(x, r_x)$.*

Moreover, given $r_0 > 0$, we may additionally require $r_x \leq r_0$ for each $x \in E$.

Proof. For each $x \in X$ choose $\lambda_x > 0$ for which there exists $\epsilon > 0$ so that $h| : B(x, \epsilon) \to B(hx, \lambda_x\epsilon)$ is a surjective λ_x-similarity. For each $x \in X$, let

$$r_x = \frac{1}{2} \sup\{\epsilon > 0 \mid h| : B(x, \epsilon) \to B(hx, \lambda_x\epsilon) \text{ is a surjective } \lambda_x\text{-similarity and } \epsilon \leq \operatorname{diam} X\}.$$

Note that $h| : \bar{B}(x, r_x) \to \bar{B}(hx, \lambda_x r_x)$ is a surjective λ_x-similarity for each $x \in X$. If $x, y \in X$ and $\bar{B}(x, r_x) \cap \bar{B}(y, r_y) \neq \emptyset$, then one of these balls contains the other (by 4.2); say, $\bar{B}(x, r_x) \subseteq \bar{B}(y, r_y)$. In this case it follows that $\bar{B}(x, r_x) = \bar{B}(y, r_y)$. Define an equivalence relation on X by declaring x and y related if and only if $\bar{B}(x, r_x) = \bar{B}(y, r_y)$. Finally, let E be a set containing exactly one representative from each equivalence class. Now, if $r_0 > 0$ is given, simply replace each r_x by $\min\{r_x, r_0\}$ in the argument above. □

5. The end space of a tree

In this section we define the functor $\mathcal{E}$ from trees to ultrametric spaces that will be the equivalence in the Main Theorem. On objects the functor takes a rooted tree to the end space of the tree, so we begin by defining the end space of a rooted $\mathbb{R}$-tree and its natural metric. After establishing that the end space functor $\mathcal{E} : \mathbf{T} \to \mathbf{U}$ is indeed a functor, we prove that it is full and faithful (it is proved to be an equivalence in §6).

The following concept is quite well-known.

Definition 5.1. The *end space* of a rooted $\mathbb{R}$-tree (T, v) is given by

$$end(T, v) = \{f : [0, \infty) \to T \mid f(0) = v \text{ and } f \text{ is an isometric embedding}\}.$$

For $f, g \in end(T, v)$, define

$$d_e(f, g) = \begin{cases} 0 & \text{if } f = g \\ 1/e^{t_0} & \text{if } f \neq g \text{ and } t_0 = \sup\{t \geq 0 \mid f(t) = g(t)\} \end{cases}.$$

Note that since T is uniquely arcwise connected,

$$\{t \geq 0 \mid f(t) = g(t)\} = \begin{cases} [0, \infty) & \text{if } f = g \\ [0, t_0] & \text{if } f \neq g. \end{cases}$$

Proposition 5.2. *If (T, v) is a rooted $\mathbb{R}$-tree, then $(end(T, v), d_e)$ is a complete ultrametric space of diameter ≤ 1.*

Proof. To check the ultrametric inequality, let $f, g, h \in end(T, v)$ and show that $d_e(f, g) \leq \max\{d_e(f, h), d_e(h, g)\}$. Without loss of generality suppose $d_e(f, h) = e^{-t_1} \geq d_e(h, g) = e^{-t_2}$. Then $t_1 \leq t_2$, $f = h$ on $[0, t_1]$ and $h = g$ on $[0, t_2]$. Thus, $f = g$ on $[0, t_1]$ and $d_e(f, g) \leq e^{-t_1}$. The statement about diameter is obvious.

To verify that $(end(T, v), d_e)$ is complete, let $\{f_i\}_{i=1}^{\infty}$ be a Cauchy sequence in $end(T, v)$. By passing to a subsequence we may assume that there is a non-decreasing sequence of integers $1 \leq i_1 \leq i_2 \leq i_3 \leq \cdots$ so that $f_i = f_j$ on $[0, n]$ whenever $i, j \geq i_n$. Define $f : [0, \infty) \to T$ by setting $f|[0, n] = f_{i_n}|[0, n]$ for each n. Then f is a well-defined isometric embedding and $\lim_{i\to\infty} f_i = f$. □

Proposition 5.3. *Let $(f, C_T, C_S) : (T, v) \to (S, w)$ be an isometry at infinity between geodesically complete, rooted $\mathbb{R}$-trees. Then there is an induced local similarity equivalence $f_* : end(T, v) \to end(S, w)$. Moreover, if (g, C'_T, C'_S) is another such isometry at infinity and $[f] = [g]$, then $f_* = g_*$.*

Proof. In order to define f_*, let $\alpha : [0, \infty) \to T$ be an element of $end(T, v)$. Since C_T is a cut set, there exists a unique $t_0 > 0$ such that $\alpha(t_0) \in C_T$. Moreover, $\alpha([t_0, \infty)) \subseteq T_{\alpha(t_0)}$. Let $\hat{\alpha} : [0, ||f\alpha(t_0)||] \to S$ be the unique isometric embedding such that $\hat{\alpha}(0) = w$ and $\hat{\alpha}(||f\alpha(t_0)||) = f\alpha(t_0)$. Define

$$f_*(\alpha)(t) = \begin{cases} \hat{\alpha}(t) & \text{if } 0 \leq t \leq ||f\alpha(t_0)|| \\ f\alpha(t - ||f\alpha(t_0)|| + t_0) & \text{if } ||f\alpha(t_0)|| \leq t. \end{cases}$$

It follows from Lemma 2.14 that $f_*(\alpha) \in end(S, w)$. To see that f_* is a local similarity equivalence, we will first show, given α as above, that there exist $\epsilon > 0$

and $\lambda > 0$ such that $f_*| : B(\alpha, \epsilon) \to B(f_*(\alpha), \lambda\epsilon)$ is a surjective λ-similarity. Let $\epsilon = e^{-t_0}$ and $\lambda = e^{t_0 - ||f\alpha(t_0)||}$. If $\beta \in end(T, v)$ with $\alpha \neq \beta$ and $d_e(\alpha, \beta) < \epsilon$, then

$$t_\beta = \sup\{t \geq 0 \mid \alpha(t) = \beta(t)\} > -\ln\epsilon = t_0.$$

In particular, $d_e(\alpha, \beta) = e^{-t_\beta}$, $\alpha(t_0) = \beta(t_0)$ and $f\alpha(t_0) = f\beta(t_0)$. It follows that

$$f_*(\beta)(t) = \begin{cases} \hat{\alpha}(t) & \text{if } 0 \leq t \leq ||f\beta(t_0)|| \\ f\beta(t - ||f\beta(t_0)|| + t_0) & \text{if } ||f\beta(t_0)|| \leq t \end{cases}$$

$$= \begin{cases} \hat{\alpha}(t) & \text{if } 0 \leq t \leq ||f\alpha(t_0)|| \\ f\alpha(t - ||f\alpha(t_0)|| + t_0) & \text{if } ||f\alpha(t_0)|| \leq t \leq t_\beta - t_0 + ||f\alpha(t_0)|| \\ f\beta(t - ||f\alpha(t_0)|| + t_0) & \text{if } t_\beta - t_0 + ||f\alpha(t_0)|| \leq t. \end{cases}$$

Hence, $\sup\{t \geq 0 \mid f_*\alpha(t) = f_*\beta(t)\} = t_\beta - t_0 + ||f\alpha(t_0)||$ and $d_e(f_*\alpha, f_*\beta) = e^{-t_\beta + t_0 - ||f\alpha(t_0)||} = e^{t_0 - ||f\alpha(t_0)||} d_e(\alpha, \beta) = \lambda d_e(\alpha, \beta)$.

To see that $f_*| : B(\alpha, \epsilon) \to B(f_*\alpha, \lambda\epsilon)$ is surjective, let $\gamma \in B(f_*\alpha, \lambda\epsilon)$. Then $d_e(\gamma, f_*\alpha) < \lambda\epsilon = e^{-||f\alpha(t_0)||}$. It follows that $\gamma(||f\alpha(t_0)||) = f\alpha(t_0)$ and

$$\gamma([||f\alpha(t_0)||, \infty)) \subseteq S_{f\alpha(t_0)}.$$

Since $f| : T_{\alpha(t_0)} \to S_{f\alpha(t_0)}$ is an isometry, we can define $\beta : [0, \infty) \to T$ by

$$\beta(t) = \begin{cases} \alpha(t) & \text{if } 0 \leq t \leq t_0 \\ (f|T_{\alpha(t_0)})^{-1}\gamma(t + ||f\alpha(t_0)|| - t_0) & \text{if } t_0 \leq t. \end{cases}$$

Lemma 2.14 implies that $\beta \in end(T, v)$. One can check that $\beta \in B(\alpha, \epsilon)$ and $f_*\beta = \gamma$ (to see that $d_e(\alpha, \beta) < \epsilon$, as opposed to just $d_e(\alpha, \beta) \leq \epsilon$, use the fact that $d_e(\gamma, f_*\alpha) < \lambda\epsilon$).

A similar construction shows $f_* : end(T, v) \to end(S, w)$ to be surjective. Here are the details. If $\gamma \in end(S, w)$, then there exists a unique $t_\gamma > 0$ such that $\gamma(t_\gamma) \in C_S$, and there exists a unique $c \in C_T$ such that $f(c) = \gamma(t_\gamma)$. Let $\hat{\gamma} : [0, ||c||] \to T$ be the unique isometric embedding such that $\hat{\gamma}(0) = v$ and $\hat{\gamma}(||c||) = c$. Define $\beta : [0, \infty) \to T$ by

$$\beta(t) = \begin{cases} \hat{\gamma}(t) & \text{if } 0 \leq t \leq ||c|| \\ (f|T_c)^{-1}\gamma(t + ||\gamma(t_\gamma)|| - ||c||) & \text{if } ||c|| \leq t. \end{cases}$$

To see that f_* is injective, suppose $f_*\alpha = f_*\beta$ for some $\alpha, \beta \in end(T, v)$. Then there eixsts $t_1 > 0$ such that $\alpha([t_1, \infty)) \cup \beta([t_1, \infty))$ is in the domain of f and $f\alpha([t_1, \infty)) = f\beta([t_1, \infty))$. Since f is a homeomorphism, it follows that $\alpha([t_1, \infty)) = \beta([t_1, \infty))$ and Lemma 2.15 implies that $\alpha = \beta$.

To show that f_* is independent of the representation of $[f]$, we need the following lemma:

Lemma 5.3.1. *If $c \in C_T$ and $x \in T_c$, then $||x|| - ||c|| = ||f(x)|| - ||f(c)||$.*

Proof. $||x| - ||c||$ is the lenght of $[c, x]$ (because $[c, x] \subseteq [v, x]$) and $||f(x)|| - ||f(c)||$ is the length of $[f(c), f(x)]$. Since $[c, x] \subseteq T_c$, these two lengths are the same. ☐

Returning to the proof of 5.3, it suffices to show that the definition of $f_*(\alpha)$ will not change if another cut set C'_T for (T, v) larger than C_T is used in place of C_T. For such a cut set, there exists a unique $t_1 > 0$ such that $\alpha(t_1) \in C'_T$. It follows that $t_1 \geq t_0$, $\alpha(t_1) \in T_{\alpha(t_0)}$ and $f\alpha(t_1) \in T_{f\alpha(t_0)}$. Let $\hat{\alpha}' : [0, ||f\alpha(t_1)||] \to S$ be the unique isometric embedding such that $\hat{\alpha}'(0) = w$ and $\hat{\alpha}'(||f\alpha(t_1)||) = f\alpha(t_1)$. The the map $f'_*(\alpha) : [0, \infty) \to S$ given by

$$f'_*(\alpha)(t) = \begin{cases} \hat{\alpha}'(t) & \text{if } 0 \leq t \leq ||f\alpha(t_1)|| \\ f\alpha(t - ||f\alpha(t_1)|| + t_1) & \text{if } ||f\alpha(t_1)|| \leq t \end{cases}$$

is how $f_*(\alpha)$ would be defined if C'_T were used in place of C_T. However, since S is an $\mathbb{R}$-tree and $\hat{\alpha}'(||f\alpha(t_0)||) = f\alpha(t_0)$, it follows that $\hat{\alpha}'|[0, ||f\alpha(t_0)||] = \hat{\alpha}$. Moreover, Lemma 5.3.1 implies that

$$t_1 - t_0 = ||\alpha(t_1)|| - ||\alpha(t_0)|| = ||f\alpha(t_1)|| - ||f\alpha(t_0)||$$

and, hence, $t_1 - ||f\alpha(t_1)|| = t_0 - ||f\alpha(t_0)||$. It follows that $f'_*(\alpha) = f_*(\alpha)$ and $f'_* = f_*$. □

Definition 5.4. Define $\mathcal{E} : \mathbf{T} \to \mathbf{U}$ by $\mathcal{E}(T, v) = \mathit{end}(T, v)$ for every geodesically complete rooted $\mathbb{R}$-tree, and $\mathcal{E}([f]) = f_*$ for every equivalence class of an isometry at infinity.

Proposition 5.5. *$\mathcal{E} : \mathbf{T} \to \mathbf{U}$ is a full and faithful functor.*

Proof. We begin with the functorial properties. Clearly $\mathcal{E}(\mathrm{id}_{(T,v)}) = \mathrm{id}_{\mathit{end}(T,v)}$. Now suppose $[f, C_R] : (R, v) \to (S, w)$ and $[g, C_S] : (S, w) \to (T, x)$ are equivalence classes of isometries at infinity. By passing to larger cut sets, we may assume that $f(C_R) = C_S$. Thus, $g \circ f$ is defined and we need to show that $g_* f_* = (gf)_* : \mathit{end}(R, v) \to \mathit{end}(T, x)$. Let $\alpha \in \mathit{end}(R, v)$ be given. Let t_0 be the unique number such that $\alpha(t_0) \in C_R$. Let $\beta : [0, ||gf\alpha(t_0)||] \to T$ be the unique isometric embedding such that $\beta(0) = x$ and $\beta(||gf\alpha(t_0)|| = gf\alpha(t_0)$. Then one may check that

$$g_* f_*(\alpha)(t) = (gf)_*(\alpha)(t) = \begin{cases} \beta(t) & \text{if } 0 \leq t \leq ||gf\alpha(t_0)|| \\ gf\alpha(t - ||gf\alpha(t_0)|| + t_0) & \text{if } ||gf\alpha(t_0)|| \leq t, \end{cases}$$

concluding the proof that $\mathcal{E}$ is a functor.

To show that $\mathcal{E}$ is full, suppose (R, v) and (S, w) are two geodesically complete, rooted $\mathbb{R}$-trees for which there exists a local similarity equivalence

$$h : \mathit{end}(R, v) \to \mathit{end}(S, w).$$

We need to find an isometry at infinity $(f, C_R, C_S) : (R, v) \to (S, w)$ such that $f_* = h$. If $\mathit{end}(R, v)$ and $\mathit{end}(S, w)$ each consist of a single point, then R and S are each isolated rays, and the desired result is immediate. Hence, we assume that $\mathit{end}(R, v)$ and $\mathit{end}(S, w)$ each contain more than a single point. By Lemma 4.10 there exist a subset $E \subseteq \mathit{end}(R, v)$ and positive numbers $\lambda_\alpha > 0$, $r_\alpha > 0$ for each $\alpha \in E$ such that

(1) $h|\bar{B}(\alpha, r_\alpha) \to \bar{B}(h\alpha, \lambda_\alpha r_\alpha)$ is a surjective λ_α-similarity for every $\alpha \in E$,
(2) if $\alpha, \beta \in E$, $\alpha \neq \beta$, then $\bar{B}(\alpha, r_\alpha) \cap \bar{B}(\beta, r_\beta) = \emptyset$,
(3) $\mathit{end}(R, v) = \cup_{\alpha \in E} \bar{B}(\alpha, r_\alpha)$,
(4) $r_\alpha < \operatorname{diam} \mathit{end}(R, v) \leq 1$ for each $\alpha \in E$.

Note that since $end(R,v)$ and (S,w) each contain more than a single point and $r_\alpha < \operatorname{diam} end(R,v)$, it follows that $\lambda_\alpha r_\alpha < \operatorname{diam} end(S,w) \le 1$. Thus, $\lambda_\alpha r_\alpha < 1$ for each $\alpha \in E$.

Let $C_R = \{\alpha(-\ln r_\alpha) \mid \alpha \in E\}$. We claim that C_R is a cut set for (R,v). To prove this, let $\beta : [0,\infty) \to R$ be an isometric embedding for which $\beta(0) = v$. Then $\beta \in end(R,v)$ so there exists $\alpha \in E$ such that $d_e(\alpha,\beta) \le r_\alpha$, which is to say $\alpha(t) = \beta(t)$ whenever $0 \le t \le -\ln r_\alpha$. In particular, $\beta(-\ln r_\alpha) = \alpha(-\ln r_\alpha) \in C_R$. To show uniqueness, suppose $t_0 > 0$ and $\beta(t_0) \in C_R$. Then $\beta(t_0) = \alpha'(-\ln r_{\alpha'})$ for some $\alpha' \in E$; hence, $t_0 = -\ln r_{\alpha'}$ and $\alpha'(t) = \beta(t)$ whenever $0 \le t \le t_0$ (by Lemma 2.15). If $t_0 \ne -\ln r_\alpha$, then either $t_0 < -\ln r_\alpha$ or $-\ln r_\alpha < t_0$. In the first case, $\alpha \in \bar{B}(\alpha', r_{\alpha'})$ and, in the second case, $\alpha' \in \bar{B}(\alpha, r_\alpha)$. In either case, $\bar{B}(\alpha, r_\alpha) \cap \bar{B}(\alpha', r_{\alpha'}) \ne \emptyset$ implying $\alpha = \alpha'$ and $t_0 = -\ln r_\alpha$. This completes the proof that C_R is a cut set for (R,v).

Let $C_S = \{(h\alpha)(-\ln \lambda_\alpha r_\alpha) \mid \alpha \in E\}$. We claim that C_S is a cut set for (S,w). First note that , under our assumptions, $0 < \lambda_\alpha r_\alpha < 1$ so that $-\ln \lambda_\alpha r_\alpha \ge 0$ for each $\alpha \in E$. Now suppose $\beta \in end(S,w)$. Then $h^{-1}\beta \in end(R,v)$ and so there exists a unique $\alpha \in E$ such that $h^{-1}\beta \in \bar{B}(\alpha, r_\alpha)$. Thus, $d_e(\alpha, h^{-1}\beta) \le r_\alpha$ and $d_e(h\alpha, \beta) = \lambda_\alpha d_e(\alpha, h^{-1}\beta) \le \lambda_\alpha r_\alpha$, which is to say $(h\alpha)(t) = \beta(t)$ whenever $0 \le t \le -\ln \lambda_\alpha r_\alpha$. In particular, $\beta(-\ln \lambda_\alpha r_\alpha) = (h\alpha)(-\ln \lambda_\alpha r_\alpha) \in C_S$. To show uniqueness, suppose $t_0 > 0$ and $\beta(t_0) \in C_S$. Then $\beta(t_0) = (h\alpha')(-\ln \lambda_{\alpha'} r_{\alpha'})$ for some $\alpha' \in E$; hence, $t_0 = -\ln \lambda_{\alpha'} r_{\alpha'}$ and $(h\alpha')(t) = \beta(t)$ whenever $0 \le t \le t_0$ (by Lemma 2.15). If $t_0 \ne -\ln \lambda_\alpha r_\alpha$, then either $t_0 < -\ln \lambda_\alpha r_\alpha$ or $-\ln \lambda_\alpha r_\alpha < t_0$. In the first case, $h\alpha \in \bar{B}(h\alpha', \lambda_{\alpha'} r_{\alpha'})$ and, in the second case, $h\alpha' \in \bar{B}(h\alpha, \lambda_\alpha r_\alpha)$. In either case, $\bar{B}(h\alpha, \lambda_\alpha r_\alpha) \cap \bar{B}(h\alpha', \lambda_{\alpha'} r_{\alpha'}) \ne \emptyset$ implying that $\bar{B}(\alpha, r_\alpha) \cap \bar{B}(\alpha', r_{\alpha'}) \ne \emptyset$ and, thus, $\alpha = \alpha'$ and $t_0 = -\ln \lambda_\alpha r_\alpha$. This completes the proof that C_S is a cut set for (S,w).

Now note that there is a bijection $C_R \to C_S$ given by

$$\alpha(-\ln r_\alpha) \mapsto (h\alpha)(-\ln \lambda_\alpha r_\alpha)$$

for $\alpha \in E$; in fact, only injectivity needs to be checked. So suppose $\alpha, \beta \in E$ and $(h\alpha)(-\ln \lambda_\alpha r_\alpha) = (h\beta)(-\ln \lambda_\beta r_\beta)$. Then Lemma 2.15 implies $-\ln \lambda_\alpha r_\alpha = -\ln \lambda_\beta r_\beta$ and $(h\alpha)(t) = (h\beta)(t)$ whenever $0 \le t \le -\ln \lambda_\alpha r_\alpha$. Thus, $d_e(h\alpha, h\beta) \le \lambda_\alpha r_\alpha$ and, hence, $d_e(\alpha,\beta) \le r_\alpha$. Thus, $\beta \in \bar{B}(\alpha, r_\alpha)$ and $\alpha = \beta$.

Now define $f : \cup\{R_c \mid c \in C_R\} \to \cup\{S_c \mid c \in C_S\}$ by first defining, for $\alpha \in E$, $f| : R_{\alpha(-\ln r_\alpha)} \to S_{(h\alpha)(-\ln \lambda_\alpha r_\alpha)}$ as follows. If $x \in R_{\alpha(-\ln r_\alpha)}$, then there exists $\beta \in end(R,v)$ such that $\beta(-\ln r_\alpha) = \alpha(-\ln r_\alpha)$ and $\beta(||x||) = x$. Set $f(x) = (h\beta)(||x|| - \ln \lambda_\alpha)$. Note that $f(x) = (h\beta)(||x|| + \ln r_\alpha - \ln \lambda_\alpha r_\alpha)$. We need to show that $f| : R_{\alpha(-\ln r_\alpha)} \to S_{(h\alpha)(-\ln \lambda_\alpha r_\alpha)}$ is (1) well-defined (i.e., does not depend on β), (2) an isometric embedding, and (3) a surjection.

For (1), suppose $\beta' \in end(R,v)$ such that $\beta'(-\ln r_\alpha) = \alpha(-\ln r_\alpha)$ and $\beta'(||x||) = x$. Then $d_e(\alpha, \beta') \le r_\alpha$ and $d_e(\beta, \beta') \le e^{-||x||}$. Thus, $d_e(h\beta, h\beta') = \lambda_\alpha d_e(\beta, \beta') \le \lambda_\alpha e^{-||x||}$. It follows that $(h\beta)(t) = (h\beta')(t)$ whenever $0 \le t \le -\ln \lambda_\alpha e^{-||x||} = -\ln \lambda_\alpha + ||x||$. In particular, $(h\beta)(||x|| - \ln \lambda_\alpha) = (h\beta')(||x|| - \ln \lambda_\alpha)$.

For (2), suppose $x, y \in R_{\alpha(-\ln r_\alpha)}$, $x \ne y$ and $\beta, \gamma \in end(R,v)$ such that $\beta(-\ln r_\alpha) = \gamma(-\ln r_\alpha) = \alpha(-\ln r_\alpha)$, $\beta(||x||) = x$ and $\gamma(||y||) = y$. If it so happens that $y = \beta(||y||)$, then $d(x,y) = |||x|| - ||y|||$, $f(x) = (h\beta)(||x|| - \ln \lambda_\alpha)$ and $f(y) = (h\beta)(||y|| - \ln \lambda_\alpha)$; this implies $d(fx, fy) = |||x|| - \ln \lambda_\alpha - ||y|| + \ln \lambda_\alpha| = d(x,y)$. Likewise, if it so happens that $x = \gamma(||x||)$, we have $d(fx, fy) = d(x,y)$. So we suppose that $x \neq \gamma(||x||)$ and $y \neq \beta(||y||)$. It follows that $t_0 \leq ||x||$, $t_0 \leq ||y||$ and $d(x,y) = ||x|| + ||y|| - 2t_0$. Note that $d_e(h\beta, h\gamma) = \lambda_\alpha e^{-t_0}$, $-\ln \lambda_\alpha r_\alpha \leq -\ln \lambda_\alpha + t_0$, $t_0 - \ln \lambda_\alpha \leq ||x|| - \ln \lambda_\alpha$ and $t_0 - \ln \lambda_\alpha \leq ||y|| - \ln \lambda_\alpha$. Thus, $d(fx, fy) = d((h\beta)(||x|| - \ln \lambda_\alpha), (h\gamma)(||y|| - \ln \lambda_\alpha)) = ||(h\beta)(||x|| - \ln \lambda_\alpha)|| + ||(h\gamma)(||y|| - \ln \lambda_\alpha)|| - 2(-\ln \lambda_\alpha + t_0) = ||x|| + ||y|| - 2t_0 = d(x,y)$. This completes the proof that $f|$ is an isometric embedding.

For (3), suppose $z \in S_{(h\alpha)(-\ln \lambda_\alpha r_\alpha)}$. Then there exists $\beta \in end(S,w)$ such that $\beta(-\ln \lambda_\alpha r_\alpha) = (h\alpha)(-\ln \lambda_\alpha r_\alpha)$ and $\beta(||z||) = z$. It follows that $\beta \in \bar{B}(h\alpha, \lambda_\alpha r_\alpha)$ so $h^{-1}\beta \in \bar{B}(\alpha, r_\alpha)$ and $(h^{-1}\beta)(||z|| + \ln \lambda_\alpha) \in R_{\alpha(-\ln r_\alpha)}$. Finally, note that

$$f((h^{-1}\beta)(||z|| + \ln \lambda_\alpha)) = \beta(||z||) = z.$$

To show that $\mathcal{E}$ is faithful, suppose $(f, C_R, C_S), (f', C'_R, C'_S) : (R,v) \to (S,w)$ are two isometries at infinity between geodesically complete, rooted $\mathbb{R}$-trees such that $f_* = f'_* : end(R,v) \to end(S,w)$. We need to show that $[f, C_R, C_S] = [f', C'_R, C'_S]$. By passing to larger cut sets, we may assume that $C_R = C'_R$. Thus, we need to show that given $c \in C_R$,

(1) if R_c is not an isolated ray, then $f|R_c = f'|R_c$,
(2) if R_c is an isolated ray, then $f(R_c) \cap f'(R_c) \neq \emptyset$.

Let $\alpha \in end(R,v)$ be such that $\alpha(||c||) = c$. Let $\hat{\alpha} : [0, ||fc||] \to S$ and $\hat{\alpha}' : [0, ||f'c||] \to S$ be the unique isometric embeddings such that $\hat{\alpha}(0) = w = \hat{\alpha}'(0)$, $\hat{\alpha}(||fc||) = fc$ and $\hat{\alpha}'(||f'c||) = f'c$. Then

$$f_*(\alpha)(t) = \begin{cases} \hat{\alpha}(t) & \text{if } 0 \leq t \leq ||fc|| \\ f\alpha(t - ||fc|| + ||c||) & \text{if } ||fc|| \leq t, \end{cases}$$

and

$$f'_*(\alpha)(t) = \begin{cases} \hat{\alpha}'(t) & \text{if } 0 \leq t \leq ||f'c|| \\ f'\alpha(t - ||f'c|| + ||c||) & \text{if } ||f'c|| \leq t, \end{cases}$$

Since $f_*(\alpha) = f'_*(\alpha)$, it follows that $f\alpha(t - ||fc|| + ||c||) = f'\alpha(t - ||f'c|| + ||c||)$ for all $t \geq \max\{||fc||, ||f'c||\}$. In particular, $f(R_c) \cap f'(R_c) \neq \emptyset$, so we are left with the case that R_c is not an isolated ray. In this case, there exists $\beta \in end(R,v)$ such that $\alpha \neq \beta$ and $\beta(||c||) = c$. Let $d_e(\alpha, \beta) = e^{-t_0}$ where $t_0 \geq ||c||$. Now $d_e(f_*\alpha, f_*\beta) = e^{-t_0 + ||c|| - ||fc||}$ and $d_e(f'_*\alpha, f'_*\beta) = e^{-t_0 + ||c|| - ||f'c||}$. since $f_* = f'_*$, it follows that $||fc|| = ||f'c||$. From this it follows that $f|R_c = f'|R_c$. □

6. The tree of an ultrametric space and the categorical equivalence

In this section we complete the proof of the Main Theorem by showing how an $\mathbb{R}$-tree T_X may be associated with any ultrametric space X of finite diameter. The construction is, in fact, quite well-known. Proposition 6.4 shows that the end space of T_X is similar to the metric completion of X.

Let (X,d) be an ultrametric space of finite diameter $d_0 > 0$. Define an equivalence relation $\sim$ on $X \times [-\ln d_0, \infty)$ by

$$(x,t) \sim (y,s) \text{ if } t = s \text{ and } d(x,y) \leq 1/e^t.$$

Definition 6.1. The *tree associated to* (X, d) is $T_X = (X \times [-\ln d_0, \infty))/\sim$.

A point in T_X is denoted by its equivalence class $[x, t]$ where $(x, t) \in X \times [-\ln d_0, \infty)$. If $[x, t], [y, s] \in T_X$, define

$$D([x,t],[y,s]) = \begin{cases} |t-s| & \text{if } x = y \\ t + s - 2\min\{-\ln d(x,y), t, s\} & \text{if } x \neq y. \end{cases}$$

Note that with the convention $-\ln 0 = \infty$,

$$D([x,t],[y,s]) = t + s - 2\min\{-\ln d(x,y), t, s\} \text{ for all } x, y, s, t.$$

Proposition 6.2. *D is a metric on T_X. Moreover, the function $q : X \times [-\ln d_0, \infty) \to T_X$, $(x,t) \mapsto [x,t]$, is continuous.*[2]

Proof. The first step is to show that D is well-defined. Suppose $[x, s] = [x', s]$; i.e., $d(x, x') \leq e^{-s}$, and show that

$$2s - 2\min\{s, -\ln d(x,y)\} = 2s - 2\min\{s, -\ln d(x',y)\}.$$

It suffices to show that for all $y \in X$,

$$\max\{e^{-s}, d(x,y)\} = \max\{e^{-s}, d(x',y)\}.$$

This can be accomplished by considering two cases:

(1) Assume $d(x, y) \geq e^{-s}$ and show that $d(x, y) = \max\{e^{-s}, d(x', y)\}$. Proposition 4.2(6) implies that either $d(x, y) = d(x', y)$, or $d(x, y) = d(x, x') = e^{-s}$ and $d(x', y) \leq e^{-s}$. It is easy to see that either of these situations implies the desired conclusion.

(2) Assume $d(x, y) \leq e^{-s}$ and show that $e^{-s} = \max\{e^{-s}, d(x', y)\}$. If $d(x', y) > e^{-s}$, then $d(x, x') < d(x', y)$ and $d(x, y) < d(x', y)$, contradicting Proposition 4.2(6). Thus, $d(x', y) \leq e^{-s}$ and the desired conclusion follows.

This completes the proof that D is well-defined.

The next step is to show that D is a metric. To see that $D \geq 0$, let $[x, s], [y, t] \in T_X$. If $s = \min\{s, t, -\ln d(x, y)\}$, then $D([x, s], [y, t]) = s + t - 2s = t - s \geq 0$. A similar statement can be made if t is the minimum. If the minimum is $-\ln d(x, y)$, then $D([x, s], [y, t]) = s + t - 2(-\ln d(x, y)) \geq s + t - s - t = 0$.

To see that $D([x, s], [y, t]) = 0$ if and only if $[x, s] = [y, t]$, first note that $D([x, s], [x, s]) = 0$. Conversely, suppose $D([x, s], [y, t]) = 0$ (i.e., $s + t = 2\min\{s, t, -\ln d(x, y)\}$) and show $s = t$ and $d(x, y) \leq e^{-s}$. On the contrary, assume that $s \neq t$. Without loss of generality assume $s < t$. Then $2s < s + t = 2\min\{s, -\ln d(x, y\}$ and $s < s$, a contradiction. Thus, $s = t$ and $2s = 2\min\{s, -\ln d(x, y)\}$. Hence, $s \leq \ln d(x, y)$, which is to say $d(x, y) \leq e^{-s}$.

It is clear that D is symmetric. In order to verify the triangle inequality[3]

$$D([x,s],[y,t]) \leq D([x,s],[z,u]) + D([y,t],[z,u])$$

[2]It need not be the case that q is a quotient map.

[3]The proof of the triangle inequality for D requires the ultrametric property of d. For example, if $X = \{x, y, z\}$ with metric d given by $d(x, y) = 3$, $d(x, z) = 1$, $d(z, y) = 2$, then D will not satisfy the triangle inequality.

for $x, y, z \in X$ and $s, t, u \in [-\ln d_0, \infty)$, let $a = -\ln d(x,z)$, $b = -\ln d(y,z)$, $c = -\ln d(x,y)$. We need to show that $\min\{s,u,a\}+\min\{t,u,b\} \leq \min\{s,t,c\}+u$. The ultrametric inequality $d(x,y) \leq \max\{d(x,z), d(z,y)\}$ becomes $e^{-c} \leq \max\{e^{-a}, e^{-b}\}$, which is equivalent to $\min\{a,b\} \leq c$. Without loss of generality assume that $a \leq b$ so that $a = \min\{a,b,c\}$. There are three cases to consider:

1. $u = \min\{s,u,a\}$ and show $u + \min\{t,u\} \leq \min\{t,u\} + u$. This is clear.
2. $s = \min\{s,u,a\}$ and show $s + \min\{t,u,b\} \leq \min\{s,t\} + u$. This becomes clear upon considering the two subcases: $s = \min\{s,t\}$ and $t = \min\{s,t\}$.
3. $a = \min\{s,u,a\}$ and show $a + \min\{t,u,b\} \leq \min\{a,t\} + u$. This becomes clear upon considering the two subcases: $a = \min\{a,t\}$ and $t = \min\{a,t\}$.

It remains to show that $q : X \times [-\ln d_0, \infty) \to T_X$ is continuous. First observe that for each $x \in X$, $q| : \{x\} \times [-\ln d_0, \infty) \to T_X$ is an isometric embedding. Now suppose $x_n \to x$ in X and $t_n \to t$ in $[-\ln d_0, \infty)$. Choose a positive integer N such that $n \geq N$ implies $|1/e^t - 1/e^{t_n}| < 1/2e^t$ and $d(x, x_n) < 1/2e^t$. Then $n \geq N$ implies $d(x_n, x) < 1/e^{t_n}$; hence, $[x_n, t_n] = [x, t_n]$. Thus, for $n \geq N$, $q(x_n, t_n) = [x_n, t_n] = [x, t_n] \to [x, t] = q(x, t)$. □

We have to make a special definition if X consists of a single point (i.e., if X has diameter 0). In this case, let $T_X = [0, \infty)$ with the usual metric and root 0.

Theorem 6.3. *If (X,d) is an ultrametric space of finite diameter, then (T_X, D) is a geodesically complete $\mathbb{R}$-tree.*

Proof. To see that T_X is an $\mathbb{R}$-tree, it suffices to show that T_X is connected and is 0-hyperbolic in the sense of Gromov [Gr1] (e.g., see [Chi, Lemma 4.13]). Since T_X is obviously connected (by Proposition 6.2 every point is in the path component of the root r_X), we proceed to show that T_X is 0-hyperbolic. Recall that if $[x,t], [y,s] \in T_X$, then the Gromov product of $[x,t]$ and $[y,s]$ with respect to the root r_X is given by

$$([x,t] \cdot [y,s])_{r_X} = \frac{1}{2}\{D([x,t], r_X) + D([y,s], r_X) - D([x,t],[y,s])\}.$$

Using the fact that $D([x,t], r_X) = t + \ln d_0$ for all $[x,t] \in T_X$ where $d_0 = \operatorname{diam} X$ (we may assume that $d_0 > 0$ because the $d_0 = 0$ case is trivial), it is easy to calculate

$$([x,t] \cdot [y,s])_{r_X} = \ln d_0 + \min\{-\ln d(x,y), t, s\}. \tag{6.3.1}$$

Given $[z,u] \in T_X$, formula (6.3.1) is to be compared with

$$\begin{aligned} &\min\{([x,t] \cdot [z,u])_{r_X}, ([y,s] \cdot [z,u])_{r_X}\} = \\ &\min\{\ln d_0 + \min\{-\ln d(x,z), t, u\}, \ln d_0 + \min\{-\ln d(y,z), s, u\}\} = \\ &\qquad \ln d_0 + \min\{-\ln d(x,z), -\ln d(y,z), t, s, u\}. \end{aligned} \tag{6.3.2}$$

To verify 0-hyperbolicity, we need to conclude that (6.3.1) is greater than or equal to (6.3.2). This amounts to checking that

$$-\ln d(x,y) \geq \min\{-\ln d(x,z), -\ln d(y,z)\}.$$

This is equivalent to

$$\ln d(x,y) \leq \max\{\ln d(x,z), \ln d(y,z)\},$$

which comes from the ultrametric inequality for (X,d).

To see that T_X is geodesically complete, let $\alpha : [0,t_0] \to T_X$ be an isometric embedding such that $\alpha(0) = r_X$. Then $\alpha(t_0) = [x_0, t_0 - \ln d_0]$ for some $x_0 \in X$, and uniqueness of arcs in T_X implies that $\alpha(t) = [x_0, t - \ln d_0]$ for $0 \leq t \leq t_0$. The same formula, but now for all $t \geq 0$, gives an extension of α to a geodesic ray. □

The tree T_X comes with a natural root r_X. If the diameter of X is $d_0 > 0$, then $r_X = [x, -\ln d_0]$ for any $x \in X$. If $d_0 = 0$, then $r_X = 0$.

Proposition 6.4. *If (X,d) is an ultrametric space of finite diameter, then the metric completion of X is similar to $end(T_X, r_X)$.*

Proof. We may assume that $d_0 = \operatorname{diam} X > 0$. Define $h : X \to end(T_X, r_X)$ by $h(x)(t) = [x, t - \ln d_0]$ for $0 \leq t < \infty$. If $x, y \in X$ with $x \neq y$, then $h(x) \neq h(y)$ and $d_e(h(x), h(y)) = e^{-t_0}$ where

$$\begin{aligned} t_0 &= \sup\{t \geq 0 \mid h(x)(t) = h(y)(t)\} \\ &= \sup\{t \geq 0 \mid [x, t - \ln d_0] = [y, t - \ln d_0]\} \\ &= \sup\{t \geq 0 \mid d(x,y) \leq e^{\ln d_0 - t}\} \\ &= \sup\{t \geq 0 \mid t \leq \ln d_0 - \ln d(x,y)\} \\ &= \ln \frac{d_0}{d(x,y)}. \end{aligned}$$

Thus, $d_e(h(x), h(y)) = \frac{1}{d_0} d(x,y)$, which is to say h is a $(1/d_0)$-similarity.

Since $end(T_X, r_X)$ is complete with respect to the metric d_e (Prop. 5.2), we must now show that $h(X)$ is dense in $end(T_X, r_X)$. For this, let $\alpha : [0,\infty) \to T_X$ be an isometric embedding such that $\alpha(0) = r_X$ and let $\epsilon > 0$ be given. Choose $t_0 > 0$ such that $e^{-t_0} < \epsilon$. As stated at the end of the proof of Proposition 6.3, there exists $x_0 \in X$ such that $\alpha(t) = [x_0, t - \ln d_0]$ for $0 \leq t \leq t_0$. It follows that $h(x_0)(t) = \alpha(t)$ for $0 \leq t \leq t_0$ and, therefore, $d_e(\alpha, h(x_0)) \leq e^{-t_0} < \epsilon$. □

Proposition 6.5. *If $h : X \to Y$ is a local similarity equivalence between complete ultrametric spaces of finite diameter, then there is an induced isometry at infinity $(\tilde{h}, C_X, C_Y) : (T_X, r_X) \to (T_Y, r_Y)$.*

Proof. Let d_0 and d_1 be the diameters of X and Y, respectively. We may assume that $d_0, d_1 > 0$. By Lemma 4.10 there exist a subset $E \subset X$ and positive numbers $\lambda_x > 0$, $r_x > 0$ for each $x \in E$ such that

1. $h| : \bar{B}(x, r_x) \to \bar{B}(hx, \lambda_x r_x)$ is a surjective λ_x-similarity for each $x \in E$,
2. if $x, y \in E$, $x \neq y$, then $\bar{B}(x, r_x) \cap \bar{B}(y, r_y) = \emptyset$,
3. $X = \cup_{x \in E} \bar{B}(x, r_x)$,
4. for each $x \in E$, $r_x < d_0$,
5. if $x \in E$ is an isolated point of X, then $\bar{B}(x, r_x) = \{x\}$ and $\lambda_x = 1$,
6. if $x \in E$ is not an isolated point of X, then $\bar{B}(x, r_x)$ is the closure of $B(x, r_x)$.

The first four items follow immediately from Lemma 4.10. We may assume item (5) simply by redefining r_x and λ_x for the isolated points. Likewise item (6) can be achieved by defining a new r_x for x non-isolated to be $\sup\{d(x,y) \mid y \in B(x, r_x)\}$.

For each $x \in E$, let $c_x = [x, -\ln r_x] \in T_X$. We verify that $C_X = \{c_x \mid x \in E\}$ is a cut set for (T_X, r_X). First note that the root $r_X \notin C_X$ because of item (4).

Next, we make use of the following fact about the special form ends in (T_X, r_X) must take.

Claim 6.5.1. *If $\alpha : [0, \infty) \to T_X$ is any isometric embedding with $\alpha(0) = r_X$, then there exists $x_0 \in X$ such that $\alpha(t) = [x_0, t - \ln d_0]$ for every $t \geq 0$.*

Proof. As in the end of the proof of Proposition 6.3, for each $n = 1, 2, 3, \ldots$ there exists $x_n \in X$ such that $\alpha(t) = [x_n, t - \ln d_0]$ for $0 \leq t \leq n$. In particular, $[x_n, n - \ln d_0] = [x_m, n - \ln d_0]$ for each $m \geq n$. Thus, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X; let $x_0 = \lim_{n\to\infty} x_n$. Hence, $\alpha(t) = [x_0, t - \ln d_0]$ for all $t \geq 0$. $\square$

Continuing with the proof that C_X is a cut set, we must show that there exists a unique $t_0 > 0$ such that $\alpha(t_0) \in C_X$. Choose $x \in E$ such that $d(x, x_0) \leq r_x$. It follows that $\alpha(\ln d_0 - \ln r_x) = c_x$, thereby establishing existence of t_0. For uniqueness, suppose $y \in E$ and $t \geq 0$ such that $\alpha(t) = [y, -\ln r_y]$. Then $[x_0, t - \ln d_0] = [y, -\ln r_y]$, implying $t = \ln d_0 - \ln r_y$ and $d(x_0, y) \leq 1/e^{-\ln r_y} = r_y$; that is, $y = x$. This completes the proof that C_X is a cut set for (T_X, r_X).

Note now that for each $x \in E$, $\lambda_x r_x < d_1$. For suppose that $\lambda_x r_x \geq d_1$ for some $x \in E$. Then $\bar{B}(h(x), \lambda_x r_x) = Y$ and so $\bar{B}(x, r_x) = X$. Since $d_0 > 0$, it follows that x is not an isolated point of X. Thus, item (7) implies that $r_x = d_0$, a contradiction to item (4).

Let $F = h(E)$ and for each $y = h(x)$ with $x \in E$, let $c_y = [h(x), -\ln(\lambda_x r_x)]$. The fact just established that for each $x \in E$, $\lambda_x r_x < d_1$, implies that each $c_y \in T_Y$ and $c_y \neq r_Y$. We claim that $C_Y = \{c_y \mid y \in F\}$ is a cut set for (T_Y, r_Y). The proof mirrors the proof above that C_X is a cut set. The main fact needed is that if $\alpha : [0, \infty) \to Y$ is an isometric embedding such that $\alpha(0) = r_Y$, then there exists $y_0 \in Y$ such that $\alpha(t) = [y_0, t - \ln d_1]$ for all $t \geq 0$. This follows from the completeness of Y.

Define an isometry at infinity $(\tilde{h}, C_X, C_Y) : (T_X, r_X) \to (T_Y, r_Y)$ as follows. If $[z, t] \in \cup\{(T_X)_{c_x} \mid x \in E\}$, then let $x(z)$ be the unique point of E such that $[z, t] \in (T_X)_{c_{x(z)}}$. It follows that $d(z, x(z)) \leq 1/e^{-\ln r_{x(z)}} = r_{x(z)}$ and $t \geq -\ln r_{x(z)}$. In particular, $x(z)$ is the unique point of E such that $z \in \bar{B}(x(z), r_{x(z)})$. Define

$$\tilde{h}([z, t]) = [h(z), t - \ln \lambda_{x(z)}] \in T_Y.$$

Note that $d(h(z), h(x(z))) \leq \lambda_{x(z)} r_{x(z)}$ and $t - \ln \lambda_{x(z)} \geq -\ln(\lambda_{x(z)} r_{x(z)})$. Hence, $\tilde{h}([z, t]) \in (T_Y)_{c_{h(x(z))}}$. It can be checked that

(1) $\tilde{h}(C_X) = C_Y$, and
(2) for any $c_x \in C_X$, $\tilde{h}| : (T_X)_{c_x} \to (T_Y)_{h(c_x)} = (T_Y)_{c_{h(x)}}$ is an isometry. $\square$

Definition 6.6. Define $\mathcal{T} : \mathbf{U} \to \mathbf{T}$ by $\mathcal{T}(X, d) = (T_X, r_X)$ for every complete ultrametric space X of finite diameter, and $\mathcal{T}(h) = h_*$ for every local similarity equivalence h, where h_* is the equivalence class of the isometry at infinity $\tilde{h}$ defined in Proposition 6.5.

Theorem 6.7. *$\mathcal{T} : \mathbf{U} \to \mathbf{T}$ is a functor.*

Proof. The functorial properties are easy to verify given the explicit construction in the proof of Proposition 6.5. Perhaps the only ambiguity lies in the local similarity constant λ_x at an isolated point x. But the isolated points in the ultrametric space X lead to isolated rays in the tree T_X; therefore, the ambiguity is eliminated by Definition 3.7(2). $\square$

The following proposition follows immediately from Proposition 6.4.

Proposition 6.8. *The composition of functors* $\mathbf{U} \xrightarrow{\mathcal{I}} \mathbf{T} \xrightarrow{\mathcal{E}} \mathbf{U}$ *takes every object in* $\mathbf{U}$ *to a similar copy of itself.* □

The following result is a restatement of the Main Theorem in the introduction.

Theorem 6.9. *The functor* $\mathcal{E} : \mathbf{T} \to \mathbf{U}$ *is an equivalence of categories.*

Proof. It is enough to know that $\mathcal{E}$ is a full and faithful functor, and that, given any object (X, d) in $\mathbf{U}$, there exists an object (T, v) in $\mathbf{T}$ such that there is a local similarity equivalence $end(T, v) \to X$ (see [MaL]). Thus, the theorem follows from Propositions 5.5 and 6.8. □

Example 6.10. It need not be the case that (T, v) and (T_X, r_X) are isometric, where $X = end(T, v)$. For example, let $T = \{(x, y) \in \mathbb{R}^2 \mid x \geq -1 \text{ and } y = 0, \text{ or } x = 0 \text{ and } y \geq 0\}$ with the length metric induced from the restriction of the standard metric on $\mathbb{R}^2$. Let $v = (-1, 0)$. Then $X = \{\alpha, \beta\}$ with $d_e(\alpha, \beta) = e^{-1}$ and T_X is isometric to $\mathbb{R}$.

7. Uniform isometries at infinity and uniform local similarity equivalences

This section contains a proof of Corollary 1 to the Main Theorem. For the category $\mathbf{T}$ of trees and equivalence classes of isometries at infinity considered above, a subcategory $\mathbf{T_u}$ is defined by allowing only those isometries with domain the complement of a metric ball about the root. These are the so-called uniform isometries at infinity, where "uniform" refers to the fact that the roots of the subtrees making up the domain of the isometry are all a constant distance from the root of the original tree.

Likewise for the category $\mathbf{U}$ of ultrametric spaces and local similarity equivalences, a subcategory $\mathbf{U_u}$ is defined by allowing only those local similarities with constant moduli of similarity.

Theorem 7.13 shows that the functor $\mathcal{E} : \mathbf{T} \to \mathbf{U}$ restricts to an equivalence of categories $\mathcal{E}_{\mathbf{u}} : \mathbf{T_u} \to \mathbf{U_u}$.

Definition 7.1. An isometry at infinity $(f, C_T, C_S) : (T, v) \to (S, w)$ between geodesically complete, rooted $\mathbb{R}$-trees is an *uniform isometry at infinity* provided there exist $\epsilon, \delta > 0$ such that $C_T = \partial B(v, \epsilon)$ and $C_S = \partial B(w, \delta)$.

Definition 7.2. Two uniform isometries at infinity from (T, v) to (S, w) are *equivalent* provided they are equivalent as isometries at infinity.

Example 7.3. Let (T, v) be a geodesically complete, rooted $\mathbb{R}$-tree and $\epsilon > 0$. If $f : T \to T$ is any isometry, then $(f, \partial B(v, \epsilon), \partial B(f(v), \epsilon)) : (T, v) \to (T, f(v))$ is a uniform isometry at infinity.

Following the discussion in §3 we know that two equivalence classes of uniform isometries at infinity can be composed to get the equivalence class of an isometry at infinity. We now observe that this is, in fact, the equivalence class of an *uniform* isometry at infinity. To this end, let $[f, C_R] : (R, v) \to (S, w)$ and $[g, C_S] : (S, w) \to (T, x)$ be equivalence classes of uniform isometries at infinity between geodesically complete, rooted $\mathbb{R}$-trees. Thus, there exist $\epsilon_1, \epsilon_2, \delta_1, \delta_2 > 0$ such that $C_R = \partial B(v, \epsilon_1)$, $f(C_R) = \partial B(w, \delta_1)$, $C_S = \partial B(w, \epsilon_2)$ and $g(C_S) = \partial B(x, \delta_2)$. Let $\max\{\delta_1, \epsilon_2\} = \delta_1 + \lambda_1 = \epsilon_2 + \lambda_2$ where one of λ_1, λ_2 is 0. Then (using notation consistent with §3) one may check that $C'_S = \max\{f(C_R), C_S\} = \partial B(w, \delta_1 + \lambda_1)$, $C'_R = f^{-1}(C'_S) = \partial B(\epsilon_1 + \lambda_1)$ and $C_T = g(C'_S) = \partial B(x, \delta_2 + \lambda_2)$. Thus, $gf(C'_R) = C_T$ is the boundary of a ball centered at $x \in T$ and $[g, C_S] \circ [f, C_R] = [g| \circ f|, C'_R]$ is the equivalence class of a uniform isometry at infinity.

Definition 7.4. If (T, v) is a geodesically complete, rooted $\mathbb{R}$-tree, let $Isom^u_\infty(T, v)$ denote the *group of equivalence classes of uniform isometries at infinity from* (T, v) *to itself*.

Proposition 7.5. *$Isom^u_\infty(T, v)$ is a subgroup of $Isom_\infty(T, v)$*

Proof. $Isom^u_\infty(T, v)$ is closed under inverses by Lemma 3.4, and is closed under composition by the preceding discussion. □

Definition 7.6. Let $\mathbf{T_u}$ be the subcategory of $\mathbf{T}$ having the same objects, but whose morphisms are the equivalence classes of uniform isometries at infinity.

Definition 7.7. A homeomorphism $h : X \to Y$ between metric spaces is a *uniform local similarity equivalence* if there exist $\epsilon > 0$ and $\lambda > 0$ such that for every $x \in X$ the restriction $h| : B(x, \epsilon) \to B(h(x), \lambda\epsilon)$ is a surjective λ-similarity.

Lemma 7.8. *The inverse of a uniform local similarity equivalence is a uniform local similarity equivalence. The composition of two uniform local similarity equivalences is a uniform local similarity equivalence.*

Proof. The proof of Lemma 4.5 for the corresponding facts about local similarity equivalences specializes to give a proof for the uniform case. □

Definition 7.9. If (X, d) is a metric space, let $LSE^u(X)$ denote the *group of uniform local similarity equivalences from* X *to itself*.

The following result follows immediately from Lemma 7.8.

Proposition 7.10. *$LSE^u(X)$ is a subgroup of $LSE(X)$.* □

Definition 7.11. Let $\mathbf{U_u}$ be the subcategory of $\mathbf{U}$ having the same objects as $\mathbf{U}$, but whose morphisms are uniform local similarity equivalences.

Proposition 7.12. *Let (T, v) be a geodesically complete, rooted $\mathbb{R}$-tree with metric d, $X = end(T, v)$ with metric d_e, and $T_X = T_{end(T,v)}$ with metric D and root r_X. Then (T, v) and (T_X, r_X) are uniformly isometric at infinity.*

Proof. Let $d_0 = \operatorname{diam} X$. If $d_0 = 0$, then T and T_X are both single isolated rays; hence, isometric. Thus, we may assume $d_0 > 0$. Let $r > -\ln d_0$ (of course, $d_0 \leq 1$ by Proposition 5.2 so $r > 0$). Define $h : T \setminus B(v, r) \to T_X$ as follows. If $z \in T \setminus B(v, r)$, let $\alpha_z : [0, \infty) \to T$ be an isometric embedding such that $\alpha_z(0) = v$ and z is in the image of α. Then $\alpha_z(d(v, z)) = z$. Let

$$h(z) = [\alpha_z, d(v, z)] \in T_X.$$

(Since $d(v, z) > -\ln d_0$, $h(z) \in T_X$.) To see that h is well-defined, suppose that $\beta : [0, \infty) \to T$ is another isometric embedding with $\beta(0) = v$ and$\beta(d(v, z)) = z$. To show that $[\alpha_z, d(v, z)] = [\beta, d(v, z)]$ we are required to show that $d_e(\alpha_z, \beta) \leq e^{-d(v,z)}$. Since $\alpha_z(t) = \beta(t)$ for $0 \leq t \leq d(v, z)$, it follows that $d_e(\alpha_z, \beta) \leq e^{-d(v,z)}$ as required.

Now let $c \in \partial B(v,r)$ and show that $h| : T_c \to T_X$ is an isometric embedding. Thus, let $z, w \in T_c$ and show $D(h(z), h(w)) = d(z,w)$. To this end, let $\alpha_z, \alpha_w : [0,\infty) \to T$ be isometric embeddings such that $\alpha_z(0) = v = \alpha_w(0)$, $\alpha_z(r) = c = \alpha_w(r)$, and $\alpha_z(d(v,z)) = z$, $\alpha_w(d(v,w)) = w$. If there exist such α_z and α_w such that $\alpha_z = \alpha_w$, then assume that we have chosen α_z and α_w such that $\alpha_z = \alpha_w$. Thus, $h(z) = [\alpha_z, d(v,z)]$, $h(w) = [\alpha_w, d(v,w)]$ and

$$D(h(z), h(w)) = d(v,z) + d(v,w) - 2\min\{-\ln d_e(\alpha_z, \alpha_w), d(v,z), d(v,w)\}$$

(we are using the convention $-\ln 0 = \infty$ here and below). Recall the Gromov product [Gr1] (as in the proof of Theorem 6.3):

$$(z \cdot w)_v = \frac{1}{2}\{d(v,z) + d(v,w) - d(z,w)\}.$$

Thus,

$$d(z,w) = d(v,z) + d(v,w) - 2(z \cdot w)_v.$$

Moreover, it is easy to verify (using the $\mathbb{R}$-tree properties) that

$$(z \cdot w)_v = \begin{cases} \min\{d(v,z), d(v,w)\} & \text{if } \alpha_z = \alpha_w \\ \sup\{t \geq 0 \mid \alpha_z(t) = \alpha_w(t)\} & \text{if } \alpha_z \neq \alpha_w \end{cases}$$

(this uses the fact $\alpha_z \neq \alpha_w$ if and only if it is not possible to choose α_z and α_w such that $\alpha_z = \alpha_w$). Of course, $-\ln d_e(\alpha_z, \alpha_w) = \sup\{t \geq 0 \mid \alpha_z(t) = \alpha_w(t)\}$. It follows that $\min\{-\ln d_e(\alpha_z, \alpha_w), d(v,z), d(v,w)\} = (z \cdot w)_v$. Hence, $D(h(z), h(w)) = d(v,z) + d(v,w) - 2(z \cdot w)_v = d(z,w)$, completing the proof that $h|T_c$ is an isometric embedding.

Next we need to show that for every $c \in \partial B(v,r)$, $h| : T_c \to (T_X)_{h(c)}$ is onto. If $[\alpha, t] \in (T_X)_{h(c)}$, then $\alpha : [0,\infty) \to T$ is an isometric embedding with $\alpha(0) = v$, $\alpha(r) = c$ and $t \geq r$. Then $h(\alpha(t)) = [\alpha, d(v, \alpha(t))] = [\alpha, t]$. Note that $\alpha(t) \in T_c$ so that $h|$ is onto.

Finally, we show that $h(\partial B(v,r)) = \partial B(r_X, \delta)$ where $\delta = r + \ln d_0$. (Note that $r + \ln d_0 > \max\{\ln d_0, 0\} \geq 0$.) If $z \in \partial B(v,r)$, then $h(z) = [\alpha_z, r]$ and $D([\alpha_z, r], r_X) = D([\alpha_z, r], [\alpha_z, -\ln d_0]) = r + \ln d_0$ as required. On the other hand, if $[\alpha, t] \in \partial B(r_X, \delta)$, then $D([\alpha, t], r_X) = \delta$, so $D([\alpha, t], [\alpha, -\ln d_0]) = \delta$. Thus, (using the fact that $t \geq -\ln d_0$ so that $t + \ln d_0 \geq 0$), we have $t + \ln d_0 = |t + \ln d_0| = r + \ln d_0$ so that $t = r$ as required.

We have shown that $(h, \partial B(v,r), \partial B(r_X, r + \ln d_0)) : (T,v) \to (T_X, r_X)$ is a uniform isometry at infinity. □

Theorem 7.13. *The functor $\mathcal{E}$ restricts to an equivalence of categories*

$$\mathcal{E}_{\mathbf{u}} : \mathbf{T}_{\mathbf{u}} \to \mathbf{U}_{\mathbf{u}}.$$

Proof. In order to show that $\mathcal{E}$ restricts to a functor $\mathcal{E}_{\mathbf{u}}$ on $\mathbf{T}_{\mathbf{u}}$ with image in $\mathbf{U}_{\mathbf{u}}$ we need to show that $\mathcal{E}$ takes equivalence classes of uniform isometries at infinity to uniform local similarity equivalences. Let $(f, \partial B(v, r_1), \partial B(w, r_2)) : (T,v) \to (S,w)$ be a uniform isometry at infinity between geodesically complete, rooted $\mathbb{R}$-trees for some $r_1, r_2 > 0$, and let $f_* : end(T,v) \to end(S,w)$ be the induced local similarity equivalence given in the proof of Proposition 5.3. Given $\alpha \in end(T, v)$ the unique $t_0 > 0$ such that $\alpha(t_0) \in \partial B(v, r_1)$ is, of course, $r_1 = t_0$ and is independent of α. Thus, the $\epsilon = e^{-r_1}$ of 5.3 is also independent of α. Moreover, $f\alpha(r_1) \in \partial B(w, r_2)$ so $||f\alpha(r_1)|| = r_2$ and the $\lambda = e^{r_1 - r_2}$ of 5.3 is also independent of α. Thus, f_* is indeed a *uniform* local similarity equivalence.

Likewise we need to show that $\mathcal{T}$ restricts to a functor $\mathcal{T}_{\mathbf{u}}$ on $\mathbf{U_u}$ with image in $\mathbf{T_u}$. For this we need to observe that $\mathcal{T}$ takes a uniform local similarity equivalence to the equivalence class of a uniform isometry at infinity. Let $h : X \to Y$ be a uniform local similarity equivalence between complete ultrametric spaces with finite diameters d_0 and d_1, respectively (which we assume positive). Suppose $\epsilon > 0$ and $\lambda > 0$ are the constants associated with h as in Definition 7.7. We need to show that the isometry at infinity $(\tilde{h}, C_X, C_Y) : (T_X, r_X) \to (T_Y, r_Y)$ constructed in the proof of Proposition 6.5 is equivalent to a uniform isometry at infinity. For this observe that the r_x's in 6.5 (which, in turn, come from Lemma 4.10) may be chosen so that $\frac{\epsilon}{2} \le r_x$ for all $x \in E$, the important fact being that the r_x's are uniformly bounded below by a positive constant. Thus, $\partial B(r_X, \ln d_0 - \ln \frac{\epsilon}{2})$ is a cut set for (T_X, r_X) larger than C_X. Now for the λ_x's appearing in 6.5, for each $x \in E$ that is not an isolated point of X, we necessarily have $\lambda_x = \lambda$. It follows that $\partial B(r_Y, \ln d_1 - \ln \lambda\frac{\epsilon}{2})$ is a cut set for (T_Y, r_Y) and $(\hat{h}, \partial B(r_X, \ln d_0 - \ln \frac{\epsilon}{2}), \partial B(r_Y, \ln d_1 - \ln \lambda\frac{\epsilon}{2})) : (T_X, r_X) \to (T_Y, r_Y)$ given by $\hat{h}([z, t]) = [h(z), t - \ln \lambda]$ for each $[z, t] \in T_X \setminus B(r_X, \ln d_0 - \ln \frac{\epsilon}{2})$, is a uniform isometry at infinity equivalent to $\tilde{h}$.

Hence, we have a commuting diagram of functors

$$\begin{array}{ccccc} \mathbf{T_u} & \xrightarrow{\mathcal{E}_{\mathbf{u}}} & \mathbf{U_u} & \xrightarrow{\mathcal{T}_{\mathbf{u}}} & \mathbf{T_u} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{T} & \xrightarrow{\mathcal{E}} & \mathbf{U} & \xrightarrow{\mathcal{T}} & \mathbf{T} \end{array}$$

where the vertical arrows are inclusion functors.

To verify that $\mathcal{E}_{\mathbf{u}}$ is full, follow the proof of Proposition 5.5 that $\mathcal{E}$ is full. Suppose (R, v) and (S, w) are geodesically complete, rooted $\mathbb{R}$-trees for which there exists a *uniform* local similarity equivalence $h : end(R, v) \to end(S, w)$. In the proof of 5.5 there is constructed an isometry at infinity $(f, C_R, C_S) : (R, v) \to (S, w)$ such that $f_* = h$. What we now need to observe is that f can be constructed to be a *uniform* isometry at infinity. The key to this is when using Lemma 4.10 in the case of a uniform local similarity equivalence, the positive numbers λ_α, r_α may be chosen to be constants; say, $\lambda_\alpha = \lambda$ and $r_\alpha = r < 1$ for each $\alpha \in E$. Then $C_R = \{\alpha(-\ln r) \mid \alpha \in E\} \subseteq \partial B(v, -\ln r)$. Since C_R is a cut set, it must be that $C_R = \partial B(v, -\ln r)$. Likewise, $C_S = \partial B(w, -\ln \lambda r)$ and, thus, (f, C_R, C_S) is a uniform isometry at infinity.

The diagram above and the faithfulness of $\mathcal{E}$ imply that $\mathcal{E}_{\mathbf{u}}$ is faithful. Finally, since a similarity is, in particular, a uniform local similarity equivalence, the completion of the proof that $\mathcal{E}_{\mathbf{u}}$ is an equivalence follows as in the proof of Theorem 6.9. □

To close this section, we point out in the next result that the obvious notion of "uniform equivalence" is no different from the notion of equivalence that we are using.

Proposition 7.14. *Suppose*

$$(f, \partial B(v, \epsilon_1), \partial B(w, \delta_1)), (f', \partial B(v, \epsilon_2), \partial B(w, \delta_2)) : (T, v) \to (S, w)$$

are two uniform isometries at infinity between geodesically complete, rooted $\mathbb{R}$*-trees such that* $[f, \partial B(v, \epsilon_1)] = [f', \partial B(v, \epsilon_2)]$. *Let* $\epsilon = \max\{\epsilon_1, \epsilon_2\}$. *Then for every* $c \in \partial B(v, \epsilon)$*:*

(1) *if* T_c *is not an isolated ray, then* $f|T_c = f'|T_c$,
(2) *if* T_c *is an isolated ray, then* $f(T_c) \cap f'(T_c) \neq \emptyset$.

Proof. Let C be a cut set for (T, v) so that (1) and (2) hold for every $c \in C$ (such a C exists because $[f] = [f']$). We may assume that C is larger than $\partial B(v, \epsilon)$. If $c \in \partial B(v, \epsilon)$, then $T_c = \cup\{T_x \cup [c, x] \mid x \in T_c \cap C\}$. If T_c is not an isolated ray, then $f(x) = f'(x)$ for each $x \in T_c \cap C$. It follows that $f|[c, x] = f'|[c, x]$ (otherwise $[w, f(c)] \cup f[c, x]$ and $[w, f'(c)] \cup f'[c, x]$ would be two different arcs between w and $f(x)$). Hence, (1) holds. (2) holds because it holds with respect to C. □

8. Isometries and local isometries

In this section we introduce two more pairs of equivalent categories and summarize the relationship among the four pairs of categories studied in this paper.

As with the other pairs of categories, each new pair consists of one category whose objects are certain $\mathbb{R}$-trees and another category whose objects are certain ultrametric spaces. The first pair of equivalent categories is more standard than the others in that the morphisms are globally defined. The morphisms are rooted isometries of $\mathbb{R}$-trees in one of the categories, and isometries of ultrametric spaces in the other. I suspect that the equivalence of these two categories is known to experts, but I am unaware of a reference.

The second pair of categories is more in line with the other two pairs already defined. The morphisms for the $\mathbb{R}$-tree category are a specialization of equivalence classes of uniform isometries at infinity, namely, equivalence classes of *strong* uniform isometries at infinity. For the category of ultrametric spaces, the morphisms are local isometry equivalences, which lie between isometries and uniform local similarity equivalences.

A commuting diagram involving the four pairs of categories is given in Corollary 8.15. The corresponding diagram involving the groups of automorphisms of objects is given in Corollary 8.16.

We begin with rooted isometries of $\mathbb{R}$-trees and isometries of ultrametric spaces.

Recall that a homeomorphism between metric spaces is an *isometry* if it preserves distances. An isometry $h : T \to S$ between rooted $\mathbb{R}$-trees (T, v) and (S, w) is a *rooted isometry* provided $h(v) = w$.

Definition 8.1. Let $\mathbf{T_{RI}}$ be the *category of geodesically complete, rooted* $\mathbb{R}$*-trees and rooted isometries.* The objects of $\mathbf{T_{RI}}$ are geodesically complete, rooted $\mathbb{R}$-trees and the morphisms are rooted isometries.

If $h : (T, v) \to (S, w)$ is a rooted isometry, then $(h, \partial B(v, \epsilon), \partial B(w, \epsilon))$ is a uniform isometry at infinity for every $\epsilon > 0$. Hence, the morphism $[h]$ in $\mathbf{T_u}$ is well-defined and there is an induced functor $i : \mathbf{T_{RI}} \to \mathbf{T_u}$ which is the identity on objects.

Example 8.2. The functor $i : \mathbf{T_{RI}} \to \mathbf{T_u}$ is not full. For example, let $T = \{(x, y) \in \mathbb{R}^2 \mid x \geq -1 \text{ and } y = 0, \text{ or } x = 0 \text{ and } y \geq 0\}$ and let $S = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ and } y = 0, \text{ or } x = 0 \text{ and } y \geq 0\}$. Give T and S the length metrics induced from the restriction of the standard metric on $\mathbb{R}^2$. Then there exists a uniform isometry at infinity between $(T, (-1, 0))$ and $(S, (0, 0))$, but no rooted isometry.

Proposition 8.3. *The functor $i : \mathbf{T_{RI}} \to \mathbf{T_u}$ is faithful.*

Proof. Suppose $g, h : (T, v) \to (S, w)$ are two rooted isometries between geodesically complete, rooted $\mathbb{R}$-trees. Further suppose that $[g] = [h]$ in $\mathbf{T_u}$. In order to show that $g = h$, let C_T be a cut set for (T, v) such that for every $c \in C_T$:

(1) if T_c is not an isolated ray, then $g|T_c = h|T_c$,
(2) if T_c is an isolated ray, then $g(T_c) \cap h(T_c) \neq \emptyset$.

It suffices to show that $g|[v, c] \cup T_c = h|[v, c] \cup T_c$ for every $c \in C_T$. If T_c is not an isolated ray, then this is clear because $g(v) = h(v)$ and $g(c) = h(c)$. If T_c is an isolated ray, then there exist $x, y \in T_c$ such that $g(x) = h(y)$. It follows that $g([v, x]) = h([v, y])$ and so $x = y$. From this we have $g(c) = h(c)$ and $g|[v, c] = h|[v, c]$. Thus, $g(T_c) = h(T_c)$. This, together with the fact that T_c is an isolated ray, implies $g|T_c = h|T_c$. □

Definition 8.4. Let $\mathbf{U_I}$ be the subcategory of $\mathbf{U_u}$ with objects complete ultrametric spaces of diameter ≤ 1, and isometries for morphisms.

Theorem 8.5. *The composition of functors $\mathbf{T_{RI}} \xrightarrow{i} \mathbf{T_u} \xrightarrow{\mathcal{E}_\mathbf{u}} \mathbf{U_u}$ has image in $\mathbf{U_I}$. Moreover, the induced functor*

$$\mathbf{T_{RI}} \xrightarrow{\mathcal{E}_\mathbf{RI}} \mathbf{U_I}$$

is an equivalence of categories.

Proof. To see that the image is in $\mathbf{U_I}$, it is enough to observe that if $h : (T, v) \to (S, w)$ is a morphism in $\mathbf{T_{RI}}$, then $h_* : end(T, v) \to end(S, w)$ (as defined in the proof of Proposition 5.3) is an isometry. For this note that $h_*(\alpha) = h \circ \alpha$ for every $\alpha \in end(T, v)$. From this it follows immediately that $d_e(h_*(\alpha), h_*(\beta)) = d_e(\alpha, \beta)$ for every $\alpha, \beta \in end(T, v)$.

That $\mathcal{E}_\mathbf{RI}$ is faithful follows from Proposition 8.3 and Theorem 7.13.

To verify that $\mathcal{E}_\mathbf{RI}$ is full, follow the proofs of Proposition 5.5 and Theorem 7.13 that $\mathcal{E}$ and $\mathcal{E}_\mathbf{u}$ are full. Suppose (R, v) and (S, w) are geodesically complete, rooted $\mathbb{R}$-trees for which there exists an isometry $h : end(R, v) \to end(S, w)$. In the proof of 7.13 (which is based on the proof of 5.5) there is constructed a uniform isometry at infinity $(f, \partial B(v, -\ln r), \partial B(w, -\ln \lambda r)) : (R, v) \to (S, w)$ where $r < 1$ such that $f_* = h$. Since, in the present setting, h is an isometry, we may take $\lambda = 1$ and r arbitrarily close to 1. Thus, the balls whose boundaries are serving as cut sets may be taken to be of arbitrarily small radius. Now note that the formula for f in the proof of 5.5 is independent of r. In other words, we may define $f : R \to S$ by $f(x) = (h\alpha)(||x||)$ where $\alpha \in end(R, v)$ has the property that $\alpha(||x||) = x$. The point is that the proof of 5.5 shows that f is an isometry and $f_* = h$.

Finally, let X be an object of $\mathbf{U_I}$, that is, X is a complete ultrametric space of diameter $d_0 \leq 1$. We need an object (T, v) of $\mathbf{T_{RI}}$ such that $end(T, v)$ is isometric to X. If $d_0 = 0$, then we may take $(T, v) = ([0, \infty), 0)$. Thus, assume $0 < d_0 \leq 1$ and let $T_X = (X \times [-\ln d_0, \infty))/\sim$ be the tree of Definition 6.1 with root

$r_X = [x, -\ln d_0]$ (for each $x \in X$). Attach the interval $I = [0, -\ln d_0]$ to T_X by identifying $-\ln d_0 \in I$ with $r_X \in T_X$ to form the tree $T = I \cup T_X$ with root 0 and the natural metric so that I and T_X are isometrically embedded in T. An isometry $h : X \to end(T, 0)$ may be defined by

$$h(x)(t) = \begin{cases} t & \text{if } 0 \le t \le -\ln d_0 \\ [x, t] & \text{if } -\ln d_0 \le t < \infty. \end{cases}$$

An argument similar to that given for Proposition 6.4 shows that h is an isometry. □

Definition 8.6.

(1) If (X, d) is a metric space, let $Isom(X)$ denote the group of all isometries from X to itself.
(2) If (T, v) is a rooted $\mathbb{R}$-tree, let $Isom(T, v)$ denote the group of all rooted isometries from (T, v) to itself.

Note that if (T, v) is a rooted $\mathbb{R}$-tree, then $Isom(T, v)$ is a subgroup of $Isom(T)$.

Of the results and ideas in this paper that are already known, perhaps the following is the most well-known.

Corollary 8.7. *Let (T, v), (S, w) be geodesically complete, rooted $\mathbb{R}$-trees and let X, Y be ultrametric spaces of diameter ≤ 1.*

(1) *$Isom(T, v)$ is isomorphic to $Isom(end(T, v))$.*
(2) *(T, v) and (S, w) are rooted isometric if and only if $end(T, v)$ and $end(S, w)$ are isometric.*
(3) *If X and Y are isometric, then (T_X, r_X) and (T_Y, r_Y) are rooted isometric.*

Proof. (1) and (2) follow from the fact that $\mathcal{E}_{\mathbf{RI}}$ is an equivalence. (3) follows immediately from Definition 6.1. □

The converse of 8.7(3) is not true. For example, if $X = \{a, b\}$ has any metric, then T_X is isometric to $\mathbb{R}$.

The discussion now turns to strong uniform isometries at infinity of $\mathbb{R}$-trees and local isometries of ultrametric spaces.

Definition 8.8. A homeomorphism $h : X \to Y$ between metric spaces is a *local isometry equivalence* if there exist $\epsilon > 0$ such that for every $x \in X$ the restriction $h| : B(x, \epsilon) \to B(h(x), \epsilon)$ is an isometry.

Definition 8.9. If (X, d) is a metric space, let $LI(X)$ denote the *group of local isometry equivalences from X to itself*.

Proposition 8.10. *$LI(X)$ is a subgroup of $LSE^u(X)$.*

Proof. The proof of Lemma 4.5 shows that inverses and compositions of local isometry equivalences are local isometry equivalences. □

Note that for any metric space (X, d), there are the following group inclusions:

$$Isom(X) \subseteq LI(X) \subseteq LSE^u(X) \subseteq LSE(X).$$

The second inclusion is often an equality, but none of the others need be. The situation is clarified by the next result and the examples below.

Proposition 8.11. *If (X,d) is a compact metric space, then $LI(X) = LSE^u(X)$.*

Proof. If X is finite, then every self-homeomorphism of X is both a local isometry equivalence and a uniform local similarity equivalence. Hence, we assume, by way of contradiction, that X is infinite and $h : X \to X$ is a uniform local similarity equivalence that is not a local isometry equivalence. Let $\epsilon > 0$ and $\lambda > 0$ be given as in Definition 7.7. We may assume $\lambda < 1$ (otherwise consider h^{-1}). Choose distinct points $x_1, \ldots, x_N \in X$ such that $\{B(x_i, \epsilon) \mid i = 1, \ldots, N\}$ covers X. It follows that for each $n = 1, 2, 3, \ldots$, $\{B(h^n(x_i), \lambda^n \epsilon) \mid i = 1, \ldots, N\}$ also covers X. It follows from this that the cardinality of X is at most N. For suppose $y_1, \ldots, y_{N+1} \in X$ are distinct and let $\delta = \min\{d(y_j, y_k) \mid j \neq k\}$. Choose n such that $\lambda^n < \delta/2$. Then there exist $i \in \{1, \ldots, N\}$ and $j, k \in \{1, \ldots, N+1\}$, $j \neq k$, such that $y_j, y_k \in B(h^n(x_i), \lambda^n \epsilon)$. Thus, $d(y_j, y_k) < 2\lambda^n \epsilon < \delta$, a contradiction. □

Examples 8.12.
(1) There is a compact ultrametric space (X,d) such that $Isom(X) \neq LI(X)$. Let $X = \{a, b, c\}$ with metric d satisfying $d(a,b) = d(a,c) = 2$ and $d(b,c) = 1$. Then every bijection of X is a local isometry equivalence, but not all are isometries.
(2) There is a (non-compact) ultrametric space (X,d) such that $LI(X) \neq LSE^u(X)$. Let $X = \{x_i\}_{i=0}^{\infty} \cup \{y_i\}_{i=1}^{\infty}$ and define d by

(i) $d(y_i, z) = 1$ whenever $i \geq 1$ and $z \neq y_i$.
(ii) $d(x_m, x_n) = e^{-n}$ whenever $m \geq n \geq 1$.
(iii) $d(x_0, x_n) = e^{-n}$ whenever $n \geq 1$.

In particular, x_0 is the only non-isolated point of X. Define $h : X \to X$ by

(i) $h(x_0) = x_0$
(ii) $h(y_i) = y_{i+1}$ for all $i \geq 1$
(iii) $h(x_i) = x_{i-1}$ for all $i \geq 2$
(iv) $h(x_1) = y_1$.

Then $h \in LSE^u(X)$ but $h \notin LI(X)$.
(3) There is a compact ultrametric space (X,d) such that $LSE^u(X) \neq LSE(X)$. Let X be the end space of the Fibonacci tree as defined in §9. Then Proposition 9.10 below shows that $LI(X)$ does not equal $LSE(X)$.

Figure 3

Definition 8.13.

(1) An isometry at infinity $(f, C_T, C_S) : (T, v) \to (S, w)$ between geodesically complete, rooted $\mathbb{R}$-trees is a *strong uniform isometry at infinity* provided there exist $r > 0$ such that $C_T = \partial B(v, r)$ and $C_S = \partial B(w, r)$.
(2) Two strong uniform isometries at infinity are *equivalent* provided they are equivalent as isometries at infinity.
(3) If (T, v) is a geodesically complete, rooted $\mathbb{R}$-tree, let $Isom^{su}_{\infty}(T, v)$ denote the *group of equivalence classes of strong uniform isometries at infinity from* (T, v) *to itself*.
(4) Let $\mathbf{T_{su}}$ be the subcategory of $\mathbf{T_u}$ having the same objects, but whose morphisms are the equivalence classes of strong uniform isometries at infinity.
(5) Let $\mathbf{U_{LI}}$ be the subcategory of $\mathbf{U_u}$ with objects complete ultrametric spaces of diameter ≤ 1, and local isometry equivalences for morphisms.

The facts that are implicitly assumed in this definition about compositions of various morphisms are readily verified.

Proposition 8.14.

(1) *The functor* $i : \mathbf{T_{RI}} \to \mathbf{T_u}$ *factors as* $i : \mathbf{T_{RI}} \xrightarrow{j} \mathbf{T_{su}} \hookrightarrow \mathbf{T_u}$.
(2) *The functor* $j : \mathbf{T_{RI}} \to \mathbf{T_{su}}$ *is faithful, but not full.*
(3) *The composition of functors* $\mathbf{T_{su}} \hookrightarrow \mathbf{T_u} \xrightarrow{\mathcal{E}_\mathbf{u}} \mathbf{U_u}$ *has image in* $\mathbf{U_{LI}}$. *Moreover, the induced functor*

$$\mathbf{T_{su}} \xrightarrow{\mathcal{E}_\mathbf{su}} \mathbf{U_{LI}}$$

is an equivalence of categories.

Proof. It is clear that the functor $i : \mathbf{T_{RI}} \to \mathbf{T_u}$ defined immediately after Definition 8.1 takes rooted isometries to equivalence classes of strong uniform isometries at infinity; thus, (1) holds. Example 8.2 shows that j is not full, and the faithfulness of j follows from (1) and Proposition 8.3; thus, (2) holds. For the first part of (3), the first paragraph of the proof of Theorem 7.13 shows that $\mathcal{E}_\mathbf{u}$ takes the equivalence class of a strong uniform isometry at infinity to a local isometry (because $r_1 = r_2$ implies $\lambda = 1$). The faithfulness of $\mathcal{E}_\mathbf{su}$ follows from the faithfulness of $\mathcal{E}_\mathbf{u}$. The fullness of $\mathcal{E}_\mathbf{su}$ follows from the proof of the fullness of $\mathcal{E}_\mathbf{u}$ in 7.13 (because $\lambda = 1$). $\square$

We can now give a commuting diagram that summarizes the various categories and functors studied in this paper.

Corollary 8.15. *There is a commuting diagram of categories and functors in which all vertical arrows are equivalences of categories:*

$$\begin{array}{ccccccc}
\mathbf{T_{RI}} & \longrightarrow & \mathbf{T_{su}} & \longrightarrow & \mathbf{T_u} & \longrightarrow & \mathbf{T} \\
\cong\big\downarrow \mathcal{E}_\mathbf{RI} & & \cong\big\downarrow \mathcal{E}_\mathbf{su} & & \mathcal{E}_\mathbf{u}\big\downarrow\cong & & \mathcal{E}\big\downarrow\cong \\
\mathbf{U_I} & \longrightarrow & \mathbf{U_{LI}} & \longrightarrow & \mathbf{U_u} & \longrightarrow & \mathbf{U}
\end{array} \qquad \square$$

There is a corresponding diagram involving the automorphism groups of objects given in the next and final result of this section.

Corollary 8.16. *Let (T, v) be a geodesically complete, rooted $\mathbb{R}$-tree and let $X = end(T, v)$. Then the following diagram of natural homomorphisms is commutative and the vertical arrows are isomorphisms:*

$$\begin{array}{ccccccc} Isom(T,v) & \longrightarrow & Isom^{su}_{\infty}(T,v) & \longrightarrow & Isom^{u}_{\infty}(T,v) & \longrightarrow & Isom_{\infty}(T,v) \\ \cong\big\downarrow & & \cong\big\downarrow & & \big\downarrow\cong & & \big\downarrow\cong \\ Isom(X) & \longrightarrow & LI(X) & \longrightarrow & LSE^{u}(X) & \longrightarrow & LSE(X) \end{array}$$

Moreover:

1. *All of the horizontal arrows are injective (on the bottom row, they are inclusions).*
2. *None of the horizontal arrows need be surjective.*
3. *If X is compact, then the only horizontal inclusions that need be equalities are the middle ones.*

Proof. The commutativity of the diagram and the fact that the vertical arrows are isomorphisms follow directly from Corollary 8.15. That the horizontal arrows are injective then follows from the fact that the arrows on the bottom row are inclusions of subgroups. The statements about surjectivity follow from Proposition 8.11 and Examples 8.12. □

9. The Cantor tree, the Fibonacci tree and final comments

In this final section we examine the Cantor tree C, the Fibonacci tree F and their end spaces. The trees C and F are bi-Lipschitz homeomorphic, but not isometric at infinity.

The group $Isom(end(C))$ acts transitively on $end(C)$; thus, we think of the groups $Isom(end(C))$, $LI(end(C))$ and $LSE(end(C))$ as being rather large in the sense that the quotient of $end(C)$ by any of these groups reduces to a single point.

On the other hand, the corresponding groups for $end(F)$ are rather small. In fact, $Isom(end(F))$ is trivial. The groups $LI(end(F))$ and $LSE(end(F))$ are non-trivial and the quotients of $end(F)$ by these groups are determined (and found to be non-trivial).

Definition 9.1. The Cantor tree C and its end space $end(C)$. The Cantor tree C, also called the infinite binary tree, is a locally finite, simply connected one-dimensional simplicial complex (with the natural length metric d so that every edge is of length 1). It has a root r of valency two (i.e., there exists exactly two edges containing r) and every other vertex is of valency three. If v is a vertex different from r, then the two edges that contain v and are separated from r by v are not labelled identically. Each edge is labelled 0 or 1 so that for every vertex v, at least one edge containing v is labelled 0 and at least one is labelled 1.

Let $end(C) = end(C, r)$ since the root r is understood. An element of $end(C)$, being an infinite sequence of successively adjacent edges in C beginning at r, can be labelled uniquely by an infinite sequence of 0's and 1's. Thus,

$$end(C) = \{(x_0, x_1, x_2, \dots) \mid x_i \in \{0, 1\} \text{ for each } i\}$$

and

$$d_e((x_i), (y_i)) = \begin{cases} 0 & \text{if } (x_i) = (y_i) \\ 1/e^n & \text{if } (x_i) \neq (y_i) \text{ and } n = \inf\{i \geq 0 \mid x_i \neq y_i\} \end{cases} .$$

Definition 9.2. The Fibonacci tree F and its end space $end(F)$. The Fibonacci tree F is a subtree of C with the same root r and labelling scheme. In F, only edges labelled 0 are allowed to follow edges labelled 1 as one moves away from the root. Thus,

$$end(F) = \{(x_0, x_1, x_2, \dots) \in end(C) \mid x_i = 1 \text{ implies } x_{i+1} = 0\}.$$

Recall the following definitions.

Definition 9.3. A homeomorphism $h : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is:

(1) a *bi-Lipschitz homeomorphism* if there exist constants $c_1, c_2 > 0$ such that for every $x, y \in X$,

$$c_1 d_X(x, y) \le d_Y(h(x), h(y)) \le c_2 d_X(x, y).$$

(2) a *bi-Hölder homeomorphism* if there exist constants $c_1, c_2, \alpha_1, \alpha_2 > 0$ such that for every $x, y \in X$,

$$c_1 [d_X(x, y)]^{\alpha_1} \le d_Y(h(x), h(y)) \le c_2 [d_X(x, y)]^{\alpha_2}.$$

(3) *quasi-conformal* if there exists a constant $k \in \mathbb{R}$ such that for every $x \in X$,[4]

$$\limsup_{r\to 0} \frac{\sup\{d_Y(h(x), h(y) \mid d_X(x, y) = r\}}{\inf\{d_Y(h(x), h(y) \mid d_X(x, y) = r\}} \le k.$$

(4) *conformal* if for every $x \in X$,

$$\limsup_{r\to 0} \frac{\sup\{d_Y(h(x), h(y) \mid d_X(x, y) = r\}}{\inf\{d_Y(h(x), h(y) \mid d_X(x, y) = r\}} = 1.$$

Proposition 9.4. *There exists a bi-Lipschitz homeomorphism $h : C \to F$ that induces a conformal bi-Hölder homeomorphism $\hat{h} : end(C) \to end(F)$.*

Since a bi-Lipschitz homeomorphism is a special case of a quasi-isometry, the fact that $\hat{h} : end(C) \to end(F)$ in the proposition is a quasi-conformal bi-Hölder homeomorphism follows from the general theory of Ghys and de la Harpe [GdH], but I include an explicit proof for purposes of illustration (and because of the slightly stronger conclusion of conformality instead of quasi-conformality in this special case).

Proof of 9.4. The homeomorphism $h : C \to F$ is defined by introducing a vertex at the midpoint of each edge e of C that is labelled 1. The new edge created from e that is closest to the root is labelled 1; the other new edge is labelled 0. This new tree created from C can be naturally identified with F and h is the resulting homeomorphism. In particular, if e is an edge of C labelled 0, then $h|e$ is an

[4]In taking these $\limsup$'s, we assume that no point of X is isolated and that we only consider those r's for which $\{y \in X \mid d_X(x, y) = r\} \ne \emptyset$. This consideration applies to the definition of conformal as well. See [GdH].

isometric embedding; if e is an edge of C labelled 1, then $h|e$ multiplies distances by 2. Thus,

$$d(x,y) \le d(h(x),h(y)) \le 2d(x,y)$$

for every $x, y \in C$, showing that h is bi-Lipschitz.

The induced map $\hat{h} : end(C) \to end(F)$ is given by

$$\hat{h}(x_0, x_1, x_2, \dots) = (x_0, \epsilon_0, x_1, \epsilon_1, x_2, \epsilon_2, \dots)$$

where

$$\begin{cases} \epsilon_i = 0 & \text{if } x_i = 1 \\ \epsilon_i \text{ is the empty symbol} & \text{if } x_i = 0. \end{cases}$$

It is clear that $\hat{h}$ is bijective. To check the other properties, suppose $x, y \in C$ and $d(x,y) = 1/e^n$. Then $x_i = y_i$ for $0 \le i \le n-1$ and $x_n \ne y_n$. It follows that

$$1/e^{2n} \le d_e(h(x),h(y)) \le 1/e^n.$$

Moreover, $d_e(h(x),h(y))$ depends only on the number of 1's in $(x_0, x_1, x_2, \dots x_{n-1})$. This implies

$$[d_e(x,y)]^2 \le d_e(h(x),h(y)) \le d_e(x,y)$$

for all $x, y \in C$, so h is a bi-Hölder homeomorphism. Moreover, $d(h(x),h(y)) = d(h(x),h(z))$ if $d(x,y) = d(x,z)$, which implies that

$$\sup\{d(h(x),h(z)) \mid d(x,z) = 1/e^n\} = \inf\{d(h(x),h(z)) \mid d(x,z) = 1/e^n\}$$

and h is conformal. □

In contrast to Proposition 9.4, the next result points out a difference between $end(C)$ and $end(F)$ by comparing their groups of isometries.

Proposition 9.5.

(1) *$end(C)$ is isometrically homogeneous; i.e., $Isom(end(C))$ acts transitively on $end(C)$.*
(2) *$end(F)$ is rigid; i.e., $Isom(end(F)) = \{1\}$.*

Proof. We begin by showing that if $x, y \in C$, then there exists an isometry $h : C \to C$ such that $h(x) = y$. For $z \in C$, define $h(z)$ by

$$(h(z))_i = \begin{cases} z_i & \text{if } x_i = y_i \\ 1 - z_i & \text{if } x_i \ne y_i. \end{cases}$$

To show that F is rigid, we need a lemma. First note that if $x \in F$ and $n \ge 0$, then

$$B(x, 1/e^n) = \{z \in F \mid z_i = x_i \text{ for all } i \le n\}.$$

Lemma 9.5.1. *If $x, y \in F$, $n \geq 0$ and $f : B(x, 1/e^n) \to B(y, 1/e^n)$ is an isometry such that $f(x) = y$, then $x_i = y_i$ for all $i \geq n$.*

Proof of 9.5.1. Suppose on the contrary that there exists $i \geq n$ such that $x_i \neq y_i$. Without loss of generality, assume that $x_i = 0$ and $y_i = 1$. Let

$$z = (x_1, x_2, \dots, x_i, 0, 0, 0, \dots)$$

and $w = (x_1, x_2, \dots, x_i, 1, 0, 0, \dots)$. Note that $z, w \in F$, $d(z, w) = 1/e^{(i+1)}$, $d(x, z) \leq 1/e^{(i+1)}$ and $d(x, w) \leq 1/e^{(i+1)}$. In particular, $z, w \in B(x, 1/e^n)$ (because $1/e^{(i+1)} < 1/e^n$). Since $f(z), f(w) \in B(y, 1/e^n) \subseteq F$, $d(y, f(z)) = d(f(x), f(z)) \leq 1/e^{(i+1)}$ and $d(y, f(w)) = d(f(x), f(z)) \leq 1/e^{(i+1)}$, we must have

$$f(z) = (y_0, y_1, \dots, y_i, 0, z'_{i+2}, z'_{i+3}, \dots)$$

and $f(w) = (y_0, y_1, \dots, y_i, 0, w'_{i+2}, w'_{i+3}, \dots)$. Thus, $d(f(z), f(w)) \leq 1/e^{(i+2)}$, contradicting the fact that f is an isometry (since $d(z, w) = 1/e^{(i+1)}$). □

To see how the lemma implies that F is rigid, suppose $h : F \to F$ is a isometry and $x \in F$. Then $h| : B(x, 1/e^0) \to B(h(x), 1/e^0)$ is an isometry, so 9.5.1 implies $x_i = h(x)_i$ for all $i \geq 0$; that is, $x = h(x)$. □

Since $Isom(end(C))$ acts transitively on $end(C)$, so do the larger groups $LI(end(C))$ and $LSE(end(C))$. The situation is different for $end(F)$ as we now begin to describe.

Lemma 9.6. *If $x \in F$ and $m \geq 0$, then there exists a similarity equivalence $h : F \to B(x, 1/e^m)$.*

Proof. Define h by

$$h(z) = \begin{cases} (x_0, \dots, x_m, z_0, z_1, z_2, \dots) & \text{if } x_m = 0 \\ (x_0, \dots, x_m, 0, z_0, z_1, z_2, \dots) & \text{if } x_m = 1. \end{cases}$$

One can check that h is a λ-similarity equivalence with

$$\lambda = \begin{cases} e^{-m} & \text{if } x_m = 0 \\ e^{-m-1} & \text{if } x_m = 1. \end{cases} \qquad \square$$

Lemma 9.7. *Let $x, y \in F$ and $m, n \geq 0$. Then there exists a unique similarity equivalence $f : B(x, 1/e^m) \to B(y, 1/e^n)$. Moreover, the similarity constant of any such similarity equivalence is given by*

$$\lambda = \begin{cases} e^{m-n} & \textit{if } x_m = y_n \\ e^{m-n-1} & \textit{if } x_m = 0, y_n = 1 \\ e^{m-n+1} & \textit{if } x_m = 1, y_n = 0. \end{cases}$$

Proof. We first show how to define the similarities in the three cases:

(1) $x_m = y_n$. If $z \in B(x, 1/\mathrm{e}^m)$, then $z = (x_0, \dots, x_m, z_{m+1}, z_{m+2}, \dots)$ Define $f(z) = (y_0, \dots, y_n, z_{m+1}, z_{m+2}, \dots)$. The condition $x_m = y_n$ guarantees that $f(z) \in F$.

(2) $x_m = 0, y_n = 1$. If $z \in B(x, 1/\mathrm{e}^m)$, then $z = (x_0, \dots, x_m, z_{m+1}, z_{m+2}, \dots) = (x_0, \dots, x_{m-1}, 0, z_{m+1}, z_{m+2}, \dots)$. Define

$$f(z) = (y_0, \dots, y_n, 0, z_{m+1}, z_{m+2}, \dots) = (y_0, \dots, y_{n-2}, 0, 1, 0, z_{m+1}, z_{m+2}, \dots).$$

(3) $x_m = 1, y_n = 0$. If $z \in B(x, 1/\mathrm{e}^m)$, then $z = (x_0, \dots, x_m, z_{m+1}, z_{m+2}, \dots) = (x_0, \dots, x_{m-2}, 0, 1, 0, z_{m+2}, z_{m+3}, \dots)$. Define

$$f(z) = (y_0, \dots, y_n, z_{m+2}, z_{m+3}, \dots) = (y_0, \dots, y_{n-1}, 0, z_{m+2}, z_{m+3}, \dots).$$

It is easy to see that the similarities just defined have similarity constants as in the statement above. But we must now observe that any similarity homeomorphism will have such a constant. In general, note that if $h : X \to Y$ is any λ-similarity equivalence between metric spaces of finite diameter, then $\lambda = \operatorname{diam} X / \operatorname{diam} Y$. It follows that, in the finite diameter case, the similarity constant is uniquely determined. Thus, the similarity constants must be given as in the statement of the lemma.

It remains to show that f is unique. Suppose $g : B(x, 1/\mathrm{e}^m) \to B(y, 1/\mathrm{e}^n)$ is another similarity homeomorphism. Since f and g must have the same similarity constants, it follows that $g^{-1}f$ is an isometry of $B(x, 1/\mathrm{e}^m)$. By Lemma 9.6 there is a similarity homeomorphism $h : F \to B(x, 1/\mathrm{e}^m)$. Hence, $h^{-1}g^{-1}fh : F \to F$ is an isometry. Rigidity of F (Proposition 9.5(2)) implies that $h^{-1}g^{-1}fh = \mathrm{id}_F$. Hence $g = f$. □

Lemma 9.8. *Let $x, y \in F$ and $m, n \geq 0$ There exists a similarity equivalence $f : B(x, 1/e^m) \to B(y, 1/e^n)$ with $f(x) = y$ if and only if*

$$\begin{cases} x_{m+k} = y_{n+k} \text{ for all } k \geq 1 & \text{if } x_m = y_n \\ x_{m+k-1} = y_{n+k} \text{ for all } k \geq 1 & \text{if } x_m = 0, y_n = 1 \\ x_{m+k} = y_{n+k-1} \text{ for all } k \geq 1 & \text{if } x_m = 1, y_n = 0. \end{cases}$$

Proof. If such a similarity f exists, then it is unique by Lemma 9.7. The explicit description given in the proof of 9.7 allows a comparison of $f(x)$ and y in each of the three cases:

(1) $x_m = y_n \Rightarrow f(x) = (y_0, \dots, y_n, x_{m+1}, x_{m+2} \dots) = y \Rightarrow y_{n+k} = x_{m+k}$ for all $k \geq 1$.

(2) $x_m = 0$, $y_n = 1 \Rightarrow f(x) = (y_0, \dots, y_n, y_{n+1} = x_m, x_{m+1}, x_{m+2} \dots) = y \Rightarrow y_{n+k} = x_{m+k-1}$ for all $k \geq 1$.

(3) $x_m = 1$, $y_n = 0 \Rightarrow f(x) = (y_0, \dots, y_{n-1}, y_n = x_{m+1}, x_{m+2} \dots) = y \Rightarrow y_{n+k-1} = x_{m+k}$ for all $k \geq 1$.

Conversely, in each of these three cases the proof of 9.7 gives an explicit unique similarity equivalence $f : B(x, 1/e^m) \to B(y, 1/e^n)$. By examination of these three cases, the given relations betweeen x and y and the definition of f show that $f(x) = y$. □

Definition 9.9. Two sequences $(x_0, x_1, x_2, \dots)$ and $(y_0, y_1, y_2, \dots)$ are *eventually equal* (or *tail equivalent*) if there exists an integer $n \geq 0$ such that $x_i = y_i$ for all $i \geq n$. They are *eventually equal with lag* (or *eventually shift equivalent*) if there exist integers $m, n \geq 0$ such that $x_{m+j} = y_{n+j}$ for all $j \geq 0$.

Proposition 9.10.

(1) *If $x, y \in end(F)$, then there exists $h \in LI(end(F))$ such that $h(x) = y$ if and only if x and y are eventually equal.*
(2) *If $x, y \in end(F)$, then there exists $h \in LSE(end(F))$ such that $h(x) = y$ if and only if x and y are eventually equal with lag.*

Proof. (1) If $x, y \in end(F)$ and there exists a local isometry equivalence $h : end(F) \to end(F)$ such that $h(x) = y$, then Lemma 9.5.1 shows that x and y are eventually equal.

Conversely, suppose $x, y \in end(F)$ and $x_i = y_i$ for all $i \geq n$. Then we can write $x = (x_0, x_1, \dots, x_{n-1}, x_n, x_{n+1}, \dots)$ and $y = (y_0, y_1, \dots, y_{n-1}, x_n, x_{n+1}, \dots)$. Let $z \in B(x, 1/e^n)$. Then $d(x, z) \leq 1/e^{(n+1)}$ and $z = (x_0, x_1, \dots, x_{n-1}, x_n, z_{n+1}, z_{n+2}, \dots)$. Define $f : B(x, 1/e^n) \to B(y, 1/e^n)$ by $f(z) = (y_0, y_1, \dots, y_{n-1}, x_n, z_{n+1}, z_{n+2}, \dots)$. Then define $h : end(F) \to end(F)$ by

$$h(w) = \begin{cases} f(w) & \text{if } w \in B(x, 1/e^n) \\ f^{-1}(w) & \text{if } w \in B(y, 1/e^n) \setminus B(x, 1/e^n) \\ w & \text{if } w \notin B(x, 1/e^n) \cup B(y, 1/e^n). \end{cases}$$

It can be checked that h is a local isometry equivalence with $h(x) = y$. For this it is helpful to recall that $B(x, 1/e^n) = B(y, 1/e^n)$ or $B(x, 1/e^n) \cap B(y, 1/e^n) = \emptyset$ (Proposition 4.2).

(2) If $x, y \in end(F)$ and there exists a local similarity equivalence $h : end(F) \to end(F)$ such that $h(x) = y$, then Lemma 9.8 shows that x and y are eventually equal with lag.

Conversely, suppose $x, y \in end(F)$ and there exist integers $m, n \geq 0$ such that $x_{m+j} = y_{n+j}$ for all $j \geq 0$. We may assume that $x \neq y$ (otherwise the result is trivial) and that m, n are so large that $B(x, 1/e^m) \cap B(y, 1/e^n) = \emptyset$. Lemma 9.8 implies that there exists a similarity equivalence $f : B(x, 1/e^m) \to B(y, 1/e^n)$ with $f(x) = y$. Then define $h : end(F) \to end(F)$ by

$$h(w) = \begin{cases} f(w) & \text{if } w \in B(x, 1/e^m) \\ f^{-1}(w) & \text{if } w \in B(y, 1/e^n) \\ w & \text{if } w \notin B(x, 1/e^m) \cup B(y, 1/e^n). \end{cases}$$

It can be checked that h is a local similarity equivalence and $h(x) = y$. □

The consequence of the difference between local similarity equivalnce for $end(C)$ and $end(F)$ is summarized in the following final result. The point is that C and F have different geometry at infinity even though they have the same asymptotic (or, large-scale) geometry.

Corollary 9.11. *There exists no local similarity equivalence $end(C) \to end(F)$, and there exists no isometry at infinity $C \to F$.*

Proof. The first part follows from the fact that, on the one hand, $LSE(end(C))$ contains $Isom(end(C))$, and so $LSE(end(C))$ acts transitively on $end(C)$ (Proposition 9.5), but, on the other hand, $LSE(end(F))$ does not act transitively on $end(F)$ (Prposition 9.10). The second part follows from the first part and the Main Theorem. □

Of course, this corollary also follows from the fact that $end(C)$ and $end(F)$ are not bi-Lipschitz equivalent. This is because every local isometry equivalence between compact metric spaces is easily seen to be a bi-Lipschitz homeomorphism. An elementary calculation of Hausdorff dimension shows that $end(C)$ and $end(F)$ are not bi-Lipschitz equivalent (Hausdorff dimension being a bi-Lipschitz invariant).

If one looks at the treatment of the space of Penrose tilings in Connes [Con], then one sees why noncommutative geometry is relevant here. The quotient

$$end(F)/LI(end(F))$$

is exactly the same as the space of Penrose tilings and Connes uses it as a motivating example. The basic idea is that the quotients $end(F)/LI(end(F))$ and $end(F)/LSE(end(F))$ are too pathological to study by classical topological methods; they should be viewed as noncommutative spaces.

The theory of groupoids is relevant because what we are detecting in $LI(end(T))$ and $LSE(end(T))$ are non-global symmetries of a tree T. Groupoids are to non-global symmetries as groups are to symmetries (see Weinstein [Wei]).

See [Hug] for more information about how to use noncommutative geometry and groupoids to analyze local isometries.

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